PETROV NONCOMMUTATIVE TOPOLOGICAL QUANTUM FIELD THEORY

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This paper gives a definition of category **NC-Einst** of noncommutative Einstein spaces, and a Petrov noncommutative topological quantum field theory (NC TQFT) is constructed. We suggest extensions of these ideas which may be useful to further NC TQFT and apply it in higher dimensions.

1. Introduction

The subjects of the double category, TQFT, and noncommutative Einstein spaces have been studied in [1–13]. Let us describe some noncommutative geometric aspects of twisted deformations. Consider a Lie algebra g over $\mathbb{C}$, and its associated universal enveloping algebra Ug. A general twist $\mathcal{F}$ is an element $\mathcal{F} \in Ug \otimes Ug$ in the tensor product of a Hopf algebra $(Ug, \cdot, \Delta, S, \varepsilon)$ given by

$$\mathcal{F} = f^\alpha \otimes f_\alpha,\ \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha\,, \tag{1}$$

and satisfying the conditions

$$\mathcal{F}_{12}(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}(id \otimes \Delta)\mathcal{F}\,, \tag{2}$$

$$(\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}\,, \tag{3}$$

where the elements $f^\alpha, f_\alpha, \bar{f}^\alpha, \bar{f}_\alpha$ belong to Ug, Δ denotes the coproduct and ϵ the co-unit of the respective Hopf algebra [14–16].

Then, the universal $\mathcal{R}$ matrix is defined by

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} = R^\alpha \otimes R_\alpha,\ \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha\,. \tag{4}$$

Using the $\mathcal{R}$ matrix, we obtain, for functions h and g,

$$h \star g = \bar{R}^\alpha(g) \star \bar{R}_\alpha(h)\,. \tag{5}$$

Our strategy is to deform a product $\circ$ of some objects A and B by replacing it with a twisted product $\circ_\star$:

$$A \circ_\star B := \bar{f}^\alpha(A) \circ \bar{f}_\alpha(B)\,. \tag{6}$$

The universal enveloping algebra of vector fields can be deformed in two different ways:

– $U\Xi_\star$

This is a Hopf algebra [16] defined by deforming the structure functions of $U\Xi$:

$$u \star v = \bar{f}^\alpha(u)\bar{f}_\alpha(v)\,, \tag{7}$$

$$\Delta_\star(u) = u \otimes \mathbf{1} + \bar{R}^\alpha \otimes \bar{R}_\alpha(u)\,, \tag{8}$$

$$\epsilon_\star(u) = \epsilon(u) = 0\,, \tag{9}$$

$$S_\star(u) = -\bar{R}^\alpha(u)\bar{R}_\alpha\,, \tag{10}$$

where $\bar{R}^\alpha(u)$ is the usual Lie derivative of u along the vector field $\bar{R}^\alpha$.

There is a natural action of $\Xi_\star$ on the algebra of functions $\mathcal{A}_\star$ given in terms of the usual undeformed Lie derivative,

$$\mathcal{L}^\star_u(h) := \bar{f}^\alpha(u)(\bar{f}_\alpha(h))\,, \tag{11}$$

which can be extended to $U\Xi_\star$.

The $\star$-Lie algebra of vector fields $\Xi_\star$ generates the Hopf algebra $U\Xi_\star$.

– $U\Xi^\mathcal{F}$

We have the following structure maps:

$$u \cdot^\mathcal{F} v = u \cdot v\,, \tag{12}$$

$$S^\mathcal{F}(u) = S(u)\,, \tag{13}$$

$$\epsilon^\mathcal{F}(u) = \epsilon(u)\,, \tag{14}$$

$$\Delta^\mathcal{F}(u) = \mathcal{F}\Delta(u)\mathcal{F}^{-1}\,. \tag{15}$$

However, $U\Xi_\star$ and $U\Xi^\mathcal{F}$ turn out to be isomorphic Hopf algebras.

The star-connection $\nabla^\star$ is defined to satisfy the following axioms:

$$\nabla^*_{u+v} z = \nabla^*_u z + \nabla^*_v z \,,$$

$$\nabla_{h\star u} v = h \star \nabla^*_u v \,,$$

$$\nabla^*_u (h \star v) = \mathcal{L}^*_u(h) \star v + \bar{R}^\alpha(h) \star \nabla^*_{\bar{R}_\alpha(u)} v \,, \tag{16}$$

where u, v and z are vector fields. Next, we define the connection coefficients by

$$\nabla^\star_\mu \hat{\partial}_\nu := \Gamma^\sigma_{\mu\nu} \star \hat{\partial}_\sigma \,, \tag{17}$$

using the basis $\{\hat{\partial}_\mu\}$. The action of the covariant derivative on a one-form can be obtained employing the star-dual pairing of a vector field v with a one-form ω,

$$\nabla^*_u \langle v, w\rangle_\star = \mathcal{L}^*_u \langle v, w\rangle_\star =$$

$$= \langle \nabla^*_u v, w\rangle_\star + \langle \bar{R}^\alpha(v), \nabla^*_{\bar{R}_\alpha(u)} w\rangle_\star, \tag{18}$$

which can be written equivalently as

$$\langle v, \nabla^*_u w\rangle_\star = \mathcal{L}_{\bar{R}^\alpha(u)} \langle \bar{R}_\alpha(v), w\rangle_\star -$$

$$-\langle \nabla^*_{\bar{R}^\alpha(u)} (\bar{R}_\alpha(v)), w\rangle_\star \,. \tag{19}$$

For a given metric

$$g = g_{\mu\nu} \star d\hat{x}^\mu \otimes_\star d\hat{x}^\nu \,, \tag{20}$$

the connection that leaves it invariant is called a Levi-Civita connection:

$$\nabla^\star_\mu g = 0. \tag{21}$$

For a general twist $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha$, the torsion and curvature tensors are given by [13]

$$T(u, v) = \nabla^*_u v - \nabla^*_{\bar{R}^\alpha(v)} \bar{R}_\alpha(u) - [u, v]_*, \tag{22}$$

$$R(u, v, z) \equiv R(u, v) z =$$

$$= \nabla^*_u \nabla^*_v z - \nabla^*_{\bar{R}^\alpha(v)} \nabla^*_{\bar{R}_\alpha(u)} z - \nabla^*_{[u,v]_*} z \,. \tag{23}$$

It is enough to calculate the tensor on a basis $\hat{\partial}_\mu$ because of the tensorial property, i.e.,

$$T(u, v) = u^\nu \star T(\hat{\partial}_\nu, \hat{\partial}_\mu) \star v^\mu. \tag{24}$$

In this frame, the star-connection is given by

$$\nabla^*_z u = \mathcal{L}^*_z(u^\nu) * \hat{\partial}_\nu + \bar{R}^\alpha(u^\nu) * \bar{R}_\alpha(z)^\mu * \Gamma^\sigma_{\mu\nu} * \hat{\partial}_\sigma \,. \tag{25}$$

We will need to compute the components of the curvature tensor in this base. They can be expressed in the following way:

$$R_{ijk}{}^l = \langle R(\hat{\partial}_i, \hat{\partial}_j, \hat{\partial}_k),\, d\hat{x}^k\rangle_* \,. \tag{26}$$

Consequently, we have, for the deformed Ricci tensor,

$$R_{ij} = R_{ijk}{}^k \,. \tag{27}$$

Classical Einstein spaces have a Ricci tensor proportional to the metric. In the noncommutative case, we are looking for spaces satisfying the same property:

$$R_{ij} = c g_{ij},$$

where c is some constant.

2. Noncommutative Einstein Spaces

2.1. Weyl–Moyal plane $\mathbb{R}^4_\theta$

The metric is the usual Minkowski or Euclidean one; the twist is Abelian [16]:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}, \tag{28}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{R}$. The covariant derivative is given by

$$\nabla^*_z u = z^\mu \star \partial_\mu(u^\nu) \star \partial_\nu + z^\mu \star u^\nu \star \Gamma^\sigma_{\mu\nu} \star \partial_\sigma \,. \tag{29}$$

In a first step, let us show that the choice $\Gamma^\sigma_{\mu\nu} = 0$ is a good choice and renders the affine connection to be a Levi-Civita connection. Thus, the expression for the covariant derivative (29) becomes

$$\nabla^*_z u = z^\mu \star \partial_\mu(u^\nu) \star \partial_\nu \,. \tag{30}$$

Let us show that axioms (16) are satisfied:

- $$\nabla^*_{u+v} z = (u+v)^\mu \star \partial_\mu(z^\nu) \star \partial_\nu = \nabla^*_u z + \nabla^*_v z \,, \tag{31}$$

- $$\nabla_{h\star u} v = (h \star u^\mu) \star \partial_\mu(v^\nu) \star \partial_\nu$$

  $$= h \star (u^\mu \star \partial_\mu v^\nu \star \partial_\nu) = h \star \nabla^*_u v \,, \tag{32}$$

- $$\nabla^*_u (h \star v) = u^\mu \star \partial_\mu(h \star v^\nu) \star \partial_\nu =$$

$$= \mathcal{L}_u^*(h) \star v + u^\mu \star h \star (\partial_\mu v^\nu) \star \partial_\nu =$$

$$= \mathcal{L}_u^*(h) \star v + \bar{R}^\alpha(h) \star \bar{R}_\alpha(u^\mu) \star (\partial_\mu v^\nu) \star \partial_\nu =$$

$$= \mathcal{L}_u^*(h) \star v + \bar{R}^\alpha(h) \star \nabla^*_{\bar{R}_\alpha(u)} v \,. \qquad (33)$$

In a next step, we show that the curvature and the torsion vanish. The torsion is given by

$$T(\partial_\mu, \partial_\nu) = \nabla^*_\mu \partial_\nu - \nabla^*_\nu \partial_\mu - [\partial_\mu, \partial_\nu]_* = 0 \,, \qquad (34)$$

since the Christoffel symbols are all zero, and the derivatives commute. Similarly, we see that the curvature tensor also vanishes:

$$R(\partial_\nu, \partial_\beta, \partial_\mu) =$$

$$= \nabla^*_\nu \nabla^*_\beta \partial_\mu - \nabla^*_{\bar{R}^\alpha(\partial_\beta)} \nabla^*_{\bar{R}_\alpha(\partial_\nu)} \partial_\mu - \nabla^*_{[\partial_\nu, \partial_\beta]_*} z = 0 \,. \quad (35)$$

At last, we consider the covariant derivative of the metric:

$$\nabla^*_\mu g = \nabla^*_\mu (g_{\alpha\beta}\, dx^\alpha \otimes_* dx^\beta) =$$

$$= \partial_\mu (g_{\alpha\beta}) dx^\alpha \otimes_* dx^\beta - g_{\alpha\beta} \Gamma^\alpha_{\mu\sigma} dx^\sigma \otimes_* dx^\beta -$$

$$- g_{\alpha\beta} dx^\alpha \otimes_* \Gamma^\beta_{\mu\sigma} dx^\sigma = 0 \,, \qquad (36)$$

since the star-dual pairing (19) yields

$$\nabla^*_\mu dx^\alpha = -\Gamma^\alpha_{\mu\sigma} \star dx^\sigma = 0 \,.$$

Among these metrics, those that are classically Einstein metrics are also shown to be noncommutative Einstein metrics.

2.2. $\mathbb{R}^5_q$

The algebra is generated by the coordinates $\hat{x}^1, \ldots, \hat{x}^5$ satisfying the relations [16]

$$\hat{x}^1 \hat{x}^2 = q\, \hat{x}^2 \hat{x}^1, \;\; \hat{x}^1 \hat{x}^4 = q^{-1} \hat{x}^4 \hat{x}^1,$$

$$\hat{x}^1 \hat{x}^5 = \hat{x}^5 \hat{x}^1, \quad \hat{x}^2 \hat{x}^4 = \hat{x}^4 \hat{x}^2,$$

$$\hat{x}^2 \hat{x}^5 = q\, \hat{x}^5 \hat{x}^2, \;\; \hat{x}^4 \hat{x}^5 = q^{-1} \hat{x}^5 \hat{x}^4. \qquad (37)$$

The coordinate $\hat{x}^3$ is central. The conjugation is given by

$$\hat{x}^{1*} = \hat{x}^5, \; \hat{x}^{2*} = \hat{x}^4, \; \hat{x}^{3*} = \hat{x}^3 \,.$$

Hence, the twist (for the symmetric ordering) reads

$$\mathcal{F} = \exp\left(\frac{ih}{2} (\chi_1 \otimes \chi_2 - \chi_2 \otimes \chi_1) \right) , \qquad (38)$$

where χ_1 and χ_2 are the following commuting vector fields:

$$\chi_1 = x^2 \partial_2 - x^4 \partial_4, \;\; \chi_2 = x^1 \partial_1 - x^5 \partial_5 \,.$$

Thus, we have, for the inverse $\mathcal{R}$ matrix,

$$\mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha = f^\alpha \bar{f}_\beta \otimes f_\alpha \bar{f}^\beta =$$

$$= \sum (-1)^{m+k-l} (\frac{h}{2})^{n+k} \frac{\binom{n}{m}\binom{k}{l}}{n!k!} \chi_1^{n-m+l} \chi_2^{m+k-l} \otimes$$

$$\otimes \chi_1^{m+k-l} \chi_2^{n-m+l} \,. \qquad (39)$$

2.2.1. *Note on Hermitian generators*

Let us introduce Hermitian generators for the algebra $\mathbb{R}^5_q$:

$$\hat{x}_1 = \hat{z}_1 + i\hat{z}_2, \;\; \hat{x}_5 = \hat{z}_1 - i\hat{z}_2$$

$$\hat{x}_2 = \hat{y}_1 + i\hat{y}_2, \;\; \hat{x}_4 = \hat{y}_1 - i\hat{y}_2 \,, \qquad (40)$$

with $\hat{y}_i^* = \hat{y}_i$ and $\hat{z}_i^* = \hat{z}_i$, $i = 1, 2$. Inserting these identifications into the commutation relations (37) yields the identical relations

$$\hat{z}_1 \hat{y}_1 = q\, \hat{y}_1 \hat{z}_1, \;\; \hat{z}_1 \hat{y}_2 = q^{-1} \hat{y}_2 \hat{z}_1,$$

$$\hat{z}_1 \hat{z}_2 = \hat{z}_2 \hat{z}_1, \quad \hat{y}_1 \hat{y}_2 = \hat{y}_2 \hat{y}_1,$$

$$\hat{y}_1 \hat{z}_2 = q\, \hat{z}_2 \hat{y}_1, \;\; \hat{y}_2 \hat{z}_2 = q^{-1} \hat{z}_2 \hat{y}_2 \,, \qquad (41)$$

in the case where q is a square root of unity.

2.2.2. Geometry

Again, we propose

$$\Gamma^{\mu}_{\alpha\beta} = 0 \tag{42}$$

and show that this definition leads to a sensible covariant derivative and geometric tensors. The covariant derivative (25) is given by

$$\nabla^*_z u = \mathcal{L}^*_z(u^\nu) \star \hat{\partial}_\nu \,. \tag{43}$$

This satisfies the axioms for a affine connection, since

- $$\nabla^*_{u+v} z = \mathcal{L}^*_{u+v}(z^\nu) \star \hat{\partial}_\nu = \mathcal{L}^*_{u}(z^\nu) \star \hat{\partial}_\nu$$
$$+\mathcal{L}^*_{v}(z^\nu) \star \hat{\partial}_\nu = \nabla^*_u z + \nabla^*_v z \tag{44}$$

- $$\nabla^*_{h\star u} v = \mathcal{L}^*_{h\star u}(v^\nu) \star \hat{\partial}_\nu$$
$$= h \star \mathcal{L}^*_{u}(v^\nu) \star \hat{\partial}_\nu = h \star \nabla^*_u(v) \tag{45}$$

- $$\nabla^*_u(h \star v) = \mathcal{L}^*_u(h \star v^\nu) \star \hat{\partial}_\nu = \mathcal{L}^*_u(h) \star v$$
$$+\bar{R}^\alpha(h) \star \mathcal{L}^*_{\bar{R}_\alpha(u)}(v^\nu) \star \hat{\partial}_\nu$$
$$= \mathcal{L}^*(h) \star v + \hat{R}^\alpha(h) \star \nabla^*_{\bar{R}_\alpha(u)}(v) \tag{46}$$

The torsion T is given by

$$T(u,v) = \nabla^*_u v - \nabla^*_{\bar{R}^\alpha(v)} \bar{R}_\alpha(u) - [u,v]_\star$$
$$= \mathcal{L}^*_u(v^\nu) \star \hat{\partial}_\nu - \mathcal{L}^*_{\bar{R}^\alpha(v)}(\bar{R}_\alpha(u)^\nu) \star \hat{\partial}_\nu - [u,v]_\star \tag{47}$$

Computing the torsion for frame elements, we see explicitly that

$$T(\hat{\partial}_\mu, \hat{\partial}_\nu) = 0. \tag{48}$$

This is due to the tensorial property and

$$[\hat{\partial}_\mu\,, \hat{\partial}_\nu]_* = [\bar{f}^\alpha(\hat{\partial}_\mu), \bar{f}_\alpha(\hat{\partial}_\nu)] = 0\,, \tag{49}$$

since the Lie derivative of $\hat{\partial}_\mu$ along $\bar{f}$, and consequently also $\bar{R}$, is again a vector field with constant coefficients: $c^\nu_\mu \hat{\partial}_\nu$, $c^\nu_\mu \in \mathbb{R}$.

Next, we compute the curvature tensor:

$$R(u,v,z) = \nabla^*_u \nabla^*_v z - \nabla^*_{\bar{R}^\alpha(v)} \nabla^*_{\bar{R}_\alpha(u)} z - \nabla^*_{[u,v]_\star} z =$$
$$= \mathcal{L}^*_u(\mathcal{L}^*_v(z^\nu)) \star \partial_\nu - \mathcal{L}^*_{\bar{R}^\alpha(v)}(\mathcal{L}^*_{\bar{R}_\alpha(u)}(z^\nu)) \star \partial_\nu -$$
$$-\mathcal{L}^*_{[u,v]_\star}(z^\nu) \star \partial_\nu = \mathcal{L}^*_{u\star v}(z^\nu) \star \partial_\nu - \mathcal{L}^*_{\bar{R}^\alpha(v)\star\bar{R}_\alpha(u)}(z^\nu) \star \partial_\nu -$$
$$-\mathcal{L}^*_{[u,v]_\star}(z^\nu) \star \partial_\nu = \mathcal{L}^*_{u\star v - \bar{R}^\alpha(v)\star\bar{R}_\alpha(u) - [u,v]_\star}(z^\nu) \star \partial_\nu = 0. \tag{50}$$

The Riemann curvature tensor vanishes identically. In a next step, we show that this connection is a metric one. We have to evaluate the covariant derivative of the metric:

$$\nabla^*_\mu g = \nabla^*_\mu(g_{\alpha\beta} d\hat{x}^\alpha \otimes_\star d\hat{x}^\beta)\,,$$

where

$$(g_{\alpha\beta}) = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}.$$

In the present case, we again obtain, from the star-dual pairing (19), that

$$\nabla^*_\mu d\hat{x}^\sigma = 0.$$

Therefore, we get

$$\nabla^*_\mu\, g = g_{\sigma\beta} \times$$
$$\times(\nabla^*_\mu d\hat{x}^\sigma \otimes_* d\hat{x}^\beta + \bar{R}^\alpha(d\hat{x}^\sigma) \otimes_* \nabla^*_{\bar{R}_\alpha(\hat{\partial}_\mu)} d\hat{x}^\beta) = 0\,. \tag{51}$$

2.3. $Gl_q(N)$

The quantum space for $Gl_q(N)$ [12] is defined by

$$\hat{x}^i \hat{x}^j = q \hat{x}^j \hat{x}^i,\ i < j\,. \tag{52}$$

Therefore, we have, for the twist,

$$\mathcal{F}^{-1} = \exp\left(-\frac{ih}{2}\sum_{i<j}(\hat{x}^j\hat{\partial}_j \otimes \hat{x}^i\hat{\partial}_i - \hat{x}^i\hat{\partial}_i \otimes \hat{x}^j\hat{\partial}_j)\right)\,. \tag{53}$$

In the same way as before, we can show that the trivial connection satisfies all requirements and defines a Levi-Civita connection with vanishing curvature tensor.

2.4. Twisted sphere

The twisted sphere is defined by relations (37) and the additional condition [15]

$$r^2 = 2(\hat{x}^1\hat{x}^5 + \hat{x}^2\hat{x}^4) + (\hat{x}^3)^2 . \tag{54}$$

With the use of the stereographic coordinates y^i, $i = 1, 2, 4, 5$, the metric is given by

$$g^* = \frac{4r^2}{(r^2+\kappa^2)^2} \star C_{ij} dy^i \otimes_* dy^j , \tag{55}$$

where

$$(C_{ij}) = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} .$$

In order to simplify the notation, we introduce the following definitions: For the vector fields, let us define

$$t_i := y^i \frac{\partial}{\partial y^i} = y^i \partial_i \tag{56}$$

(we note that no summation over the index i is implied). Hence, we write, for the twist,

$$\mathcal{F} = \exp\left(-\frac{ih}{2}\varphi_{ij} t_i \otimes t_j\right) \tag{57}$$

with

$$\varphi_{ij} = -\varphi_{ji} = -\varphi_{ij'} , \tag{58}$$

$$\varphi_{12} = 1,\ \varphi_{ii} = \varphi_{ii'} = 0 , \tag{59}$$

and $i' = 6 - i$. Furthermore, let us introduce P_{ij} and its square,

$$P_{ij} = e^{\frac{ih}{2}\varphi_{ij}},\ q_{ij} = P_{ij}^2 . \tag{60}$$

Using these definitions, we can write, for the metric,

$$g^* = \sum_{ij} g_{ij} dy^i \otimes_* dy^j = \frac{4r^2}{(r^2+\kappa^2)^2} \sum_{i,j} C_{ij} P_{ij}\, dy^i \otimes dy^j . \tag{61}$$

The Levi-Civita connection can be obtained by demanding the vanishing torsion and the vanishing covariant derivative of the metric. The former condition reads

$$\Gamma^{*\ k}_{ij} = q_{ij} \Gamma^{*\ k}_{ji} . \tag{62}$$

The latter condition then leads to

$$\Gamma^{*\ k}_{ij} = \frac{1}{2} g^{lk} \left(q_{ij}\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) . \tag{63}$$

As a result, the universal connection is the same as that in the undeformed case:

$$\nabla^* = \nabla . \tag{64}$$

The converse is also true: Assuming (64), we obtain (63) for the connection coefficients.

Similarily, we obtain, for the Riemann curvature,

$$R^* = R \tag{65}$$

and, in terms of components,

$$R^* = R^{*\ \ m}_{ikl} dy^i \otimes_* dy^j \otimes_* dy^k \otimes_* \partial_m , \tag{66}$$

$$R^*_{ijkl} = \frac{1}{r^2}\left(g_{li} g_{jk} - q_{ik}\, g_{lj} g_{ik}\right) . \tag{67}$$

Now let us consider a possible transformation between a 5d theta-deformed plane (see Section 2.1) and a 5d q-deformed one (see Section 2.2). The theta-deformed space is chosen in the following way: $[x_i, x_j] = i\theta_{ij}$ with the coordinate x_3 commuting with all other coordinates and

$$\theta_{ij} = \begin{pmatrix} 0 & h & -h & 0 \\ -h & 0 & 0 & h \\ h & 0 & 0 & -h \\ 0 & -h & h & 0 \end{pmatrix} .$$

Then, with the map $y_i = \exp(x_i)$, we obtain the correct commutation relations (37). But, unfortunately, this map does not respect the complex structure, and the induced metric seems not to be the proper metric for the q-deformed plane. But another possible map is from the q-deformed sphere to a plane, via a stereographic projection. Starting with the q-deformed sphere, commutation relations (37), and the constraint $r^2 = 2(x^1x^5 + x^2x^4) + (x^3)^2$, we define a map to the plane in the usual way by $y^3 = x^3$, $y^i = (x^i r)/(r - x^3)$, $i = 1, 2, 4, 5$. The induced metric is then given by (55).

3. Double (Bi-)Category

Definition 1. *A category is a quadruple* (**Obj**, **Mor**, id, $\circ$) *consisting of:*
(C1) a class **Obj** *of objects;*
(C2) a set **Mor**(A, B) *of morphisms for each ordered pair* (A, B) *of objects;*

(C3) a morphism $\mathrm{id}_A \in \mathbf{Mor}(A, A)$ *for each object* A*: the identity of* A*;*
(C4) a composition law associating, to each pair of morphisms $f \in \mathbf{Mor}(A, B)$ *and* $g \in \mathbf{Mor}(B, C)$*, a morphism* $g \circ f \in \mathbf{Mor}(A, C)$*;*
which is such that:
(M1) $h \circ (g \circ f) = (h \circ g) \circ f$ *for all* $f \in \mathbf{Mor}(A, B)$, $g \in \mathbf{Mor}(B, C)$ *and* $h \in \mathbf{Mor}(C, D)$*;*
(M2) $\mathrm{id}_B \circ f = f \circ \mathrm{id}_A = f$ *for all* $f \in \mathbf{Mor}(A, B)$*;*
(M3) the sets $\mathbf{Mor}(A, B)$ *are pairwise disjoint.*

Example 1. The category NC Einst. Objects of the category **NC Einst** are noncommutative Einstein spaces NC Einst defined in Sections 2.1–2.4 by the induced metric (55). For a morphisms s, t: NC Einst $\to$ NC Einst$'$, we define a map to the plane in the usual way by $y^3 = x^3$, $y^i = (x^i r)/(r - x^3)$, $i = 1, 2, 4, 5$.

Definition 2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two categories. A* **functor** *from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a family of functions* $\mathcal{F}$ *which associates, to each object* A *in* $\mathbf{X}$*, an object* $\mathcal{F}A$ *in* $\mathbf{Y}$ *and, to each morphism* $f \in \mathbf{Mor}_{\mathbf{X}}(A, B)$*, a morphism* $\mathcal{F}f \in \mathbf{Mor}_{\mathbf{Y}}(\mathcal{F}A, \mathcal{F}B)$ *which is such that*
(F1) $\mathcal{F}(g \circ f) = \mathcal{F}g \circ \mathcal{F}f$ *for all* $f \in \mathbf{Mor}_{\mathbf{X}}(A, B)$ *and* $g \in \mathbf{Mor}_{\mathbf{Y}}(B, C)$*;*
(F2) $\mathcal{F}\,\mathrm{id}_A = \mathrm{id}_{\mathcal{F}A}$ *for all* $A \in \mathbf{Obj}(\mathbf{X})$.

Definition 3. *A double category* D *consists of:*
(1) A category D_0 *of objects* $Obj(D_0)$ *and morphisms* $\mathbf{Mor}(D_0)$ *of* 0*-level.*
(2) A category D_1 *of objects* $Obj(D_1)$ *of* 1*-level and morphisms* $\mathbf{Mor}(\mathbf{D_1})$ *of* 2*-level.*
(3) Two functors $d, r : D_1 \rightrightarrows D_0$.
(4) A composition functor

$$* : D_1 \times_{D_0} D_1 \to D_1,$$

where the bundle product is defined by the commutative diagram

$$\begin{array}{ccc} D_1 \times_{D_0} D_1 & \xrightarrow{\pi_2} & D_1 \\ \pi_1 \downarrow & & \downarrow d \\ D_1 & \xrightarrow{r} & D_0 \end{array} .$$

(5) A unit functor $ID : D_0 \to D_1$ *which is a section of* d, r.

The above data is subject to **Associativity Axiom** *and* **Unit Axiom.** *If both of them are fulfilled only up to the equivalence, then the double category is called a* **weak** *double category, and if they are fulfilled strictly, then it is a* **strong** *double category.*

Here, we see that, for two objects $A, B \in Obj(D_0)$, there are 0-level morphisms $D_0(A, B)$ which are noted by ordinary arrows $f : A \to B$, and 1-level morphisms $D_{(1)}(A, B)$ which are noted by the arrows $\xi : A \Rightarrow B$, for $A = d(\xi)$ and $B = r(\xi)$. So, with a 2-level morphism $\alpha : \xi \to \xi'$, where $\xi : A \Rightarrow B$ and $\xi' : A' \Rightarrow B'$, we can associate the diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rightarrow} & B & & \xi \\ d(\alpha) \downarrow & & \downarrow r(\alpha) & \longmapsto & \downarrow \alpha \\ A' & \overset{\xi'}{\Rightarrow} & B' & & \xi' \end{array}$$

and the arrow $\alpha : d(\alpha) \Rightarrow r(\alpha)$

The composition on 2-level is associated with the diagram

$$\begin{array}{ccccc} A & \overset{\xi}{\Rightarrow} & B & & \xi \\ d(\alpha) \downarrow & & \downarrow r(\alpha) & & \downarrow \alpha \\ A' & \overset{\xi'}{\Rightarrow} & B' & \longmapsto & \xi' \\ d(\alpha') \downarrow & & \downarrow r(\alpha') & & \downarrow \alpha' \\ A'' & \overset{\xi''}{\Rightarrow} & B'' & & \xi'' \end{array}$$

Now we can define, for double categories, **double (category) functors** and their **morphisms**, **double subcategories**, the category *DCat* of double categories, **equivalence** of double categories, **dual double categories** (changed direction of 1-level morphisms, i.e. d, r are transposed), and so on [1, 22].

Definition 4. [4] *The theory of bicategories is the category (with finite limits)* **Th**(**Bicat**) *given by the following data:*

- *Objects* Ob*,* Mor*,* 2Mor
- *Morphisms* $s, t : \mathrm{Ob} \to \mathrm{Mor}$ *and* $s, t : \mathrm{Mor} \to 2\mathrm{Mor}$
- ***composition*** *maps* $\circ$: $\mathrm{MPairs} \to \mathbf{Mor}$ *and* $\cdot$: $\mathrm{BPairs} \to \mathbf{2Mor}$*, satisfying the interchange law (the requirement that this be a functor means that the* ***interchange law*** *holds):*

$$(\alpha \circ \beta) \cdot (\alpha' \circ \beta') = (\alpha \cdot \alpha') \circ (\beta \cdot \beta') \,, \tag{68}$$

where MPairs $=$ Mor $\times_{\mathrm{Ob}}$ Mor *and* **BPairs** $=$ 2Mor $\times_{\mathrm{Mor}}$ 2Mor *are the equalizers of diagrams of the form:*

$$\begin{tikzcd} & & \mathrm{Mor} \arrow[dr, "t"] & \\ \mathbf{MPairs} \arrow[r, "i"] & \mathrm{Mor}^2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & \mathrm{Ob} \\ & & \mathrm{Mor} \arrow[ur, "s"'] & \end{tikzcd} \tag{69}$$

and similarly for BPairs.

- *the **associator** map* $a : \mathrm{Triples} \to \mathbf{2Mor}$*, where* $\mathrm{Triples} = \times_{\mathrm{Ob}} \mathrm{Mor} \times_{\mathrm{Ob}} \mathrm{Mor}$ *is the equalizer of a similar diagram for involving* Mor^3 *such that* a *satisfies* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$

- ***unitors*** $l, r : \mathrm{Ob} \to \mathrm{Mor}$ *with* $s \circ l = t \circ l = \mathrm{id}_{\mathrm{Ob}}$ *and* $s \circ r = t \circ r = \mathrm{id}_{\mathrm{Ob}}$

These data are subject to the conditions that the associator is subject to the pentagon identity [23], and the unitors obey certain unitor laws

$$\begin{tikzcd} (g \circ 1_y) \circ f \arrow[r, "a_{g,1_y,f}"] \arrow[d, "r_g \times 1_f"'] & g \circ (1 \circ f)\ . \arrow[dl, "1_g \times l_f"] \\ g \circ f & \end{tikzcd} \qquad (70)$$

Definition 5. [4] *A double bicategory consists of:*

- *bicategories* **Obj** *of **objects**,* **Mor** *of **morphisms**,* **2Mor** *of **2-morphisms***

- ***source** and **target** maps* $s, t : \mathbf{Mor} \to \mathbf{Obj}$ *and* $s, t : \mathbf{2Mor} \to \mathbf{Mor}$

- *partially defined **composition** functors* $\circ : \mathbf{Mor^2} \to \mathbf{Mor}$ *and* $\cdot : \mathbf{2Mor^2} \to \mathbf{2Mor}$*, satisfying the interchange law (68)*

- *partially defined **associator*** $a : \mathbf{Mor^3} \to \mathbf{2Mor}$ *with* $s(a(f,g,h)) = (f \circ g) \circ h$ *and* $t(a(f,g,h)) = f \circ (g \circ h)$

- *partially defined **unitors*** $l, r : \mathbf{Obj} \to \mathbf{Mor}$ *with* $s(l(x)) = t(l(x)) = x$ *and* $s(r(x)) = t(r(x)) = x$ *.*

All the partially defined functors are defined for *composable* pairs or triples, for which the source and target maps coincide in the obvious way. The associator should satisfy the pentagon identity [23], and the unitors should satisfy the unitor laws (70).

4. Action of a Double Category

Double categories are the categorical variants of usual monoids (and groups), and thus we have the corresponding variant for their actions. Below, the definition of action of a double category $d, r : D_1 \to D_0$ on categories over D_0 is given. Thus, we get an analog of group-theoretic methods in categorical frames.

Definition 6. *(Left) action of a double category* $d, r : D_1 \to D_0$ *on a category* $p : M \to D_0$ *over* D_0 *is a functor* φ *such that*
(1)The diagram is commutative

$$\begin{tikzcd} D_1 \times_{D_0} M \arrow[r, "\varphi"] \arrow[dr, "r \circ \pi_1"'] & M \arrow[d, "p"] \\ & D_0 \end{tikzcd}\ ,$$

where the bundle product $D_1 \times_{D_0} M$ *is defined by the diagram*

$$\begin{tikzcd} D_1 \times_{D_0} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M \arrow[d, "p"] \\ D_1 \arrow[r, "r"] & D_0 \end{tikzcd}\ .$$

(2)The diagram is commutative to within an isomorphism

$$\begin{tikzcd} (D_1 \times_{D_0} D_1) \times_{D_0} M \arrow[r, "\cong"] \arrow[d, "\otimes \times_{D_0} id_M"'] & D_1 \times_{D_0} (D_1 \times_{D_0} M) \arrow[r, "id_{D_1} \times_{D_0} \varphi"] & D_1 \times_{D_0} M \arrow[d, "\varphi"] \\ D_1 \times_{D_0} M \arrow[rr, "\varphi"] & & M \end{tikzcd}\ ,$$

and there exists a functor isomorphism φ *such that*

$$\forall\, \xi, \xi' \in Obj(D_1),\ m \in Obj(M_1)$$

$$\varphi_{\xi,\xi',m} : (\xi * \xi') * m \to \xi * (\xi' * m).$$

(3) For the unit functor, we have a functor isomorphism $\chi : \varphi \circ (ID \times id_M) \xrightarrow{\sim} id_M$ *or for objects*

$$\forall\, A \in Obj(D_0),\ m \in Obj(M_1) \quad \chi_{A,m} : ID_A * m \xrightarrow{\sim} m.$$

So we have the map of a pair of objects $\xi \in Obj(D_1)$, $m \in Obj(M)$ $(A \overset{\xi}{\Rightarrow} p(m), m) \mapsto \varphi(\xi, m)$ such that

$p(\varphi(\xi, m)) = A$, and of morphisms $\alpha \in D_1(\xi, \xi'), u \in M(m, m')$

$$\begin{array}{ccccccc} \xi & A & \overset{\xi}{\Rrightarrow} & p(m) & & \varphi(\xi, m) & \\ \alpha\downarrow & f = d(\alpha)\downarrow & & \downarrow r(\alpha) = p(u) & \longmapsto & \downarrow \varphi(\alpha, u) & , \\ \xi' & A' & \overset{\xi'}{\Rrightarrow} & p(m') & & \varphi(\xi', m') & \end{array}$$

where $p(\varphi(\alpha, u)) = f$.

The definition of a **right action** is evident.

5. Cobordism and Double Categories

Let M_d be the category of oriented compact d-dimensional smooth manifolds (with boundary) and piecewise smooth maps (we do not define the sense of the condition more exactly here; this may be such continuous maps $f : M \to Y$ that are smooth on a dense open subset $U_f \subset M$), let CM_d be its subcategory of closed (with empty boundary) manifolds and smooth maps, $CM_d \subset M_d$.

There are the following functors:
(1) Disjoint union

$$\cup : M_d \times M_d \to M_d : (X, Y) \mapsto X \cup Y.$$

(2) Changing of the orientation of manifolds on the opposite one

$$(-) : M_d \to M_d : X \mapsto -X.$$

(3) Boundary operator

$$\partial : M_{d+1} \to CM_d : X \mapsto \partial X.$$

(4) Multiplication on the unit segment $I = [0, 1]$

$$I \times\ . : CM_d \to M_{d+1} : X \mapsto I \times X.$$

Now we define a double category $\mathbf{C(d)}$ with
(1) $\mathbf{C(d)_0} = \mathbf{CM_d}$.
(2) 1-level morphisms $\mathbf{C(d)_{(1)}(X, X')}$ are a set of pairs (Y, f), where Z is an oriented compact $(d+1)$-dimensional smooth manifold with the boundary ∂Y, and f is a diffeomorphism

$$f : (-X) \cup X' \to \partial Y,$$

where $\cup$ stands for the disjoint union of $-X$ and X'. Thus, we write $(Y, f) : X \Rightarrow X'$.
(3) The composition of $(Y, f) : X \Rightarrow X'$ and $(Y', f') : X' \Rightarrow X''$ is the morphism

$$(Y \cup_{X'} Y', (f|_X) \cup (f'|_{X'})) : X \Rightarrow X'',$$

where $(Y \cup_{X'} Y')$ denotes the union $(Y \cup Y')$ after the identification of each point $f(y) \in f(Y)$ with the point $f'(y) \in f'(Y)$ for all $y \in Y$ and smoothing this topological manifold.
(4) The 1-level identical morphism ID_X is $(X \times [0; 1], id_{(-X)\cup X})$, because $\partial(X \times [0; 1]) = (-X) \cup X$.
(5) 2-level morphisms of $\mathbf{C(d)_1}(\xi, \xi')$ from $\xi = (Y, f : X' \cup (-X) \to \partial Y) : X \Rightarrow X'$ to $\xi' = (Y', f' : X'' \cup (-X') \to \partial Y') : X' \Rightarrow X''$ are such triples of smooth maps (f_1, f_2, f_3) that the following diagram is commutative:

$$\begin{array}{cccc} (-X) \cup X' & \xrightarrow{f} & \partial Y & \subset Y \\ \downarrow f_1 \cup f_2 & & & \downarrow f_3 \\ (-X') \cup X'' & \xrightarrow{f'} & \partial Y' & \subset Y' \end{array}.$$

It is easy to see that the functors $\cup$ and $(-)$ may be expanded to double category functors

$$\cup : \mathbf{C(d)} \to \mathbf{C(d)},$$
$$(-) : \mathbf{C(d)} \to \mathbf{C(d)}^\circ,$$

and $(-)$ is an equivalence of the double categories.

Remark. Two following formulas for 1-level morphisms in algebras and cobordisms [18–20] are of interest:

$$f : A \otimes_k B^\circ \to End_k(N) \qquad f : (-X) \cup Y \to \partial Z,$$

where we have correspondence between the functors

$$\begin{array}{ccc} (\_)^\circ & \longleftrightarrow & -(\_), \\ \otimes_k & \longleftrightarrow & \cup, \\ End_k & \longleftrightarrow & \partial. \end{array}$$

6. Petrov Noncommutative Topological Quantum Field Theory

The Petrov Noncommutative Topological Quantum Field Theory (NC TQFT) is a 2-functor Z from a certain bicategory of double cobordisms [7] $CM(d)$ of d-dimensional manifolds into the double bicategory **NC Einst** of noncommutative Einstein spaces, and some axioms are satisfied [1, 17, 22, 23].

Thus, Petrov NC TQFT in dimension d is a 2-functor,

$$Z : \mathbf{C(d)} \to \mathbf{Mor(NC\,Einst)},$$

between double bicategories such that
(1) the disjoint union in $\mathbf{C(d)}$ goes to the tensor product

$$\cup \mapsto \otimes,$$

where $(\_)^* : NC\,Einst \to NC\,Einst^{\circ}$ is a dualization of noncommutative Einstein spaces,
(2) changing the orientation in $\mathbf{C(d)_0}$ goes to the dualization

$(-) \mapsto (.)^*$.

Thus, as a consequence of double bicategorical functorial properties, we get

(1) for each compact closed oriented smooth d-dimensional manifold $X \in Obj(\mathbf{C(d)_0})$, the value of the functor $Z(X)$ is a noncommutative Einstein space over the field $\mathbb{C}$ of the complex numbers,

(2) for each $(Y, f) : X \Rightarrow X'$ from $Obj(\mathbf{C(d)_1})$, the value of the functor $Z(Y, f)$ is a homomorphism $Z(X) \to Z(X')$ of noncommutative Einstein spaces,

and the following axioms of Petrov NC TQFT are satisfied:

A(1) (involutivity) $Z(-X) = Z(X)^*$, where $-X$ denotes the manifold with the opposite orientation, and $*$ denotes the dual noncommutative Einstein space.

A(2) (multiplicativity) $Z(X \cup X') = Z(X) \otimes Z(X')$, where $\cup$ denotes a disconnected union of manifolds.

A(3) (associativity) For the composition $(Y'', f'') = (Y, f) * (Y', f')$ of cobordisms, the following relation holds:

$$Z(Y'', f'') =$$

$$= Z(Y', f') \circ Z(Y, f) \in \mathbf{Mor_C(Z(X), Z(X''))}.$$

(Usually, the identifications

$$Z(X' - X) \cong Z(X)^* \otimes Z(X') \cong \mathbf{Mor_C(Z(X), Z(X'))}$$

allow one to identify $Z(Y, f)$ with the element $Z(Y, f) \in Z(\partial Y)$.

A(4) For the initial object, $\emptyset \in Obj(\mathbf{C(d)_0})$ $\quad \mathbf{Z(\emptyset) = C}$.

A(5) (trivial homotopy condition) $Z(X \times [0, 1]) = id_{Z(X)}$.

7. Conclusions

We have studied the noncommutative counterparts of the so-called Einstein spaces (such as twisted 4-spheres) in the framework of twisted gravity. Their Ricci tensor is proportional to the metric. We have computed the deformed Riemannian tensor and the scalar curvature in the formalism of twisted gravity. We could already see, for some examples, the remarkable property that being an Einstein space seems to be stable under deformation, using a Killing vector field in the twist. The deformed Levi-Civita connection and the deformed Riemann tensor are just the undeformed ones. Deformed spherical symmetric spaces are very important with respect to, e.g., the Black-Hole solutions and are related to cosmological problems. As a generalization, one should study star geometries, where the vector fields are not Killing vectors. On the other hand, the main result of this paper can be summarized as that the construction of Petrov NC TQFT is a 2-functor from a certain bicategory of double cobordisms $CM(d)$ of d-dimensional manifolds into the double bicategory **NC Einst** of noncommutative Einstein spaces.

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НЕКОМУТАТИВНА ТОПОЛОГІЧНА КВАНТОВА ТЕОРІЯ ПОЛЯ ТИПУ ПЕТРОВА

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Р е з ю м е

У статті дано означення категорії некомутативних просторів Ейнштейна **NC Einst** та побудовано некомутативну топологічну квантову теорію поля (НКТП) типу Петрова. Автори вважають корисним ознайомлення з ідеями даної роботи з метою дальшого розвитку НКТП та її застосування у просторах вищої розмірності.