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Analysis of compressor functions for Laplacian source's scalar compandor construction

A simple and complete analysis of nonuniform scalar quantizers based on the companding technique, so-called compandors, is presented. The performance of the scalar compandors for different definition of compressor functions are considered and compared with the performance of optimal Lloyd-Max's scalar quantizers. There are several definitions of compressor functions. Two of them, that are functions of the support region of the compandor, are presented in this paper. The support region of the compandor is defined with the maximum amplitude of the input signal that enables optimal compandor's load. In order to design the scalar compandor having the best performances i.e. with the smallest distortion, it is necessary to determine optimally its support region.

Key words: *compressor functions, scalar compandor, support region.*

Introduction

A vast amount of research has been made in the area of quantization. The optimal quantization problem, even for the simplest case-uniform scalar quantization, is very actual in contemporary signal processing [1, 2]. Namely, optimal designing of scalar quantizers requires optimal determining of the support region. Therefore, optimal determining of the support region has been considered by a lot of researchers [1, 2]. A full understanding of optimal quantization is not possible without a clear understanding of quantizer's support because there are different definitions of the support region [1, 2]. In this paper we consider nonuniform scalar quantizers based on the companding technique. Namely, nonuniform quantizer consisting of a compressor, a uniform quantizer, and expandor in cascade is called compandor Fig. 1 [3]. The companding technique is easily realized, therefore it has wide application. Because of its usefulness in analyzing and optimizing nonuniform quantizers having a large number of levels the companding technique has been considered in [3, 4]. Moreover, in this paper we determine the support region of the scalar compandor, $[-t_{\max}, t_{\max}]$, defined with the maximum amplitude of the input signal $t_{\max}$. Optimal determining of the support region of the compandor depends on the compressor function used for the compandor designing. Currently, determination of the support region for Lloyd-Max's quantizers is primarily of theoretical importance. Lloyd-Max's style algorithms for designing optimal scalar quantizers [4, 5] begin with an estimate of the support region — the better the estimate, the more rapid the algorithm convergence. In this paper the compandors' distortions for different compressor functions are calculated and compared with the Lloyd-Max's quantizers distortions. In such a way it is possible to chose the compressor function which provides designing of scalar compandor having the best performances.

Nonuniform scalar quantization

Let us consider an N-level nonuniform scalar quantizer Q for the Laplacian input signals. Scalar quantizer Q is defined with Q: $R \rightarrow C$, as a functional mapping of the set of real numbers R onto the set of the output representation. The set of the output representation constitutes the code book:

$$C \equiv \{y_1, y_2, y_3, ..., y_N\} \subset R \tag{1}$$

that has the size $|C| = N$. The output values, yj, are called the representation levels. The nonuniform scalar quantizer Q is defined with the set of output values and with the partition of the input range of values onto N cells i.e. intervals α_j, $j = 1,2,..., N$. Cells α_j are defined with the decision thresholds $\{t_0, t_1,\ldots, t_N\}$, such that $\alpha_j = (t_{j-1}, t_j]$, $j = 1,2,..., N$. Cells $\alpha_2,..., \alpha_{N-1}$ are referred to as the inner cells, while α_1 and α_N are referred to as the outer cells. The negative thresholds and the representation levels are symmetric to their nonnegative counterparts. A quantized signal has value y_j when the original signal belongs to the quantization cell α_j. Hence, N-level scalar quantizer is defined as a functional mapping of an input value x onto an output representation, such as:

$$Q(x) = y_j, \quad x \in \alpha_j. \tag{2}$$

When the inner cells are equally sized, the quantizer is called uniform quantizer. Otherwise, the quantizer is nonuniform. A general model for any nonuniform quantizer with a finite number of levels can be structured as illustrated in Fig. 1 [1], where $c(x)$ and $c^{-1}(x)$ are compressor and expandor functions respectively. Namely, nonuniform quantization can be achieved by compressing the signal x using nonuniform compressor characteristic $c(\cdot)$ (also called *companding law*), by quantizing the compressed signal $c(x)$ employing a uniform quantizer, and by expanding the quantized version of the compressed signal using a nonuniform transfer characteristic $c^{-1}(\cdot)$ inverse to that of the compressor. The overall structure of a nonuniform quantizer consisting of a compressor, a uniform quantizer, and expandor in cascade is called *compandor*.

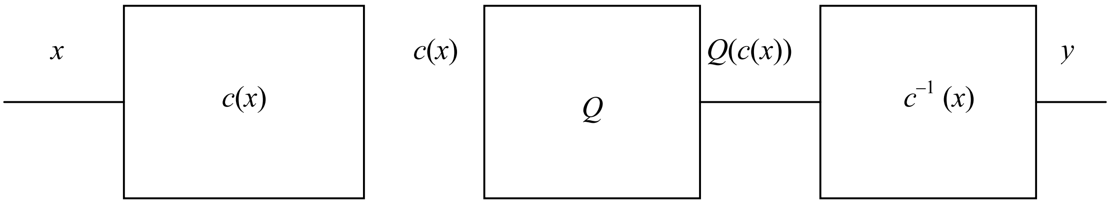

Fig. 1. Block diagram of the companding technique

The quantizer distortion

An optimal N-level nonuniform scalar quantizer, for a source characterized as a continuous random variable with probability density $p(x)$ is a quantizer that minimizes total distortion. The quantizer distortion is defined as the expected mean square error between original and quantized signal. Total distortion consists of two components, inner and outer distortion. Symbolically,

$$D = D_i + D_o\,, \tag{3}$$

where the inner and outer distortions are defined as:

$$D_i = \frac{1}{12(N-2)^2}\left(\int_{-t_{N-1}}^{t_{N-1}} p^{1/3}(x)dx\right)^3, \tag{4}$$

$$D_o = 2\int_{t_{N-1}}^{\infty}(x - y_N)^2\, p(x)dx\,. \tag{5}$$

The support region of the scalar compandor, denoted here $[-t_{\max}, +t_{\max}]$, is the interval where quantization errors are small, or at least bounded. In this paper we consider the Laplacian input signals with unrestricted amplitude range. Determination of the support region enables quantizers to be adapted to the amplitudes of input signals. Laplacian probability density function of the original random variable x with unit variance can be expressed by:

$$p(x) = \frac{1}{\sqrt{2}}e^{-\sqrt{2}|x|}\,. \tag{6}$$

By substituting (6) in (4) and (5), the expressions for determining inner and outer distortions are derived as follows:

$$D_i = \frac{9}{2(N-2)^2}\left(1 - \exp\left(-\frac{\sqrt{2}t_{N-1}}{3}\right)\right)^3, \tag{7}$$

$$D_o = \exp\left(-\sqrt{2}t_{N-1}\right)\left(t_{N-1}^2 + \sqrt{2}t_{N-1} + 1 - y_N\left(2t_{N-1} + \sqrt{2}\right) + y_N^2\right), \tag{8}$$

and the total distortion of the compandor may be rewritten such as:

$$D = \frac{9}{2(N-2)^2}\left(1 - \exp\left(-\frac{\sqrt{2}t_{N-1}}{3}\right)\right)^3 + \\ + \exp\left(-\sqrt{2}t_{N-1}\right)\left(t_{N-1}^2 + \sqrt{2}t_{N-1} + 1 - y_N\left(2t_{N-1} + \sqrt{2}\right) + y_N^2\right). \tag{9}$$

A quantizer is optimal in the sense that no other N-point scalar quantizer can obtain lower distortion.

Definition and determining of the compandor's support region

Let us consider different definitions of compressor functions. First, denoted $c_0(x)$, is similar to the one in [3]:

$$c_0\left(t_j\right) = -1 + 2\frac{\int_{-\infty}^{t_j} p^{1/3}(x)\,dx}{\int_{-\infty}^{+\infty} p^{1/3}(x)\,dx}, \tag{10}$$

$$c_0\left(y_j\right) = -1 + 2\frac{\int_{-\infty}^{y_j} p^{1/3}(x)\,dx}{\int_{-\infty}^{+\infty} p^{1/3}(x)\,dx}. \tag{11}$$

For the values of input signal x within $(-\infty, \infty)$ range, the values of $c_0(t_j)$ range $[-1,1]$. Thresholds t_j $j = 1,2,..., N-1$ and quantization points y_j $j = 1,2,..., N$ can be determined from:

$$c_0\left(t_j\right) = -1 + \frac{2j}{N}, \tag{12}$$

$$c_0\left(y_j\right) = -1 + \frac{2\left(j - 1/2\right)}{N}. \tag{13}$$

Therefore t_{N-1}, denoted $t_{N-1}^{(0)}$, can be determined by equating (10) and (12):

$$t_{N-1}^{(0)} = \frac{3}{\sqrt{2}} \ln\left(\frac{N}{2}\right). \tag{14}$$

Also, y_N, denoted $y_N^{(0)}$, can be determined by equating (11) and (13):

$$y_N^{(0)} = \frac{3}{\sqrt{2}} \ln(N). \tag{15}$$

By using this definition of compressor function it is not possible to optimize the quantizer's load, because it is not possible to determine $t_{\max}$. By generalizing compressor function $c_0(x)$, it is easy to achieve another one, denoted here $c(x)$:

$$c(t_j) = -1 + 2\frac{\int_{-t_{\max}}^{t_j} p^{1/3}(x)dx}{\int_{-t_{\max}}^{+t_{\max}} p^{1/3}(x)dx}. \tag{16}$$

In such a case, when the values of input signal x are within $[-t_{\max}, +t_{\max}]$ range, the values of $c(t_j)$ range $[-1,1]$. It is very easy to show that the same quantizers will be designed by using generalized compressor function $c(x)$ and compressor function $c_1(x)$ [3]:

$$c_1(t_j) = -t_{\max} + 2t_{\max}\frac{\int_{-t_{\max}}^{t_j} p^{1/3}(x)dx}{\int_{-t_{\max}}^{+t_{\max}} p^{1/3}(x)dx}, \tag{17}$$

$$c_1(y_j) = -t_{\max} + 2t_{\max}\frac{\int_{-t_{\max}}^{y_j} p^{1/3}(x)dx}{\int_{-t_{\max}}^{+t_{\max}} p^{1/3}(x)dx}. \tag{18}$$

Now, it is obvious that when the values of input signal x are within the $[-t_{\max}, +t_{\max}]$ range, the values of $c_1(t_j)$ are copied into the $[-t_{\max}, +t_{\max}]$ range by using thus defined compressor function. Thresholds t_j $j = 1,2,..., N-1$ and quantization points y_j $j = 1,2,..., N$ can now be determined from:

$$c_1(t_j) = -t_{\max} + \frac{2j}{N} t_{\max}, \tag{19}$$

$$c_1(y_j) = -t_{\max} + \frac{2\left(j - \frac{1}{2}\right)}{N} t_{\max} \,. \tag{20}$$

Therefore t_{N-1}, denoted $t_{N-1}^{(1)}$, can be determined by equating (17) and (19):

$$t_{N-1}^{(1)} = \frac{3}{\sqrt{2}} \ln\left(\frac{N}{2 + (N-2)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}\right)} \right), \tag{21}$$

and y_N, denoted $y_N^{(1)}$, by equating (18) and (20):

$$y_N^{(1)} = \frac{3}{\sqrt{2}} \ln\left(\frac{N}{1 + (N-1)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}\right)} \right). \tag{22}$$

Good approximation of Lloyd-Max quantizers inner distortion can be achieved by using Bennett's integral [6, 7] ranging $[-t_{N-1}, +t_{N-1}]$ which is the expression (4) for the inner compandor distortion. Therefore, a good heuristic hypothesis will be based on equating the thresholds $t_{N-1}^{(1)} = t_{N-1}^{dif}$ for the compandor and Lloyd-Max quantizer. Let us mark the Lloyd-Max quantizer threshold t_{N-1} with t_{N-1}^{dif}. The dependence of this threshold on the number of quantizaion levels was determined in [2] by minimizing the total distortion such as:

$$t_{N-1}^{dif} = \frac{3}{\sqrt{2}} \ln\frac{N}{3} \,. \tag{23}$$

Therefore, considering the expression (21) and relation $t_{N-1}^{(1)} = t_{N-1}^{dif}$, the values of $t_{\max}$ for different N, can be calculated from:

$$t_{\max} = \frac{3}{\sqrt{2}} \ln\frac{N-2}{N\exp\left(-\frac{\sqrt{2}}{3} t_{N-1}^{(1)}\right) - 2} \,. \tag{24}$$

In such a way, the estimate of the quantizer's optimal load, i.e. the realization of optimal compandor can be achieved.

Let us consider another method of finding the support region of the compandor. By putting the first derivative of compressor function $c_0(x)$, it is easy to calculate the slope of this compressor characteristic. Particularly, for $x = y_N^{(0)}$ we get:

$$c_0'\left(y_N^{(0)}\right) = \frac{\sqrt{2}}{3N}. \tag{25}$$

By using the following relation that is valid for large number of quantization cells N [4]:

$$c_0'\left(y_N^{(0)}\right) \approx \frac{2}{N\Delta_N^{(0)}}, \tag{26}$$

the width of the N-th cell α_N, denoted here $\Delta_N^{(0)}$, can be determined such as $\Delta_N^{(0)} \approx 3\sqrt{2}$. Therefore, the initial value for $t_{\max}$, denoted $t_{\max}^{(0)}$, can be estimated from:

$$t_{\max}^{(0)} = t_{N-1}^{(0)} + \Delta_N^{(0)}. \tag{27}$$

Now, let us calculate the slope of compressor characteristic $c_1(x)$, in $x = y_N^{(1,i)}$:

$$c_1'\left(y_N^{(1,i)}\right) = \frac{\sqrt{2}}{3} \frac{t_{\max}^{(i)}}{\left(1 - \exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)}\right)\right)} \exp\left(-\frac{\sqrt{2}}{3} y_N^{(1,i)}\right), \tag{28}$$

where $y_N^{(1,i)}$ is defined as:

$$y_N^{(1,i)} = \frac{3}{\sqrt{2}} \ln\left(\frac{N}{1 + (N-1)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)}\right)}\right). \tag{29}$$

The iterative process starts for $i = 0$, and we take the initial value for $t_{\max}^{(0)}$ from (27). Similar to (26) the following relation is valid:

$$c_1'\left(y_N^{(1,i)}\right) \approx \frac{2 t_{\max}^{(i)}}{N\Delta_N^{(1,i)}}. \tag{30}$$

Combining (28), (29) and (30) we get:

$$\Delta_N^{(1,i)} \approx 3\sqrt{2} \frac{1 - \exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)}\right)}{1 + (N-1)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)}\right)}. \tag{31}$$

Namely, $t_{\max}^{(i+1)}$ can now be calculated as a sum of $t_{N-1}^{(1,i)}$ and $\Delta_N^{(1,i)}$:

$$t_{\max}^{(i+1)} = \frac{3}{\sqrt{2}} \ln\left(\frac{N}{2 + (N-2)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)} \right)} \right) + 3\sqrt{2} \frac{1 - \exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)} \right)}{1 + (N-1)\exp\left(-\frac{\sqrt{2}}{3} t_{\max}^{(i)} \right)}. \tag{32}$$

Let the expression (9) has the values for t_{N-1} and y_N determined in (14) and (15) respectively; then the distortion marked as Dc_0 can be calculated. When the values for t_{N-1} and y_N in (9) are determined by using (21), (23) and (24), the distortion is marked as Dc_1. If the total distortion value is calculated by using the values of t_{N-1} and y_N, which are obtained iteratively, the total distortion will be marked as $Dc_1^{(1,7)}$. Namely, the iterations are interrupted after the seventh iteration because for $i = 7$ we get $Dc_1^{(1,i+1)} \geq$ $\geq Dc_1^{(1,i)}$. Consequently, the next iteration does not allow further reduction of distortion. Finally, the distortion marked as D^{LM} stands for the Lloyd-Max quantizer distortion. For the calculation of the last distortion we observe the values for t_{N-1} that are given in [1] and take into the calculation the fact that the distance between t_{N-1} and y_N is $1/\sqrt{2}$, also shown in [1].

Numerical results

Table 1 compares the values of distortions Dc_0, Dc_1, $Dc_1^{(1,7)}$ and D^{LM}. From the Table 1 it is obvious that the values of Lloyd-Max distortion for $N = 128$, $N = 256$ and $N = 512$ are the nearest to the corresponding values of the compandor distortion when compressor function $c_1(x)$ is used and the support region is obtained iteratively. Also, the distortion values are nearly equal to the Lloyd-Max distortion values when we applied distortion minimizing method and compressor function $c_1(x)$. Hence, it is obvious that $c_1(x)$ enables designing of the scalar compandor having the best performance. Additionaly, the values of maximum input signals amplitudes, i.e. the values of the support regions, are listed in Table 2 and Table 3. Finally, the maximal amplitudes dependence on the number of bits per sample R ($R = \log_2 N$) is given in Fig. 2.

Table 1. Distortions Dc_0, Dc_1, $Dc_1^{(1,7)}$ and D^{LM}

N	Dc_0	Dc_1	$Dc_1^{(1,7)}$	D^{LM}
128	$2{,}7450\ 10^{-4}$	$2{,}7071\ 10^{-4}$	$2{,}7062\ 10^{-4}$	$2{,}7042\ 10^{-4}$
256	$6{,}8644\ 10^{-5}$	$6{,}8168\ 10^{-5}$	$6{,}8157\ 10^{-5}$	$6{,}8132\ 10^{-5}$
512	$1{,}7164\ 10^{-5}$	$1{,}7104\ 10^{-5}$	$1{,}7103\ 10^{-5}$	$1{,}7099\ 10^{-5}$

Table 2. Distortion minimized Compandor Parameters for Compressor Function $c_1(x)$

N	$t_{N-1}^{(1)} = t_{N-1}^{dif}$	t_{max}	$y_N^{(1)}$	Dc_1
128	7,9622	10,2593	8,8139	$2{,}7069\ 10^{-4}$
256	9,4326	11,7465	10,2886	$6{,}8167\ 10^{-5}$
512	10,9030	13,2253	11,7611	$1{,}7104\ 10^{-5}$

Table 3. Iteratively obtained Compandor Parameters for Compressor Function $c_1(x)$

N	$y_N^{(1,7)}$	$\Delta_N^{(1,7)}$	$t_{N-1}^{(1,7)}$	$t_{max}^{(8)}$	$Dc_1^{(1,7)}$
128	8,6268	1,9164	7,8361	9,7525	$2{,}7062\ 10^{-4}$
256	10,0895	1,9185	9,2981	11,2161	$6{,}8157\ 10^{-5}$
512	11,5561	1,9196	10,7643	12,6839	$1{,}7103\ 10^{-5}$

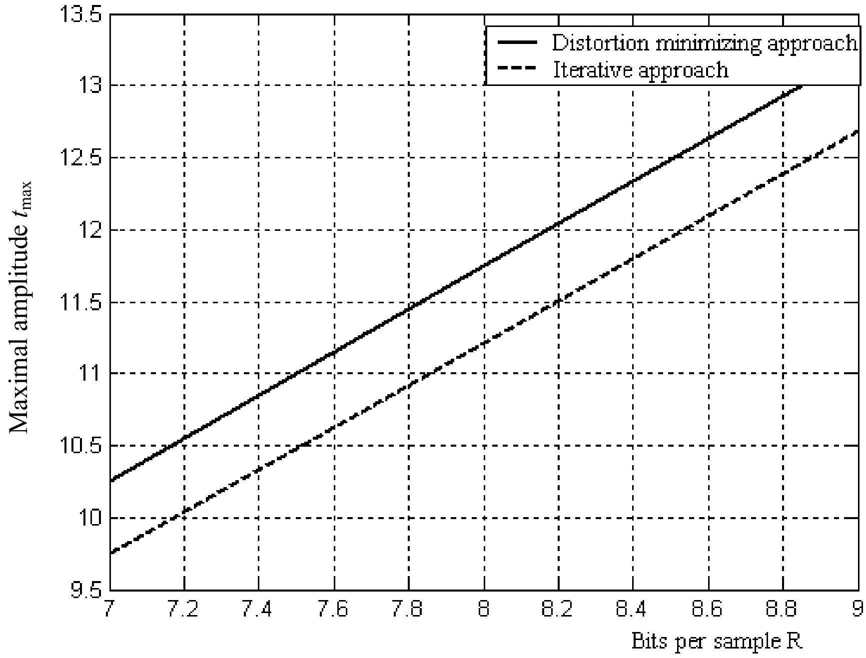

Fig. 2. The maximal amplitudes dependence on the number of bits per sample

Conclusion

In this paper the systematic analysis of different definitions of compressor functions for scalar compandor construction are carried out. Simply expressions in closed forms for the support region of the scalar compandor for Laplacian source are obtained. Also, the expression for the determining of the total distortion is derived. Hence, numerical values of the total distortion are calculated when the scalar compandor is realised by using different definitions of compressor functions. The results demonstrate that by using the *compressor* function $c_1(x)$ the calculated distortion the least differs from the optimal Lloyd-Max's quantizers distortion. Therefore, choosing the compressor function $c_1(x)$ it is possible to design the scalar compandor having the best performance.

1. *Sangsin Na and David L. Neuhoff.* On the Support of MSE-Optimal, Fixed-Rate, Scalar Quantizers // IEEE Transactions on Information Theory. — 2001, Nov. — Vol. 47, N 7. — P. 2972–2982.

2. *Sangsin Na.* On the Support of Fixed-Rate Minimum Mean-Squared Error Scalar Quantizers for a Laplacian Source // IEEE Transactions on Information Theory. — 2004, May. — Vol. 50, N 5. — P. 937–944.

3. *Neil Judell and Louis Scharf.* A Simple Derivation of Lloyd's Classical Result for the Optimum Scalar Quantizer // IEEE Transactions on Information Theory. —1986, March. — Vol. 32, N 2. — P. 326–328.

4. *Jayant N.S., Peter Noll.* Digital Coding of Waveforms. — New Jersey: Prentice-Hall, 1984. — Chapter 4. — P. 129–139.

5. *Max J.* Quantizing for Minimum Distortion // IRE Transactions on Information Theory. — 1960, March. — Vol. IT-6. — P. 7–12.

6. *Bucklew J.A. and Wise G.L.* Multidimensional Asymptotic Quantization Theory // IEEE Transactions on Information Theory. — 1982, March. — Vol. IT-28. — P. 239–247.

7. *Sangsin Na and David L.* Neuhoff, Bennett's Integral for Vector Quantizers // IEEE Transaction on Information Theory. — 1995, July. — Vol. 41. — P. 886–900.

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