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PECULIARITIES OF APPLICATION OF PERTURBATION TECHNIQUES IN PROBLEMS OF NONLINEAR OSCILLATIONS OF LIQUID WITH A FREE SURFACE IN CAVITIES OF NON-CYLINDRICAL SHAPE

ОСОБЛИВОСТІ ЗАСТОСУВАННЯ МЕТОДІВ ЗБУРЕНЬ У ЗАДАЧАХ ПРО НЕЛІНІЙНІ КОЛИВАННЯ РІДИНИ З ВІЛЬНОЮ ПОВЕРХНЕЮ В ПОРОЖНИНАХ НЕЦИЛІНДРИЧНОЇ ФОРМИ

We consider the problem about nonlinear oscillations of ideal incompressible liquid in a tank of revolution. It is shown that the ordinary way of application of perturbation techniques results in the violation of solvability conditions of the problem. To avoid this contradiction we state some additional conditions and revise previously used approaches. The construction of a discrete nonlinear model of the investigated problem is done on the basis of the Hamilton – Ostrogradsky variational formulation of the mechanical problem with preliminary satisfying of kinematical boundary conditions and solvability conditions of the problem. Numerical examples are evidence of effectiveness of the constructed model.

Розглядається задача про нелінійні коливання ідеальної нестисливої рідини в резервуарі в формі тіла обертання. Показано, що звичайний шлях застосування методів збурень приводить до порушення умов розв'язності задачі. Для уникнення цієї суперечності вводяться додаткові умови і переглядаються підходи, які використовувалися раніше. Побудова дискретної нелінійної моделі виконується на основі формулювання механічної задачі у вигляді варіаційного принципу Гамільтона – Остроградського з попереднім виконанням кінематичних граничних умов і умов розв'язності задачі. Числові приклади підтверджують ефективність побудованої моделі.

1. Introduction. The problem about oscillations of ideal incompressible liquid with a free surface in a cavity of arbitrary geometrical shape is stated in the form of the following boundary problem [1 – 5, 8, 12, 18, 23, 29]:

$$\Delta\varphi = 0 \quad \text{in} \quad \tau, \tag{1}$$

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Sigma, \tag{2}$$

$$\frac{\partial\varphi}{\partial n} = -\frac{\dfrac{\partial\eta}{\partial t}}{\|\vec{\nabla}\eta\|} \quad \text{on} \quad S, \tag{3}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}\left(\vec{\nabla}\varphi\right)^2 + U = 0 \quad \text{on} \quad S, \tag{4}$$

here motion is described in the Cartesian reference frame $Oxyz$ connected with the tank, φ is the velocity potential of liquid, τ is the domain occupied by liquid, $\dfrac{\partial}{\partial n}$ is the external normal derivative to a surface, Σ is the boundary of contact of liquid with tank walls in perturbed motion (for convenience we introduce also Σ_0, which corresponds to the boundary of contact of liquid with tank walls in unperturbed motion and $\Delta\Sigma$ variation of the contact boundary caused by liquid perturbation, $\Sigma = \Sigma_0 + \Delta\Sigma$), S is a free surface of liquid in its perturbed motion (S_0 is a free surface of liquid in unperturbed motion), $\eta(x, y, z, t)$ is the equation of a free surface of liquid, $\|\vec{\nabla}\eta\|$ is the norm of gradient of the function η, U is the function of the potential energy of liquid, t is time. For simplification of the following analysis of some general properties of the boundary problem (1) – (4) initially we limit our consideration by the case of an immovable tank.

Practical investigations showed that the unique possible way of analytical investigation of the boundary value problem (1) – (4) is connected with application of perturbation technique [5, 18, 26]. In this case we suppose that elevations of a free surface are small values, and further we can use both power expansions of nonlinear conditions (3), (4) by elevations of a free surface and projection of these conditions onto unperturbed free surface S_0. At the same time it is necessary to pay additional attention to the boundary condition (2), which, being linear by the form of relations, is set on variable boundary Σ that depends on liquid motion. Thus, in spite of linearity of the relation (2) this condition in essence is nonlinear similar to the conditions (3), (4).

In the case when the tank cavity represents a cylindrical domain one succeeded to resolve the equation of the free surface relative to the variable z, which corresponds to vertical direction. Then the equation of a free surface takes the form of

$$\eta(x, y, z, t) = z - \xi(x, y, t),$$

the corresponding linear problems admit analytical solution on the basis of the method of variables separation, and the problem (1) – (4) becomes essentially simpler. In the case of cylindrical domains occupied by liquid it is possible to construct effective algorithms for solving nonlinear problems of dynamics tanks with liquid with a free surface, including the case of translational and rotational motion of the carrying body [1, 5, 7 – 20, 23, 25, 27, 29, 30]. The most substantial results in this direction were obtained on the basis of variational algorithms of statement and solving of nonlinear problems of dynamics of tanks with liquid. Thus, for tanks of cylindrical shapes various problems of dynamics of steady and transient modes of motion of reservoirs with liquid were investigated.

In spite of the fact that the question about spreading results obtained for cylindrical tanks on the cases of tanks of complex geometry seemed for many researchers as a question of numerical realization, until now cases of tanks of non-cylindrical shape are investigated rather superficially. Attempt of creation of unified highly universal approach for solving problems of nonlinear dynamics of liquid in cavities with inclined walls was made in publications of I. Lukovsky [21, 29], where the problem about motion of liquid in tank was formulated for non-Cartesian parametrization of the liquid domain. However, until now this method have not found practical application and investigations in this direction are not continued.

Further attempts of solving problems of dynamics of liquid in tanks of non-cylindrical shapes showed that there is a number of unclarified problems, which fundamentally make difficult solving of the problem. Most clearly this become apparent on application of methods of formal point-wise discretization [6, 7, 9, 31], when during one period of oscillations violation of laws of mass and energy conservation was about 20 %. Taking into account that laws of conservation of mass and energy are not only physically evident for this class of problems, but they practically coincide with the mathematical condition of solvability of the problem (1) – (4), these methods as well as some analytical methods collapse even for small time intervals. Moreover, most frequently methods of point-wise approximation are applied for nonlinear 2D or axis-symmetrical problems, because they are based on essential usage of limited computer resources, which is insufficient for investigation of most complicated nonlinear processes.

For further progress in solving applied problems of this class it is necessary to analyze deeper mechanical and mathematical essence of the problem. This analysis will be done from the point of view of further application of the variational method of the problem

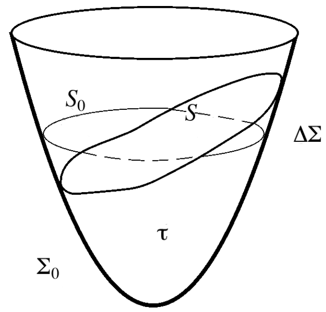

Fig. 1. General scheme of denotations.

solving, which is based on formulation of the mechanical problem on the basis of the Hamilton – Ostrogradsky variational principle. It is known that this approach was sufficiently successfully applied for solving nonlinear problems of dynamics of reservoirs of cylindrical shape with liquid with a free surface [12, 14 – 20, 25]. The mechanical analysis of this approach shows that a part of the problem conditions (kinematical constraints) should be satisfied before solving the variational problem of the stage of construction of decompositions of desired variables, and dynamic boundary conditions and motion equations for the carrying body are obtained from the variational relation. It is significant to note that in the subsequent procedure of problem solving no other increase of accuracy takes place in satisfying kinematical requirements of the problem. For cylindrical domains the solvability condition of the Neumann problem for the Laplace equation (for the problem about oscillations of liquid with a free surface (1) – (4)) is quite trivial. This condition is equivalent to requirement of conservation of a liquid volume in its perturbed motion for every natural mode of oscillations. The analysis conducted in the present article shows that simple transfer of this form of the solvability condition to the case of oscillations of liquid in non-cylindrical cavities is insufficient.

Hence, construction of a nonlinear discrete resolving model for the problem about oscillations of ideal liquid with a free surface in a tank of non-cylindrical shape will be done according to the following scheme:

1. The analysis of the solvability condition.

2. Construction of decompositions of desired variables, which hold linear kinematic boundary conditions.

3. Construction of decompositions of desired variables, which hold nonlinear kinematic boundary conditions.

4. Construction of a resolving system of motion equations relative to amplitude parameters of liquid motion and parameters of translational motion of the carrying body.

2. Problem statement. We consider a problem about oscillations of liquid with a free surface in a reservoir, which cavity is of revolution shape. We consider the case when reservoir is movable and can perform finite translational movements. Basic denotations, which were described in Introduction, are shown in Fig. 1.

For specification of liquid motion we introduce non-Cartesian parametrization of the domain, occupied by liquid, according to the following scheme:

$$\alpha = \frac{r}{f(z)}, \qquad \beta = \frac{z}{H}, \tag{5}$$

where $r = f(z)$ is the equation of the generatrix of a body of revolution, H is filling depth of liquid. Here we suppose that the origin of the reference frame is in the center of the undisturbed free surface of liquid, the axis Oz is directed upward, (r, θ, z) is the system of cylindrical coordinates, which according to the relations (5) is substituted for the new non-Cartesian system of coordinates (α, θ, β) ($\alpha \in [0, 1]$; $\theta \in [0, 2\pi]$ and for unperturbed state $\beta \in [-1, 0]$). For the accepted system of parametrization the domain of liquid takes cylindrical shape and the equation of a free surface can be resolved relative to the coordinate β and it takes the form of

$$\beta = \frac{1}{H}\xi(\alpha, \theta, t). \tag{6}$$

The problem about motion of the bounded volume of liquid takes now the form of

$$\Delta\varphi = 0, \tag{7}$$

$$\frac{\partial\varphi}{\partial n} = \frac{1}{\sqrt{1+f'^2}}\left(\frac{\partial\varphi}{\partial r} - f'\frac{\partial\varphi}{\partial z}\right) = 0 \quad \text{for} \quad r = f(z), \tag{8}$$

$$\frac{\partial\xi}{\partial t} + \frac{1}{f^2}\frac{\partial\xi}{\partial\alpha}\frac{\partial\varphi}{\partial\alpha} + \frac{1}{\alpha^2 f^2}\frac{\partial\xi}{\partial\theta}\frac{\partial\varphi}{\partial\theta} - \underline{\frac{\alpha f'}{f}\frac{\partial\xi}{\partial\alpha}\frac{\partial\varphi}{\partial z}} - \frac{\partial\varphi}{\partial z} = 0 \quad \text{for} \quad \beta = \frac{1}{H}\xi(\alpha, \theta, t). \tag{9}$$

We note that the underlined term appeared in the kinematical boundary condition (9) owing to non-cylindrical shape of the domain occupied by liquid and reflects non-Cartesian property of the accepted parametrization.

The equations (7) – (9) don't present the complete formulation of the problem about motion of liquid with a free surface, but they present the system of kinematical restrictions of the problem. Similar to the approach used for cylindrical tanks we shall obtain dynamical boundary conditions and motion equations for the carrying body from the Hamilton – Ostrogradsky variational principle.

Translational motion of the reservoir relative to a conventionally immovable reference frame is described by the vector of displacements $\vec{\varepsilon}$.

In contrast to publications [21, 29] further for realization of transformations in an invariant form we apply vector calculus instead of tensor calculus.

3. Analysis of the solvability condition of the problem. Taking into account that the problem (1) – (4) as well as the corresponding linear boundary problem [2, 8, 12, 22, 24, 28] has no exact analytical solution for arbitrary cavities of revolution, we must come from the fact that boundary conditions of the problem (1) – (4) will be realized approximately. According to the general theory of solvability of the Neumann boundary problems for the Laplace equation the solvability condition for the problem (1) – (4) can be given as

$$\int\limits_{\Sigma_0} \frac{\partial\varphi}{\partial n}\, d\Sigma + \int\limits_{\Delta\Sigma} \frac{\partial\varphi}{\partial n}\, d\Sigma + \int\limits_{S} \frac{\partial\varphi}{\partial n}\, dS = 0. \tag{10}$$

Let us analyze term by turn the expression (10). The first addend represents requirement of satisfying (in weak sense) the non-flowing condition on an unperturbed boundary of contact of liquid with tank walls Σ_0. Therefore, holding of this boundary condition of non-flowing on Σ_0 should be performed with improved accuracy. In our appearance on realization of different procedures of solving this class of problems insufficient attention was paid to this question. Sometimes natural modes of oscillations with errors of satisfying of boundary non-flowing condition about 20% and more were applied for numerical realization. Correspondingly, such violation of the non-flowing conditions, and, therefore, the solvability condition, results in instability of realization of numerical procedures.

The second addend corresponds to the requirement of realization in weak sense of the non-flowing condition on the boundary $\Delta\Sigma$, i.e., on wave crests of liquid over level of the undisturbed free surface of liquid. This physically evident kinematic boundary condition is not consequence of statement of the linear problem about oscillations of liquid in a tank, which is usually applied for construction of decompositions of desired variables. Normally this condition is not taken into consideration at all on analysis of nonlinear oscillations of liquid in tanks of non-cylindrical shape, although we suppose that this condition is the dominant one in the analysis of the physical sense of the considered problem. In accordance with the maximum principle for harmonic functions the solution tends to violate realization of non-flowing condition, and this corresponds to overflow of liquid through the tank wall (namely this causes “loss” of liquid in methods of pointwise discretization). In spite of the property that this condition is expressed by linear mathematical relation, according to its nature it is nonlinear, because it corresponds to realization of the kinematic condition on a nonlinear perturbed surface, and it is evident that this condition does not enter the linear statement of the problem.

In order to analyze the third addend in the solvability condition of the problem we perform its transformation. It is known that one of the form of the kinematic boundary condition on a free surface has the form

$$\frac{\partial\varphi}{\partial n} = \frac{1}{\sqrt{1+\left(\vec{\nabla}\xi\right)^2}}\frac{\partial\xi}{\partial t} \quad \text{on} \quad S.$$

After substituting this condition into the third addend of the condition (7) we obtain

$$\int\limits_S \frac{\partial\varphi}{\partial n}\,ds = -\int\limits_{S_0} \frac{\partial\xi}{\partial t}\,dS = -\frac{\partial}{\partial t}\int\limits_{S_0} \xi\,dS,$$

i.e., the integral over a perturbed free surface of liquid S can be transformed to the unperturbed free surface S_0. The latter form of the third addend corresponds to requirement of the liquid volume conservation in its perturbed motion. Realization of this requirement will be considered below, where we shall show that realization of the requirement of liquid volume conservation for every separately taken natural mode of liquid oscillation, which corresponds to linearized requirement, is not sufficient for realization on a whole of the requirement of volume conservation in its perturbed motion.

Thus, the analysis of solvability conditions of the nonlinear boundary problem shows, that for correct solving of the nonlinear problem it is necessary to

a) realize with high precision the non-flowing requirement of liquid on the moisten in the unperturbed state boundary of contact liquid – tank;

b) satisfy requirement of non-flowing of liquid on tank walls over the level of an unperturbed free surface where crests of nonlinear waves reach;

c) satisfy requirements of liquid volume conservation in its perturbed nonlinear motion;

d) realize all these requirements on the stage of construction of decompositions of desired variables, which satisfy all kinematic boundary conditions of the problem, before solving the variational problem.

4. Construction of decompositions of desired variables, which hold linear kinematic boundary conditions. As it follows from the analysis of solvability conditions of the problem for successful realization of an algorithm of construction of the nonlinear finite-dimensional model of dynamics of liquid with a free surface in cavity of revolution it is necessary to construct a system of coordinate functions satisfying with high accuracy requirements of non-flowing on the moisten boundary Σ, which consists of the unperturbed moisten boundary Σ_0 and its certain prolongation $\Delta\Sigma$, until which wave crests can rise.

Traditionally the problem about determination of this system of coordinate functions was identified with the classical problem about determination of natural frequencies and modes of oscillations of ideal liquid with a free surface in cavities of different geometrical shape. As it is follows from the mentioned above analysis the system of coordinate functions for solving the nonlinear problem does not coincide with natural modes of oscillations, since it must supplementary satisfy nonflowing conditions on $\Delta\Sigma$. At the same time the problem about determination of natural frequencies and modes of oscillations of liquid with a free surface has independent theoretical and applied significance. First of all, namely on the basis of this problem calculation of natural frequencies of oscillations of a free surface of liquid and hydrodynamic coefficients of the motion equations (associated or virtual masses) is realized. Here virtual masses are expressed in an open form as quadratures of natural modes of oscillations. Natural frequencies and virtual masses of liquid are the basic parameters for construction of linear dynamic systems for control of bodies with liquid. At present different methods for determination of natural frequencies and modes of oscillations of liquid in reservoirs are developed [1, 4, 5, 12, 22, 24, 28], which give suitable results for practice. At that practically all approaches use the traditional boundary eigenvalue problem

$$\Delta\varphi = 0 \quad \text{in} \quad \tau_0, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad \Sigma_0, \quad \frac{\partial\varphi}{\partial z} = \lambda\varphi \quad \text{on} \quad S_0, \tag{11}$$

or its variational analog

$$\delta I = 0, \quad \text{where} \quad I = \int\limits_{\tau_0} (\vec{\nabla}\varphi)^2 d\tau - \lambda \int\limits_{S_0} \varphi^2 dS. \tag{12}$$

As it was shown in [22], the solution of the problem (11) or its variation analog (12) contains an analytical singularity on the contour L_0. Availability of this singularity mainly clarifies sufficiently rapid manifestation of numerical instability on increase on the number of coordinate functions, which approximate a solution of the problem (11). Preliminary analytical isolation of the singularity on the contour L_0 and further search of the problem solution in the form of sum of singular and regular components essentially complicates the algorithm of solving the problem and reduces its practical applicability. Presence of this mathematical singularity is predetermined by existence of angular contour on L_0 and alteration of the form of the boundary condition in a vicinity of L_0. Partial cause of manifestation of this singularity is caused by linearization of the problem, as a result of which certain mechanical contradiction appears, i.e., the problem is solved for the immutable domain τ_0, however, for points of the boundary S_0 of the domain τ_0 non-zero

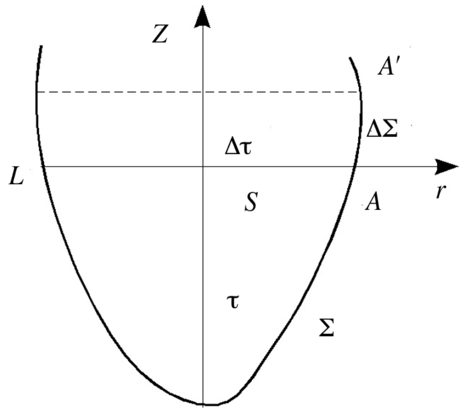

Fig. 2. General scheme of clarification of supplementary conditions and the method of an auxiliary domain.

normal velocities are admitted. This contradiction is absent in the nonlinear statement of the problem where singular properties of the solution are manifested weaker.

In connection with development of analytical and numerical-analytical methods for solving nonlinear problems of dynamics of bodies with liquid, where application of natural modes of oscillations as coordinate functions is assumed, solutions of the problem similar to (11) must meet requirements of not only integral character (as in the case of the linear theory), but of differential character too. Sense of supplementary requirements consists in the property that the nonflowing boundary condition on $\Delta\Sigma$, i.e., on prolongation of the surface Σ_0 outside the domain τ (Fig. 2), must hold. Practically this must be provided in the mentioned methods of solving nonlinear problems by existence and vanishing derivatives of the following type $\dfrac{\partial^{k+1}\varphi}{\partial n \partial l^k}$, where $\vec{l}$ is the unit tangent vector to the surface Σ_0 on the angular contour ($k = 1, 2, \ldots$ depends upon order of maximally taken nonlinearities on simulation of motiuon of liquid with a free surface).

It is evident that in the general case solutions of the problem (11) does not hold this requirements, since the initial statement of the boundary problem admits only conditions with differential operators of the first order. In this connection it is expedient to certain extent to refuse from the traditional problem of determination of natural frequencies and modes and construct approximately the system of coordinate functions $\overline{\psi}_i$, which is close to solutions of the problem (11) ψ_i with correspondingly close parameters $\overline{\lambda}_i$ and λ_i, but which in addition holds with high accuracy nonflowing condition on Σ_0 and on the surface $\Delta\Sigma$.

For realization of this goal we suggest two techniques, i.e., successive refinement of the solution of the problem (11) and the method of an auxiliary domain for reduction of influence of the singular points on behavior of the solution.

As it is known, success of application of direct methods of solving variational problems depends essentially on properties of coordinate functions. However, in most cases it is not possible to construct effective coordinate functions, which in advance hold exactly a part of conditions of the problem (11). So, mainly researchers use for solving problems for different simply connected domains the harmonic polynomials $w_k^{(m)}$ as coordinate functions, which do not hold specificity of geometrical shape of a cavity at all.

For partial account of geometrical and physical characteristics of natural functions (desired solution) we suggest the following algorithm of successive refinement of coordinate functions (it is necessary to note that in non-successive form the method with some similar elements was suggested in [6]). After solving the variational problem (12) and the algebraic eigenvalue problem we propose to select the obtained eigenfunctions (natural modes of oscillations of a free surface of liquid) as the new coordinate functions and repeat the procedure of solving the problem (12). In the case of necessity this step can be repeated in addition. However, for next step of successive refinement of coordinate functions it is not necessary to calculate quadratures again, since the above described techniques is equivalent to the following calculation scheme.

Let

$$(A - \lambda B)\, x = 0 \tag{13}$$

be the algebraic eigenvalue problem, which is obtained by numerical realization of the variational problem (12) and $\mathcal{D}$ is the matrix composed of eigen-vectors, i.e., solutions of the algebraic problem (13). Then, one step of the described successive process is equivalent to transition from the problem (13) to the problem

$$\left[\mathcal{D}^T (A - \lambda B)\mathcal{D}\right] x = 0. \tag{14}$$

Moreover, with considering non-degeneracy of the matrix $\mathcal{D}$ (otherwise solutions will be dependent) eigenvalues of the problems (13) and (14) coincide. Since solutions of the problem (11) possess property of orthogonality, then the matrix $B^* = \mathcal{D}^T B \mathcal{D}$ tends to a diagonally scattered structure (non-zero elements are located only on the main diagonal, part of rows are completely zero). This property of the matrix B^* created additional preconditions for more precise numerical solving the problem (13). We note that according to the method of the book [6] the matrix of eigen-vectors of the matrix B or A was used as the matrix $\mathcal{D}$ (in our case we use eigen-vectors of the problem (13) as a whole).

Actually effectiveness of the described technique is based on imperfections of the numerical solution of (13). Moreover, advantages of this approach manifest vividly on tending to a boundary of divergence of calculation procedure of the problem (12) on increase of the number of coordinate functions. Thus, in the case of solving the problem about oscillations of liquid with a free surface in the spherical cavity with $H = 0{,}5R$ for $N = 12$ coordinate functions with the given calculation accuracy $\varepsilon = 10^{-6}$ and for single iteration we obtain results, which are practically coincide with solutions of the problem with $\varepsilon = 10^{-13}$ without iterations. However, for $\varepsilon = 10^{-19}$ successive refinement of solving the problem (13) for $N = 12$ is not practically manifested. This technique becomes apparent most effectively for $23 \leq N \leq 28$, when for filling levels of liquid close to $H = R$ calculation convergence of solutions of the problem (13) appears. In this case results of calculations testify the described iteration procedure possesses properties of contracting mapping for errors of solving the problem (13), i.e., for certain parameters after reaching calculation instability on the initial step as early as on the first step of the iteration process recovery to a domain of stable calculation takes place. Results of numerical experiments in the domain of divergence of calculations ($23 \leq N \leq 28$ for sphere) application of the iterative scheme makes it possible to improve the solution of the problem (13) obtained according to the classical scheme 1,5 – 2 times, and in the distance from a free surface it is improved 3 – 10 times (in some cases 50 times). Let us note that the property of solution continuability along the surface $\Delta\Sigma$, as a rule, become worst after application of iteration procedure.

For the purpose of construction of coordinate functions for solving the nonlinear problem of dynamics of liquid with a free surface we apply the following technique for reduction of influence of singularity on angular contour on character of behavior of the solution. Numerical experiments show that solving the problem by the variational method with application of regular coordinate functions presence of singularity in statement of the boundary problem (11) becomes apparent in poor convergence, which further results in instability of numerical procedure.

Hence, we propose to solve the problem about searching coordinate functions close to natural modes of oscillations but having the mentioned above supplementary properties in the following way. We solve the problem (12) for the domain $\tau + \Delta\tau$ (Fig. 2), where $\Delta\tau$ is selected on the basis of requirements for validity range of the nonlinear theory for desirable boundaries of satisfying of nonflowing condition of wave crests on $\Delta\Sigma$. Here singularities in the constructed solution become apparent in a vicinity of the point A'. However, in a vicinity of the point A, which is an internal point for the domain $\tau + \Delta\tau$, the solution possesses regular properties. Next important stage consists in a method of projection of the obtained values of ψ_i' and λ_i' onto the surface S_0, i.e., with reference to the domain τ. If we use regular solutions ψ_i' for the domain τ' as coordinate functions of the problem (12) for the domain τ, then singularities in the problem statement (12) result in weak convergence for the domain $\Delta\Sigma$ including the point A. Therefore, we propose to consider values ψ_i' on the surface S_0 as $\overline{\psi}_i$. Moreover, as numerical experiments show according to character of behavior $\overline{\psi}_i$ are close to ψ_i' and ψ_i, and eigenvalues, determined for $\overline{\psi}_i$ by the Rayleigh method, differ from λ_i very slightly.

Let us show application of methods of successive refinement and auxiliary domains on examples of numerical realization. We consider a spherical reservoir filled by liquid with the level $H = 0{,}5R$. Table 1 shows results of determination of first natural modes of oscillations ($k = 1$) for peripheral numbers $m = 1$ and $m = 2$. The solution of the problem was constructed on the basis of decompositions by $N = 28$ harmonic polynomials. Algebraic eigenvalue problem was solved for $\varepsilon = 10^{-19}$. Here λ are eigenvalues, δ_c is ratio error of realization of non-flowing condition at angular point, δ_b is ratio error of realization of non-flowing condition at the point $\Delta z = 0,2R$ higher the level of a free surface of liquid. Ratio error was calculated according to the following formula

$$\delta = \frac{\partial\varphi}{\partial n}\Big|_{\Sigma} \Big/ \max \frac{\partial\varphi}{\partial n}\Big|_{S_0}.$$

The variant *1* corresponds to classical realization of the variational method, the variant *2* corresponds to the method of successive refinement and the variant *3* corresponds to the method of an auxiliary domain applied in the aggregate with the method of successive refinement. As it follows from Table 1 application of the method of successive refinement does not influence the frequency, and application of the method of an auxiliary domain for $\Delta z = \Delta H = 0{,}2R$ raises the frequency no more than on 10^{-4}. Moreover, in the case $\Delta H = R$; $\lambda = 1{,}210119$, i.e., distinction by frequency does not exceed $0{,}2\%$. This shows potential of approximate determination of frequencies in tanks of revolution according to the following scheme: we solve the problem for maximally required filling level with improved determination of natural modes; then, by known eigenfunctions for the maximal level we determine the frequency for arbitrary intermediate leven on the basis of the the

Table 1

Parameter	Variant		
	1	2	3
	$m = 1$		
δ_c	$1{,}6 \cdot 10^{-4}$	$7{,}4 \cdot 10^{-5}$	$6{,}1 \cdot 10^{-5}$
δ_b	0,172	4,64	$0{,}956 \cdot 10^{-3}$
λ	1,20772	1,20772	1,20782
	$m = 2$		
δ_c	$4{,}7 \cdot 10^{-5}$	$7{,}2 \cdot 10^{-5}$	$3{,}2 \cdot 10^{-5}$
δ_b	0,212	0,942	$1{,}4 \cdot 10^{-3}$
λ	2,30753	2,30753	2,30781

Rayleigh method, i.e., by calculation of two quadratures from similar coordinate function for the maximal level taken on the cross-section, which corresponds to the required free surface. Here it is necessary to note that the coordinate function satisfies the non-flowing condition on Σ_0 with high accuracy.

On the basis of analysis of errors of realization of non-flowing condition presented in Table 1 it is possible to note that in cases of the classical method and the method of successive refinement behavior of the solution above the free surface is unsatisfactory from the point of view of potentials of application of this solution as coordinate functions for solving the nonlinear problem. This is perfectly explicable, since statement of the boundary problem (11) obtained on the basis of the linear theory and its hypotheses does not impose restrictions on character of behavior of the solution above the free surface of liquid. The solution obtained by the method of auxiliary domain with the acceptable accuracy “follows” the contour above the free surface and values of deviation δ on Σ_0 also decrease.

Graph results reflecting behavior of the solution on $\Delta\Sigma$ are showed in Fig. 3. Here curves are enumerated in the following way: 1 corresponds to the classical method; 2 is associated with the method of an auxiliary domain. In the case of the spherical tank with liquid filling $H = 0{,}5R$ we take into consideration $N = 22$ functions $w_k^{(m)}$. For defining the auxiliary domain it was accepted $\Delta H = 0{,}25R$ for the mode $m = 1,\, k = 1$ and $\Delta H = 0{,}15R$ for the mode $m = 2,\, k = 1$. The law of alteration of ratio error δ of non-flowing condition on Σ_0 for the mode $m = 1,\, k = 1$ is schematically shown in Fig. 3. Character of alteration of δ along the surface Σ_0 is such, that δ reaches the maximum in a vicinity of L_0, i.e., errors on the contour of the generatrix are much smaller than at the corner point. So, errors obtained on the basis of the classical method and the method of an auxiliary domain are approximately of the same order. However, behavior of the solutions on $\Delta\Sigma$ differs fundamentally. For the classical method errors δ on the upper boundary $\Delta\Sigma$ are much greater than errors obtained by the method of auxiliary domain

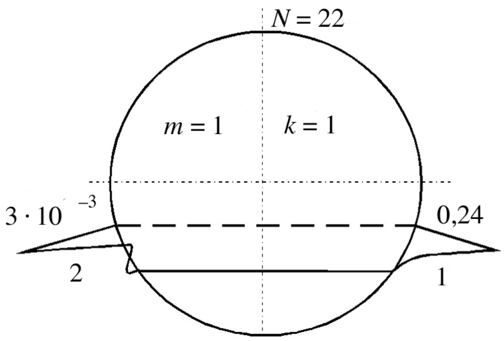

Fig. 3. Character of behavior of the solution above the free surface of liquid.

(in 800 times for the mode $m = 1$; $k = 1$ and in 1800 times for the mode $m = 2$; $k = 1$), in spite of closeness of both solutions by frequency parameter and values of δ on Σ_0.

We note that as a result of application of the auxiliary procedure of projection of function ψ_i' onto S_0 the property of orthogonality of functions $\bar{\psi}_i$, which correspond to same m and different k, is violated.

Closeness of the obtained coordinate functions and the corresponding frequency parameters to natural modes and natural frequencies of the problem (11) is fundamentally significant for application of methods of solving nonlinear problems owing to widely used division of natural modes of liquid oscillations into classes [17, 19, 20]. This division is realized on the basis of mechanical analysis of potential contribution of coordinate functions into formation of wave processes with taking into account values of the corresponding frequencies. Later on this division is used for introduction of hypothesis about ways of simulation of nonlinear terms for amplitudes, which correspond to different natural modes. This division into classes was analyzed qualitatively and quantitatively in [17, 20], where effectiveness of this techniques was shown.

In contrast to the classical approach, when the problem statement does not contain information about character of variation of the surface $\Delta\Sigma$ above the level of the undisturbed free surface, the method of an auxiliary domain reflects dependence of the solution of the problem about determination of frequencies and coordinate functions on characteristics of $\Delta\Sigma$.

The suggested methods of successive refinement of coordinate functions and the method of an auxiliary domain according to their essence are engineering approaches for construction of coordinate functions, which do not possess mathematical stringency. At the same time simplicity of numerical realization of the desribed approaches for arbitrary cavities of revolution, their mechanical clearness and obtaining good final results of realization of the non-flowing condition more precise than by classical methods of solving the problem (11) make it possible to recommend the proposed methods for construction of coordinate functions of nonlinear problem, search of reductive dependence of the frequency on filling depth.

For realization of the described techniques of improved determination of coordinate functions close to natural modes of oscillation of liquid in cavity of revolution we suggest the following algorithm:

1. The initial eigenvalue problem is solved by the method of an auxiliary domain.

2. Refinement of the problem solution is realized according to the successive approach.

3. In the case, when we include into the system of coordinate function several functions with the same circular number, then we produced their orthogonalization (functions with different circular numbers are automatically orthogonal).

4. For every function we determine the frequency parameter by the Rayleigh method (these parameters are mainly used for verification and estimation of frequency characteristics).

For solving the nonlinear problem of dynamics of combined motion of a reservoir and liquid, which partially fills cavity of revolution, we applied the following discretization parameters $n_1 = 10$, $n_2 = 6$, $n_3 = 3$. Moreover, the coordinate functions are ordered in the following way:

$$\psi_1 = \psi^*_{1,1}\sin\theta, \quad \psi_2 = \psi^*_{1,1}\cos\theta, \quad \psi_3 = \psi^*_{0,1},$$
$$\psi_4 = \psi^*_{2,1}\sin 2\theta, \quad \psi_5 = \psi^*_{2,1}\cos 2\theta, \quad \psi_6 = \psi^*_{0,2}, \quad \psi_7 = \psi^*_{3,1}\sin 3\theta, \tag{15}$$
$$\psi_8 = \psi^*_{3,1}\sin 3\theta, \quad \psi_9 = \psi^*_{1,2}\sin\theta, \quad \psi_{10} = \psi^*_{1,2}\cos\theta,$$

where $\psi^*_{m,k}$ is the solution of the problem about refined determination of the coordinate function closed to the natural mode of oscillations for the circular number m, to which the k-the eigenvalue corresponds (if eigenvalues are put in ascending order).

Tables 2 – 4 shows results of numerical determination of coordinate functions (15) for cavities of spherical (two filling levels) and conic shapes. Here we use the following denotations Δz is the level, which determines the value of auxiliary increase of the domain τ. We see that all coordinate functions with sufficiently high accuracy “follow” the profile of $\Delta\Sigma$. Moreover, it is necessary to note that on solving of the nonlinear problem amplitudes of oscitation of natural modes have order 0,1 – 0,3, which reduces in 3 – 10 times errors of realization of the non-flowing condition on the surface $\Sigma + \Delta\Sigma$.

The suggested technique of determination of coordinate functions is based on solving the linear problem, but it is supplemented by a number of requirements, which lie outside the scope of the linear statement of the problem and reflect a part of kinematic requirements and solvability conditions of the nonlinear problem. Finally this makes it possible to construct the system of coordinate functions, which in improved way holds the non-flowing condition on the perturbed moisten boundary of the domain occupied by liquid. Further this system of functions was successfully used for solving the nonlinear problem.

5. Construction of decompositions of desired variables, which hold nonlinear kinematic boundary conditions. According to results of [19, 20] we select decompositions of the desired variables of excitation of a free surface of liquid ξ and the velocity potential φ in the following form:

$$\xi = \bar{\xi}(t) + \sum_i a_i \bar{\psi}_i(\alpha) T_i(\theta), \quad \Phi = \dot{\vec{\varepsilon}} \cdot \vec{r} + \varphi_0,$$
$$\varphi = \sum b_i \psi_i(\alpha, \beta) T_i(\theta), \tag{16}$$

where

$$\bar{\psi}_i(\alpha) = \frac{\partial \psi_i}{\partial z}\Big|_{\beta=0}. \tag{17}$$

Table 2

Sphere $H = 0{,}5R,\ \Delta z = 0{,}25$						
$m; k$	$1;\ 1$	$1;\ 2$	$0;\ 1$	$0;\ 2$	$2;\ 1$	$3;\ 1$
λ	$1{,}2079$	$4{,}5120$	$3{,}1057$	$5{,}2670$	$2{,}3080$	$3{,}3703$
δ_i	$2{,}4 \cdot 10^{-4}$	$7{,}5 \cdot 10^{-5}$	$1{,}7 \cdot 10^{-4}$	$4{,}3 \cdot 10^{-4}$	$2{,}8 \cdot 10^{-4}$	$3{,}1 \cdot 10^{-4}$
δ_c	$2{,}1 \cdot 10^{-4}$	$7{,}1 \cdot 10^{-5}$	$1{,}6 \cdot 10^{-4}$	$4{,}1 \cdot 10^{-4}$	$3{,}2 \cdot 10^{-4}$	$3{,}9 \cdot 10^{-4}$
δ_b	$3{,}1 \cdot 10^{-3}$	$9{,}1 \cdot 10^{-4}$	$2{,}6 \cdot 10^{-3}$	$6{,}7 \cdot 10^{-3}$	$4{,}1 \cdot 10^{-3}$	$5{,}2 \cdot 10^{-3}$

Table 3

Sphere $H = R,\ \Delta z = 0{,}2$						
$m; k$	$1; 1$	$1; 2$	$2; 1$	$0; 1$	$0; 2$	$3; 1$
λ	$1{,}5622$	$5{,}2315$	$2{,}8246$	$3{,}7345$	$6{,}8658$	$4{,}0020$
δ_i	$2{,}9 \cdot 10^{-3}$	$7{,}2 \cdot 10^{-3}$	$3{,}1 \cdot 10^{-3}$	$5{,}9 \cdot 10^{-3}$	$1{,}4 \cdot 10^{-2}$	$4{,}0 \cdot 10^{-3}$
δ_c	$2{,}3 \cdot 10^{-3}$	$4{,}1 \cdot 10^{-3}$	$1{,}2 \cdot 10^{-3}$	$1{,}7 \cdot 10^{-3}$	$2{,}1 \cdot 10^{-3}$	$3{,}4 \cdot 10^{-4}$
δ_b	$8{,}9 \cdot 10^{-2}$	$1{,}1 \cdot 10^{-1}$	$4{,}8 \cdot 10^{-2}$	$7{,}3 \cdot 10^{-2}$	$1{,}7 \cdot 10^{-1}$	$5{,}8 \cdot 10^{-2}$

Table 4

Cone $H = R,\ \Delta z = 0{,}25$						
$m;\ k$	$1; 1$	$1; 2$	$0; 1$	$0; 2$	$2; 1$	$3; 1$
λ	$1{,}0000$	$2{,}0272$	$1{,}2972$	$2{,}9844$	$1{,}7675$	$2{,}5054$
δ_i	$4 \cdot 10^{-6}$	$4{,}2 \cdot 10^{-4}$	$8{,}0 \cdot 10^{-6}$	$5{,}2 \cdot 10^{-4}$	$6 \cdot 10^{-6}$	$9 \cdot 10^{-6}$
δ_c	10^{-6}	$5{,}2 \cdot 10^{-5}$	$2{,}0 \cdot 10^{-6}$	$3{,}9 \cdot 10^{-5}$	$1 \cdot 10^{-6}$	$3 \cdot 10^{-6}$
δ_b	$1 \cdot 10^{-6}$	$5{,}2 \cdot 10^{-4}$	$1{,}9 \cdot 10^{-5}$	$6{,}2 \cdot 10^{-4}$	$7 \cdot 10^{-6}$	$2 \cdot 10^{-5}$

Due to specificity of a cavity shape we separate the circular coordinate in the decompositions (16). Here $T_i(\theta)$ are trigonometric functions, which selection is predetermined by distribution of functions (15). The functions $\psi_i(\alpha, \beta)$ are constructed by the described above technique, therefore, they are harmonic and satisfy with high accuracy the non-flowing conditions on the moisten border Σ including a part of this domain, where waves can reach above the level of the unperturbed free surface. The functions $\bar{\psi}_i(\alpha)$ possess the property of completeness on S_0 [22, 28]. We note also that in contrast to the case of cylindrical domains decompositions (16) contain the term $\bar{\xi}(t)$, which is determined from the requirement of conservation of the liquid volume in its perturbed motion.

Following the approach of publications [15 – 20, 27] we select the amplitude parameters a_i as independent variables. Therefore, we shall determine interdependence $b_i = b_i(a_j, \dot{a}_k)$ from the kinematic boundary condition on a free surface (9), and the dependence $\overline{\xi} = \overline{\xi}(a_i, t)$ from the requirement of conservation of the liquid volume (in the case of absence of liquid outflow this dependence transforms into the form $\overline{\xi} = \overline{\xi}(a_i)$). To this end we represent $\overline{\xi}$ and b_i as

$$\overline{\xi} = \overline{\xi}^{(1)} + \overline{\xi}^{(2)} + \overline{\xi}^{(3)}; \quad b_i = b_i^{(1)} + b_i^{(2)} + b_i^{(3)} + b_i^{(4)}, \tag{18}$$

where upper indexes correspond to the order of smallness of values, if we accept the order of a_i as a small value.

On computation of variation of the liquid volume in its perturbed motion we make use of the above introduced non-Cartesian parameterization α, θ, β

$$\Delta V = \int\limits_{\tau} d\tau - \int\limits_{\tau_0} d\tau = \int\limits_0^{2\pi}\int\limits_0^1\int\limits_{-1}^{\xi/H} [f(H\beta)]^2 d\beta \alpha H\, d\alpha\, d\theta -$$

$$-\int\limits_0^{2\pi}\int\limits_0^1\int\limits_{-1}^{0} [f(H\beta)]^2 d\beta \alpha H\, d\alpha\, d\theta =$$

$$= \int\limits_0^{2\pi}\int\limits_0^1 \left[\int\limits_{-1}^{\xi/H} f^2(H\beta)\, d\beta\right] \alpha H d\alpha\, d\theta.$$

For calculation of the last integral we make use of the formula for integration of expressions over variable volume, which is based on application of the Taylor expansion of the integral with variable limits of integration [20]. After realization of this type of integration in the analytical form and substitution of decompositions for ξ from requirements of volume conservation we obtain the following expressions for $\overline{\xi}^{(i)}$:

$$\overline{\xi}^{(1)} = 0, \qquad \overline{\xi}^{(2)} = -\frac{f'(0)}{\pi f(0)} \sum_{i,j} a_i a_j \beta_{ij}^v,$$
$$\overline{\xi}^{(3)} = -\frac{f'^2(0) + f(0)f''(0)}{3\pi f^2(0)} \sum_{i,j,k} a_i a_j a_k \gamma_{ijk}^v. \tag{19}$$

As it is seen from the relations (19), the requirement of volume conservation in amplitude parameters represents certain holonomic constraint, which makes it possible to eliminate parameters $\overline{\xi}$. Moreover, the linear term in representation of $\overline{\xi}$ is absent. Character of variation of coefficients β_{ij}^v (a positively defined matrix for the non-orthogonalized system and a diagonal one with positive elements for the orthogonal system) shows that $\overline{\xi}^{(2)}$ is a function of constant sign and its sign is opposite to sign of $f'(0)$, i.e., $\overline{\xi}_2$ is determined by amplitudes a_i^2 and inclination angle of tank walls in a vicinity of a free surface. The value $\overline{\xi}^{(3)}$ depends on curvature of the surface Σ_0 in a vicinity of L_0.

Realization of the procedure of elimination of the kinematic boundary condition on a free surface is similar to the procedure for a cylindrical tank [15–20]. Distinction of this procedure consists in the property that in derivations it is necessary to keep terms $\overline{\xi}(a_i)$ presented in (19), take into account additional terms of (9), which appear owing to non-cylindrical shape of the tank cavity, and produce decompositions into series with respect to ξ not only unknown functions, but their products with the Jacobian of transition to non-Cartesian parametrization, which is also a function of β.

By omitting intermediate derivation we write down dependencies $b_i = b_i(a_j, \dot{a}_k)$, which are similar to the case of cylindrical cavity represent non-holonomic constraints. However, by application of methods of nonlinear mechanics and the Galerkin method they admit elimination from consideration the set of dependent parameters b_i :

$$b_p^{(1)} = \dot{a}_p, \qquad b_p^{(2)} = \sum_{i,j} \dot{a}_i a_j \gamma_{ijk}^c,$$
$$b_p^{(3)} = \sum_{i,j,k} \dot{a}_i a_j a_k \delta_{ijkp}^c, \qquad b_p^{(4)} = \sum_{i,j,k,l} \dot{a}_i a_j a_k a_l h_{ijklp}^c. \tag{20}$$

The relations (19) and (20) include different coefficients, which are determined by quadratures from functions ψ_i and T_i over the unknown free surface of liquid S_0. Their explicit form will be given below.

After determination of dependencies (19) and (20) parameters a_i can be supposed as the complete independent system of variables, which characterizes motion of the limited volume of liquid. Now we can pass to immediate realization of the Hamilton – Ostrogradsky variational principle for unconstrained mechanical system, which motion is set by the generalized coordinates a_i and ε_k.

6. Construction of resolving system of motion equations relative to amplitude parameters of liquid motion and parameters of translational motion of the carrying body. We shall derive the motion equations of the system on the basis of the Hamilton – Ostrogradsky variational principle applied to dynamics of bounded liquid volume and a rigid body with cavity of revolution. For transition from continuum structure of the initial model of the system rigid body – liquid to its discrete model we make use of the Kantorovich method. Here spatial and surface integration in separate terms of the Lagrange function is realized in variables α, θ, β. We note that calculation of volumetric integrals in variables α, θ, β can be reduced to successive integration with analytical derivation of integrals over liquid depth [20].

Finally this volumetric integrals are reduced to surface integrals over the undisturbed free surface S_0 from expressions in the form of expansions relative to powers of ξ. In contrast to the case of the cylindrical cavity occupied by liquid not only terms, which contain the velocity potential φ are differentiated by β, but also terms, which contain products of the velocity potential and the Jacobian of transition to non-Cartesian parametrisation of the liquid domain τ_0.

On the whole the general procedure of transition from the continuum structure of the initial mechanical system body – liquid to its discrete model (it is based on application of the Kantorovich method to the variational formulation of the problem in the form of the Hamilton – Ostrogradsky variational principle) differs insignificantly from the case of the cylindrical domain occupied by liquid [15 – 20]. Therefore, we do not describe this procedure in details. Main distinction consists not in the techniques of such a transition, but in the following two properties: first, decomposition of the velocity potential relative to coordinate function is realized with respect to the system holds the non-flowing condition approximately, second, a group of geometrical nonlinearities defined by the relation (19) is supplemented, which predetermine additional dependence of all natural modes of oscillations.

As the result of application of the proposed technique we obtain the following Lagrange equation of the second kind, i.e., the motion equations of the system body – liquid in amplitude parameters a_i and parameters of translational motion of the carrying body ε_i :

$$\sum_i \ddot{a}_i\left(V^1_{ir} + \sum_j a_j V^2_{irj} + \sum_{j,k} a_j a_k V^3_{irjk}\right) +$$

$$+\ddot{\vec{\varepsilon}}\cdot\left(\vec{U}^1_r + \sum_i a_i \vec{U}^2_{ri} + \sum_{j,k} a_i a_j \vec{U}^3_{rij} + \sum_{i,j,k} a_i a_j a_k \vec{U}^4_{rijk}\right) =$$

$$= \sum_{i,j} \dot{a}_i \dot{a}_j V^{2*}_{ijr} + \sum_{i,j,k} \dot{a}_i \dot{a}_j a_k V^{3*}_{ijrk} +$$

$$+\dot{\vec{\varepsilon}}\cdot\left(\sum_i \dot{a}_i \vec{U}^{2*}_{ir} + \sum_{i,j} \dot{a}_i a_j \vec{U}^{3*}_{ijr} + \sum_{i,j,k} \dot{a}_i a_j a_k \vec{U}^{4*}_{ijkr}\right) -$$

$$-g\left(\sum_i a_i W^2_{ir} + \frac{3}{2}\sum_{i,j} a_i a_j W^3_{ijr} + 2\sum_{i,j,k} a_i a_j a_k W^4_{ijkr}\right), \tag{21}$$

$$r = 1, 2, \ldots, N,$$

$$\frac{\rho}{M_{\mathrm{r}} + M_{\mathrm{l}}} \sum_i \ddot{a}_i\left(\vec{U}^1_i + \sum_j a_j \vec{U}^2_{ij} + \sum_{j,k} a_j a_k \vec{U}^3_{ijk}\right) + \ddot{\vec{\varepsilon}} =$$

$$= \frac{\vec{F}}{M_{\mathrm{r}} + M_{\mathrm{l}}} + \vec{g} - \frac{\rho}{M_{\mathrm{p}} + M} \sum_{i,j} \dot{a}_i \dot{a}_j \left(\vec{U}^2_{ij} + 2\sum_k a_k \vec{U}^3_{ijk}\right). \tag{22}$$

The mentioned system of equation represents the nonlinear model of dynamics of a body and liquid, which partially fills its cavity, in the case of translational motion of the carrying body. For construction of the system of equations (21), (22) it is necessary to compute the following quadratures from the coordinate functions ψ_i and $\overline{\psi}_i$:

$$N_p = \int_0^{2\pi}\int_0^1 \overline{\psi}_i^2 T_i^2 \alpha d\alpha\, d\theta,$$

$$\beta^v_{ij} = N_i \delta_{ij},$$

$$\gamma^v_{ijk} = \int_0^{2\pi}\int_0^1 \overline{\psi}_i \overline{\psi}_j \overline{\psi}_k T_i T_j T_k \alpha\, d\alpha\, d\theta,$$

$$\delta^v_{ijkl} = \int_0^{2\pi}\int_0^1 \overline{\psi}_i \overline{\psi}_j \overline{\psi}_k \psi_l T_i T_j T_k T_l \alpha\, d\alpha\, d\theta,$$

$$\gamma^{b0}_{ijp} = \int_0^{2\pi}\int_0^1 \left[\left(\alpha A^0_{1i}\frac{\partial\overline{\psi}_j}{\partial\alpha} + \alpha^2 A^0_{3i}\frac{\partial\overline{\psi}_j}{\partial\alpha} - \alpha A^1_{4i}\overline{\psi}_j\right) T_i T_j +\right.$$

$$+\frac{1}{\alpha}A_{2i}^{0}\overline{\psi}_{j}T_i'T_j'\Bigg]\overline{\psi}_{p}T_{p}\,d\alpha d\theta,$$

$$\gamma_{ijp}^{b1}=\int_0^{2\pi}\int_0^1\left[\left(\alpha A_{1i}^{1}\frac{\partial\overline{\psi}_j}{\partial\alpha}+\alpha^2A_{3i}^{1}\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^{2}\overline{\psi}_j\right)T_iT_j+\right.$$

$$\left.+\frac{1}{\alpha}A_{2i}^{1}\overline{\psi}_{j}T_i'T_j'\right]\overline{\psi}_{p}T_{p}d\alpha\,d\theta,$$

$$\delta_{ijkp}^{b1}=\int_0^{2\pi}\int_0^1\left[\left(A_{1i}^{1}\alpha\frac{\partial\overline{\psi}_j}{\partial\alpha}+\alpha^2A_{3i}^{1}\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^{2}\overline{\psi}_j\right)T_iT_j+\right.$$

$$\left.+\frac{1}{\alpha}A_{2i}^{1}\overline{\psi}_{j}T_i'T_j'\right]\overline{\psi}_{j}k\overline{\psi}_{p}T_kT_{p}d\alpha\,d\theta,$$

$$h_{ijklp}^{b2}=\int_0^{2\pi}\int_0^1\left[\left(\alpha A_{1i}^{2}\frac{\partial\overline{\psi}_j}{\partial\alpha}+\alpha^2A_{3i}^{2}\frac{\partial\overline{\psi}_j}{\partial\alpha}-\alpha A_{4i}^{3}\overline{\psi}_j\right)T_iT_j+\right.$$

$$\left.+\frac{1}{\alpha}A_{2i}^{2}\overline{\psi}_{j}T_i'T_j'\right]\overline{\psi}_{k}\overline{\psi}_{l}\overline{\psi}_{p}T_kT_lT_{p}\,d\alpha d\theta,$$

$$\beta_{ip}^{b1}=\int_0^{2\pi}\int_0^1 A_{4i}^{1}\overline{\psi}_{p}T_iT_p\alpha\,d\alpha d\beta,$$

$$\beta_{ij}^{f2}=\int_0^{2\pi}\int_0^1\overline{\psi}_i\overline{\psi}_jT_iT_j\alpha\,d\alpha d\theta f^2(0),$$

$$\beta_{i}^{f3}=\int_0^{2\pi}\int_0^1\left[\alpha\left(\frac{\partial\psi_j}{\partial\alpha}\right)^2T_i^2+\frac{1}{\alpha}\psi_i^2T_i'^2+\alpha\,f^2\overline{\psi}_i^2T_i^2\right]\Bigg|_{\beta=0}d\alpha d\theta, \tag{23}$$

$$\gamma_{ijk}^{f3}=\int_0^{2\pi}\int_0^1\left[\alpha\frac{\partial\psi_i}{\partial\alpha}\frac{\partial\psi_j}{\partial\alpha}T_iT_j+\frac{1}{\alpha}\psi_i\psi_jT_i'T_j'+\alpha\,f^2\overline{\psi}_i^2\overline{\psi}_jT_iT_j\right]\Bigg|_{\beta=0}\overline{\psi}_kT_kd\alpha d\theta,$$

$$\delta_{ijkl}^{f4}=\int_0^{2\pi}\int_0^1\left[\frac{\alpha}{2H}\left(\frac{\partial^2\psi_i}{\partial\alpha\partial\beta}\frac{\partial\psi_j}{\partial\beta}+\frac{\partial\psi_i}{\partial\alpha}\frac{\partial^2\psi_j}{\partial\alpha\partial\beta}\right)T_iT_j+\right.$$

$$+\frac{1}{2\alpha H}\left(\frac{\partial\psi_i}{\partial\beta}\psi_j+\psi_i\frac{\partial\psi_j}{\partial\beta}\right)T_i'T_j'+$$

$$\left.+\frac{\alpha}{2H}\left(2Hff'\frac{\partial\psi_i}{\partial z}\frac{\partial\psi_j}{\partial z}+f^2\frac{\partial^2\psi_i}{\partial z\partial\beta}+f^2\frac{\partial\psi_i}{\partial z}\frac{\partial^2\psi_j}{\partial z\partial\beta}\right)T_iT_j\right]\Bigg|_{\beta=0}\overline{\psi}_k\overline{\psi}_lT_kT_ld\alpha d\theta,$$

$$\alpha_i^{x1} = \int_0^{2\pi}\int_0^1 \alpha^2 f\big|_{\beta=0}\overline{\psi}_i T_i \cos\theta d\alpha d\theta,$$

$$\alpha_i^{x2} = \int_0^{2\pi}\int_0^1 \left(\alpha f\frac{\partial\psi_i}{\partial\alpha}T_i\cos\theta - f\psi_i T_i'\sin\theta\right)\Big|_{\beta=0} d\alpha d\theta,$$

$$\beta_{ij}^{x2} = \int_0^{2\pi}\int_0^1 \left(\alpha f\frac{\partial\psi_i}{\partial\alpha}T_i\cos\theta - f\psi_i T_i'\sin\theta\right)\overline{\psi}_j T_j\, d\alpha d\theta,$$

$$\gamma_{ijk}^{x3} = \frac{1}{2H}\int_0^{2\pi}\int_0^1 \Bigg[\left(\alpha H f'\frac{\partial\psi_i}{\partial\alpha} + \alpha f\frac{\partial^2\psi_i}{\partial\alpha\partial\beta}\right)T_i\cos\theta -$$

$$-\left(Hf'\psi_i + f\frac{\partial\psi_i}{\partial\beta}\right)T_i'\sin\theta\Bigg]\Bigg|_{\beta=0}\overline{\psi}_j\overline{\psi}_k T_j T_k d\alpha d\theta,$$

$$\delta_{ijkl}^{x4} = \frac{1}{6H^2}\int_0^{2\pi}\int_0^1 \Bigg[\left(H^2 f''\alpha\frac{\partial\psi_i}{\partial\alpha} + 2Hf'\alpha\frac{\partial^2\psi_i}{\partial\alpha\partial\beta} + \alpha f\frac{\partial^3\psi_i}{\partial\alpha\partial\beta^2}\right)T_i\cos\theta -$$

$$-\left(H^2 f''\psi_i + 2Hf'\frac{\partial\psi_i}{\partial\beta} + f\frac{\partial^2\psi_i}{\partial\beta^2}\right)T_i'\sin\theta\Bigg]\Bigg|_{\beta=0}\overline{\psi}_j\overline{\psi}_k\overline{\psi}_p T_j T_k T_p d\alpha d\theta,$$

$$\beta_{ij}^{z2} = f^2(0)N_i\delta_{ij},$$

$$\gamma_{ijk}^{z3} = \frac{1}{2H}\int_0^{2\pi}\int_0^1 \left[2ff'H\overline{\psi}_i + f^2\frac{\partial^2\psi_i}{\partial z\partial\beta}\right]\Bigg|_{\beta=0}\overline{\psi}_j\overline{\psi}_k T_i T_j T_k \alpha\, d\alpha d\theta,$$

$$\delta_{ijkl}^{z4} = \frac{1}{6H^2}\int_0^{2\pi}\int_0^1 \Bigg[4ff'Hf^2\frac{\partial^2\psi_i}{\partial z\partial\beta} + 2H^2\big(f'^2 + f''f\big)f^2\frac{\partial\psi_i}{\partial z} +$$

$$+f^2\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\Bigg]\Bigg|_{\beta=0}\overline{\psi}_j\overline{\psi}_k\overline{\psi}_k T_i T_j T_k T_l \alpha\, d\alpha d\theta.$$

We note that in the mentioned quadratures components of integration by α and θ are always separated, y-components of vectors are calculated similar to x-components, but it is necessary to substitute for $-\cos\theta$ instead of $\sin\theta$ and $\sin\theta$ instead of $\cos\theta$ (in relations (23) expressions for y-components are omitted). On calculation of integrals we use the following denotations, i.e., δ_{ij} is the Kronecker symbol, $f^{(i)} = f^{(i)}(0)$, where i is the order of differentiation. Numerical algorithm of determination of quadrature was based on the Gauss quadrature formula with 96 points of division. We introduced also the following functions of the argument α in the relations (23)

$$A_{1i}^0 = \frac{1}{f^2}\overline{\psi}_i, \qquad A_{1i}^1 = \frac{1}{Hf^2}\left(\frac{\partial^2\psi_i}{\partial\alpha\partial\beta} - \frac{2Hf'}{f}\frac{\partial\psi_i}{\partial\alpha}\right)\Bigg|_{\beta=0}$$

$$A_{1i}^2 = \frac{1}{2H^2 f^2}\left[\frac{\partial^3 \psi_i}{\partial\alpha\partial\beta^2} - \frac{4Hf'}{f}\frac{\partial^2\psi_i}{\partial\alpha\partial\beta} + H^2\left(6\frac{f'^2}{f^2} - 2\frac{f''}{f}\right)\frac{\partial\psi_i}{\partial\alpha}\right]\Bigg|_{\beta=0},$$

$$A_{2i}^0 = \frac{1}{f^2}\psi_i\Big|_{\beta=0}, \qquad A_{2i}^1 = \frac{1}{Hf^2}\left(\frac{\partial\psi_i}{\partial\beta} - \frac{2Hf'}{f}\psi_i\right)\Bigg|_{\beta=0},$$

$$A_{2i}^2 = \frac{1}{2H^2 f^2}\left[\frac{\partial^2\psi_i}{\partial\beta^2} - 4H\frac{f'}{f}\frac{\partial\psi_i}{\partial\beta} + H^2\left(6\frac{f'^2}{f^2} - 2\frac{f''}{f}\right)\psi_i\right]\Bigg|_{\beta=0},$$

$$A_{3i}^0 = -\frac{f'}{f}\overline{\psi}_i, \qquad A_{3i}^1 = \frac{1}{Hf}\left[\left(-Hf'' + H\frac{f'^2}{f}\right)\frac{\partial\psi_i}{\partial z} - \frac{f'}{f}\frac{\partial^2\psi_i}{\partial z\partial\beta}\right]\Bigg|_{\beta=0}, \tag{24}$$

$$A_{3i}^2 = \frac{1}{2H^2}\Bigg[\left(-H^2 + 3H^2\frac{f''f'}{f^2} - 2H^2\frac{f'^3}{f^3}\right)\frac{\partial\psi_i}{\partial z} +$$

$$+H\left(-\frac{f'}{f} + \frac{f'^2}{f^2}\right)\frac{\partial^2\psi_i}{\partial z\partial\beta} - \frac{f'}{f}\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\Bigg]\Bigg|_{\beta=0},$$

$$A_{4i}^1 = \frac{1}{H}\frac{\partial^2\psi_i}{\partial z\partial\beta}\bigg|_{\beta=0},$$

$$A_{4i}^2 = \frac{1}{2H^2}\frac{\partial^3\psi_i}{\partial z\partial\beta^2}\bigg|_{\beta=0}, \qquad A_{4i}^3 = \frac{1}{6H^3}\frac{\partial^4\psi_i}{\partial z\partial\beta^3}\bigg|_{\beta=0}.$$

On the basis of the mentioned quadratures (23), which are calculated with application of denotations (24), the coefficients of the motion equations (21) and (22) are determined according to the following algorithm:

$$\gamma_{ijp}^c = \frac{1}{N_p}\gamma_{ijp}^{b0},$$

$$\delta_{ijkp}^c = \frac{1}{N_p}\left(\sum_i \gamma_{ijm}^c\gamma_{mkp}^{b0} + \delta_{ijkp}^{b1} + \frac{g_2}{\pi g_1}\beta_{ip}^{b1}N_j\delta_{jk}\right),$$

$$h_{ijklp}^c = \frac{1}{N_p}\Bigg[\sum_m\Big(\delta_{ijkm}^c\gamma_{mlp}^{b0} + \delta_{mlkp}^{b1}\gamma_{ijm}^c +$$

$$+\frac{g_2}{\pi g_1}\gamma_{ijm}^c\beta_{mp}^{b1}N_k\delta_{kl}\Big) + \beta_{ip}^{b1}\gamma_{jkl}^v\frac{g_3}{\pi g_1}\gamma_{ijp}^{b1}N_p\delta_{kl}\Bigg],$$

$$V_{ij}^1 = \beta_{ij}^{f2}, \qquad V_{ijk}^2 = \gamma_{ijk}^{f3} + 2\sum_m \gamma_{ikm}^c\beta_{mj}^{f2},$$

$$V_{ijkl}^3 = \delta_{ijkl}^{f4} - \frac{g^2}{\pi g_1}N_k\delta_{kl}\beta_i^{f3}\delta_{ij} +$$

$$+\sum_m\left(2\gamma_{mjk}^{f3}\gamma_{ilm}^c + 2\beta_{mj}^{f2}\delta_{iklm}^c + \gamma_{jkm}^c\gamma_{ilm}^c\beta_{mm}^{f2}\right),$$

$$\vec{U}_i^1 = \alpha_i^1, \qquad U_{ij}^2 = \vec{\beta}_{ij}^2 + \sum_m \gamma_{ijm}^c\alpha_m^1,$$

$$\vec{U}^3_{ijk} = \vec{\gamma}^3_{ijk} + \sum_m \left(\delta^c_{ijkm}\vec{\alpha}^1_m + \gamma^c_{ijm}\vec{\beta}^2_{mk} \right) - \frac{g_2}{\pi g_1}\vec{\alpha}^2_i N_j \delta_{jk},$$

$$\vec{U}^4_{ijkl} = \sum_m \Bigg(h^c_{ijklm}\vec{\alpha}^1_m -$$

$$-\frac{g_2}{\pi g_1}\gamma^c_{ijm}\vec{\alpha}^2_m N_k \delta_{kl} + \delta^c_{ijkm}\vec{\beta}^2_{ml} + \gamma^c_{ilm}\vec{\gamma}^c_{mjk} \Bigg) - \tag{25}$$

$$-\frac{g_3}{\pi g_1}\gamma^v_{jkl}\vec{\alpha}^2_i + \vec{\delta}^4_{ijkl},$$

$$W^2_{ij} = e_1 N_i \delta_{ij}, \qquad W^3_{ijk} = e_2 N_i \gamma^v_{ijk},$$

$$W^4_{ijkl} = e_3 \delta^v_{ijkl} + N_i N_k \delta_{ij}\delta_{kl} \left(\frac{e_1 g_2^2}{\pi g_1^2} - \frac{3g^2 e_2}{\pi g_1} \right),$$

$$V^{2*}_{ijr} = \frac{1}{2}V^2_{ijr} - V^2_{irj}, \qquad V^{3*}_{ijkr} = V^3_{ijkr} - 2V^3_{irjk},$$

$$\vec{U}^{2*}_{ir} = \vec{U}^2_{ir} - \vec{U}^2_{ri}, \qquad \vec{U}^{3*}_{ijr} = 2\big(\vec{U}^3_{ijr} - \vec{U}^3_{rij}\big),$$

$$\vec{U}^{4*}_{ijkr} = 3\big(\vec{U}^4_{ijkr} - \vec{U}^4_{rijk}\big), \qquad e_1 = f^2, \qquad e_2 = \frac{4}{3}f'f,$$

$$e_3 = \frac{1}{2}(f''f + f'^2), \qquad g_1 = \pi e_1, \qquad g_2 = \pi f f',$$

$$g_3 = \frac{\pi}{3}\,(f'^2 + ff'').$$

Basic stages of numerical realization of the suggested approach mainly coincide with the case of cylindrical tank. However, distinction consists in the property that approximate determination of coordinate functions close to natural modes of oscillations becomes significant component of the procedure. For determination of parameters of an auxiliary domain $\Delta\tau$ we conventionally select $\Delta z = 0{,}25$, i.e., excess of the filling level of liquid z related to the radius of the undisturbed free surface. This parameter defines the domain $\Delta\tau$ and the surface $\Delta\Sigma$ on application of the method of an auxiliary domain (for comparison we consider also variants $\Delta z = 0$, $\Delta z = 0{,}2$, $\Delta z = 0{,}3$).

We consider the problem about determination of coefficients of the motion equations for nonlinear oscillation of liquid in circle cylindrical tank as a testing one. Good concordance with results of publications [10, 20, 23] was obtained, i.e., coefficients of the motion equations coincide accurate to four significant digits.

7. Numerical examples. For verification of the suggested approach we investigate problems about transient processes in the system tank — liquid in the case when cavity partially filled by liquid is of revolution shape. We realize the following three variants: the conic tank with the half-angle $\alpha = \dfrac{\pi}{4}$ and $H = R$; spherical tank for $H = 0{,}5R$ and $H = R$. Calculations were realized on the basis of the suggested scheme with the following parameters of the nonlinear model $n_1 = 10$, $n_2 = 6$, $n_3 = 3$ and with application of the above determined coordinate functions for $\Delta z = 0{,}25$. It is necessary to note that for comparison we consider also a variant, when coordinate functions were determined according to the classical approach, which corresponds to $\Delta z = 0$. In

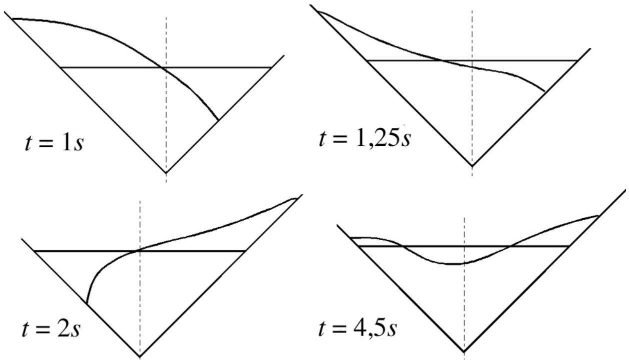

Fig. 4. Free oscillations of liquid in the immovable conic tank.

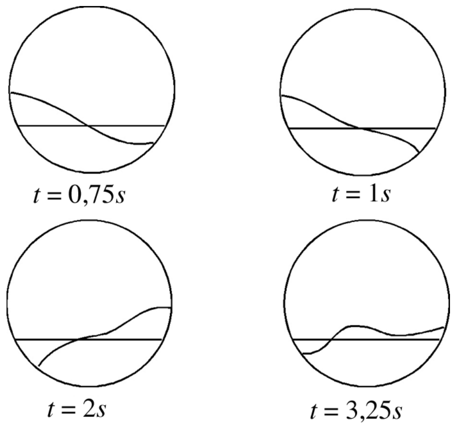

Fig. 5. Free oscillations of liquid in the immovable spherical tank.

these cases for sufficiently short time interval numerical instability takes place (for amplitudes of waves about $0{,}2R$ and $0{,}3R$ we have stable numerical solution only until 5 s and 2 s, correspondingly). Obviously, the reason of manifestation of this instability consists in violation of requirement of conservation of liquid volume, what was analyzed above.

For determination of peculiarities of development of wave generation on a free surface of liquid three groups of problems were considered:

1) free oscillations of liquid in the immovable conic (Fig. 4) and spherical $H = 0{,}5R$ (Fig. 5) tanks, caused by initial perturbations of a free surface of liquid relative to the natural mode $a_1(0) = 0{,}25R$;

2) free oscillations of liquid in movable conic (Fig. 6), spherical $H = 0{,}5R$ (Fig. 7) and spherical $H = R$ (Fig. 8) tanks, which can perform translational motion in the horizontal plan, caused by the initial perturbation of a free surface of liquid relative to the natural mode $a_1(0) = 0{,}25R$;

3) forced oscillations of liquid in movable conic (Fig. 9) and spherical $H = 01)\ 5R$ (Fig. 10) tanks, which can perform translational motion in the horizontal plane, caused by

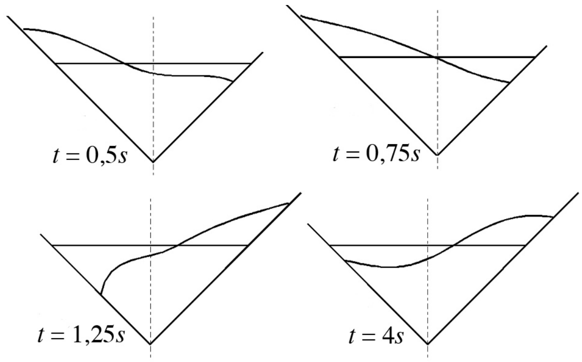

Fig. 6. Free oscillations of liquid in the movable conic tank.

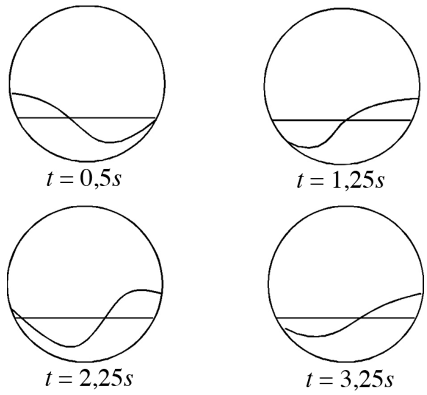

Fig. 7. Free oscillations of liquid in the movable spherical tank $H = 0{,}5R$.

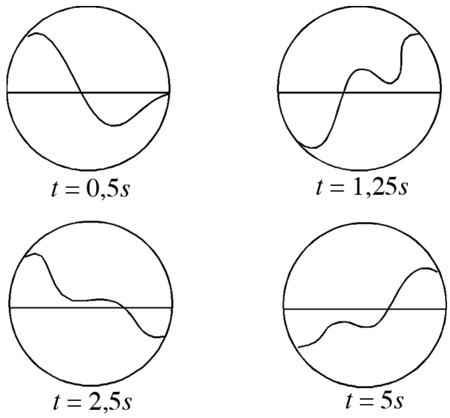

Fig. 8. Free oscillations of liquid in the movable spherical tank $H = R$.

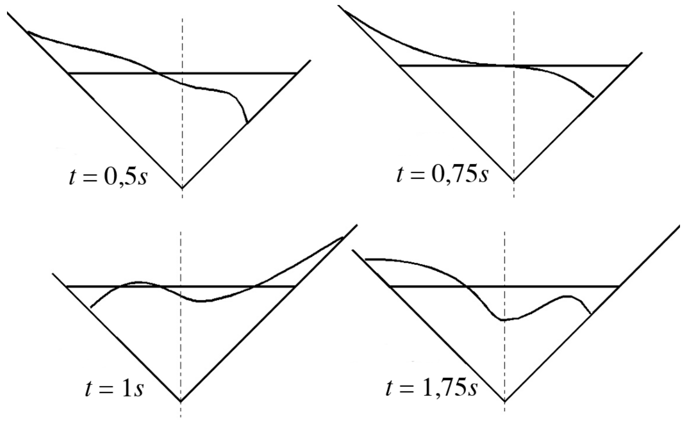

Fig. 9. Forced oscillations of liquid in movable conic tank.

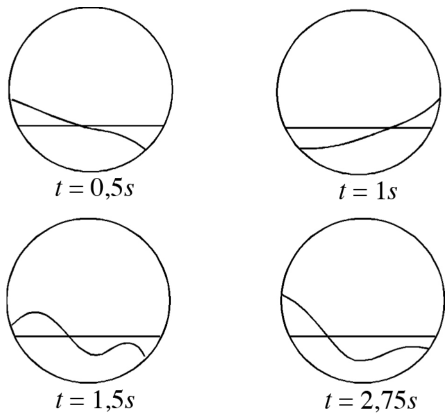

Fig. 10. Forced oscillations of liquid in movable conic tank.

sudden application of rectangular force impulse with the amplitude $F = 1{,}2(M_{\mathrm{r}} + M_{\mathrm{l}})$ and duration $\tau = 0{,}5$ s to the quiescent system.

In all cases we accept $M_{\mathrm{r}} = 0,2M_{\mathrm{l}}$; $R = 1$ m; σ and ρ were selected for water. Figures also include time, which corresponds to the observed states of a free surface of liquid.

From the analysis of behavior of a free surface of liquid in the case of free oscillations of liquid in immovable conic and spherical reservoirs it is seen that the property of excess of the wave crest over the depth of trough is manifested not so notably as in the case of a cylindrical tank, and in some time instants it is violated at all. This is caused first of all by the property that $\overline{\xi}_2 < 0$ as well as by the property that relatively large inclination of tank walls in outside direction promotes conditions for liquid displacement sideways over the undisturbed free surface larger than below it. It is seen from Figures 4 and 5 that for certain time instants perturbation of axis-symmetric natural modes is manifested essentially, which is only consequence of presence of internal nonlinear constraints in the system.

In the case of free oscillations of liquid in movable tank (Figures 6 and 7) we observe

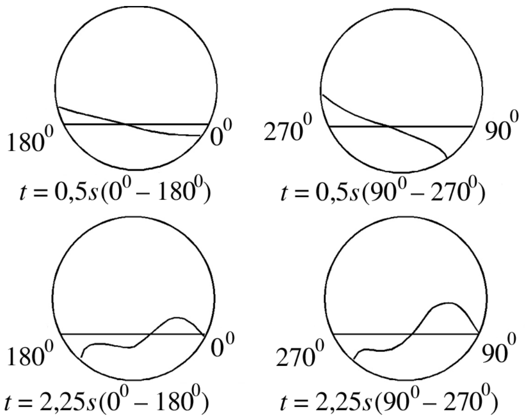

Fig. 11. Development of spatial wave generation in spherical tank.

approximately the same qualitative picture of development of wave generation. Here contribution of natural modes ψ_9 and ψ_{10}, which correspond to $m = 1$ and $k = 2$, is manifested essentially. This is caused by the property that in movable tank perturbation of high anti-symmetric modes occur even within the framework of the linear model. Especially contribution of modes ψ_9 and ψ_{10} is noticeable for the sphere $H = R$ (Fig. 8).

On impulse force excitation of tank motion (Figures 9 and 10) nonlinear interdependencies are manifested in greater extent, i.e., contribution of axis-symmetrical and high anti-symmetric modes is more substantial. The effect of excess of height of wave crest over depth of trough becomes apparent more clearly than in preceding examples.

For the analysis of influence of liquid filling on motion of a carrying body we investigate alteration in time of the reservoir velocity and the main vector of liquid pressure on tank walls. As it follows from numerical results in general influence of liquid filling on motion of a carrying body is essential. We note that at certain time instants after force disturbance the carrying body performs motion in the direction opposite to the direction of the applied force. This motion is caused by internal liquid sloshing and is manifested most clearly for the sphere at $t \approx 0{,}8$ s, i.e., on the second half-period of oscillations of a liquid free surface. Moreover, by character oscillations differ from harmonic ones. Variation of the horizontal component of the main vector of forces of liquid pressure on tank walls is close to harmonic law. however, for conic cavity in a vicinity of time 2,5 s we observe significant distortions of the harmonic law.

Results of investigation of development of spatial wave generation in movable spherical reservoir $(H = 0{,}5R)$ are shown in Fig. 11. We assume that initially a free surface of liquid is excited relative to the second natural mode with the amplitude $a_2(0) = 0{,}1R$ and the force impulse $F = 1{,}1(M_\mathrm{r} + M_\mathrm{l})$ with duration $\tau = 0{,}5$ s is applied to the tank. Figure 11 shows the picture of waves observed for time instants 0,5 s and 2,25 s in the form of cross-sections in mutually orthogonal planes. It is easy to see that in this case wave crests exceed depth of wave trough, contribution of axis-symmetric and high anti-symmetric natural modes of oscillations is noticeable (they are excited only owing to manifestation of internal nonlinear constraints in the system).

Results of the comparative analysis of behavior of reservoirs with spherical $H = 0{,}5R$, conical $H = R$ and cylindrical cavities under perturbation of their motion from the quiescent state by the rectangular force impulse $F = 1{,}2(M_r + M_\mathrm{l})$ with duration

Table 5

Parameter	Sphere $H = 0{,}5\,R$	Cone $H = R$	Cylinder $H = R$
$\xi_{\max}$	0,25	0,331	0,230
$t(\xi_{\max})$	0,55	0,60	0,56
$a_{1\,\max}$	0,188	0,249	0,205
$V_{\max}$	0,760	0,846	0,704
$t(V_{\max})$	0,35	0,50	0,50
$R_{\max}$	1,716	1,753	1,210
$t(R_{\max})$	0,50	0,50	0,54

$\tau = 0{,}5$ s and $M_r = 0{,}2M_l$ are presented in Table 5. Here $\xi_{\max}$ is the maximal perturbation of a liquid free surface on the considered time interval (until 3 s); $t(\xi_{\max})$ is time when $\xi = \xi_{\max}$; $a_{1\max}$ is the maximal value of amplitude of the first anti-symmetric mode; $V_{\max}$ is the maximal velocity of motion of the reservoir; $t(V_{\max})$ is time when $V = V_{\max}$; $R_{\max}$ is the maximal value of the horizontal component of the main vector of pressure forces on tank walls; $t(R_{\max})$ is time when $R = R_{\max}$. The realized comparison makes it possible to state that for the same relative loading and the same ratio of masses of a reservoir and liquid the case of cylindrical reservoir results in minimal manifestation of effects connected with internal sloshing of liquid, which becomes apparent for all considered parameters. So, the cylindrical shape of the tank superimposes greater restriction on liquid mobility when spherical $(H = 0{,}5R)$ and conical $(H = R)$ shapes. Comparison of spherical $(H = 0{,}5R)$ and conic $(H = R)$ tanks testifies to the property that for the considered filling levels the spherical shape superimposes greater restriction on liquid mobility when the conical one. It is interestingly to note that, apparently, these restrictions are determined first of all by inclination of tank walls of a reservoir in a vicinity of the undisturbed free surface.

8. Conclusions. We state the problem about oscillations of ideal liquid with a free surface in a tank of non-cylindrical shape. Owing to specificity of the the problem we introduce non-Cartesian coordinate system for description of the system motion. Due to the property that it is impossible to find the exact solution of the problem about determination of natural modes of liquid oscillations in arbitrary vessel, we focus our attention on approximate construction of coordinate functions of the problem, which are close to natural modes of liquid oscillation but in addition hold solvability conditions of the Neumann boundary problem for the Laplace equation describing the problem about determination of natural modes of oscillations. We propose to apply the method of successive refinement of the solution coupled with the method of an auxiliary domain. The suggested approach makes it possible to avoid some singular properties (of non-physical sense) of the initial problem, remove some mechanical contradictions in the mathematical statement of the linear problem. This approach was generated on the basis of the analysis of the mechanical nature of the statement of the problem about free oscillations of liquid with a free boundary and on properties of the solution of this problem obtained by different analytical and numerical approaches.

The constructed coordinate functions satisfy boundary conditions with the required accuracy, they "follow" tank walls above the level of the undisturbed free surface of liquid. Application of this system of boundary function results in providing stability of numerical procedures.

We construct decompositions of unknown variables and realize the procedure of analytical elimination of the kinematical boundary condition on the free surface of liquid. So, in this way we construct the complete system of amplitude parameters suitable for description of dynamics of the system and which possesses property of minimality (the number of dynamic parameters is equal to the number of mechanical degrees of freedom). For construction of the system of motion equations we make use of the Hamilton – Ostrogradsky variational principle. Realization of the suggested approach is conducted for arbitrary number of coordinate functions entrained into construction of discrete model of the system liquid — tank.

The suggested procedure was numerically realized for modes of free and forced oscillations. The obtained results about wave generation of a free surface, alteration of the tank velocity and dynamic interaction of liquid with tank walls are evidence of good reflection of general regularities of the system behavior. Comparison of properties of dynamic mobility of liquid in conic, spherical and cylindrical reservoirs made it possible to draw conclusion about factors, which determine restriction of liquid mobility in tanks.

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