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Investigations of the properties of solutions of impulsive differential systems in the linear approximation

Проведено якісний аналіз властивостей розв'язків нелінійних імпульсних систем диференціальних рівнянь із збуреннями на деяких гіперповерхнях $\{t_i(x)\}$. Отримано нові умови обмеженості, практичної стійкості за Четаєвим (рівномірної, стискуючої), стійкості за Ляпуновим (рівномірної, асимптотичної), притягування розв'язків збурених систем за лінійним наближенням.

In work [1], two-parametric scale increasing functions were first considered to investigate the problem of stability of solutions of the nonlinear system $dx/dt = A(t)x + F(t,x)$ in the linear approximation $dy/dt = A(t)y$ (shorted system). The Cauchy matrix of the shorted system satisfies such an estimate: $\|C(t,\tau)\| \leqslant c\exp[\overline{\alpha}(t-\tau)][t/\tau]^{\overline{\beta}}$, for $\tau \geqslant 1$, where $\overline{\alpha} > \alpha^* + \varepsilon$, $\varepsilon > 0$, $\alpha^* = \max\limits_k \alpha_k (k = 1, 2, \ldots, n)$, $\alpha_k = \overline{\lim\limits_{t\to\infty}} t^{-1} \ln\|y_k(t)\|$, $k = \overline{1,n}$, (α_k is the characteristic index of a Lyapunov nontrivial solution $y_k(t)$ of the shorted system), $\overline{\beta} > \beta = \max\limits_k \beta_k$, $k = \overline{1,n}$, where the characteristic degree by Lyapunov $\beta_k = \overline{\lim\limits_{t\to\infty}}[(\ln t)^{-1} \ln\{\|y_k(t)\| \exp[-\alpha_k t]\}]$, $k = \overline{1,n}$.

In work [2], the author investigated the problem of stability by Lyapunov of solutions of a nonlinear system in the linear approximation by using an estimate of the Cauchy matrix of the shorted system of such a type: $\|C(t,\tau)\| \leqslant \eta(t)l(\tau)$, $\eta(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\eta(t) \in C(\mathbb{R}^+)$, $l(\tau) \in C(\mathbb{R}^+)$. Later on, Borysenko and Martynyuk (Mat. Fizika, 1980, N 2) used this estimate to investigate, in the linear approximation, the problem of practical stability (by Chetaev; uniform, attractive) of solutions of a nonlinear regular system with nonlinearities on the right-hand side of the system either the Lipschitz or Hölder type (the evolution of processes which describe the system can be either finite or infinite). The estimate from [2] was also used to estimate the Cauchy matrix of a system of variations and to investigate the problem of stability of solutions of a nonlinear system in the nonlinear approximation [see Mat. Fizika, 1981, N 1 (Borysenko)]. These results are generally based on the method of integral inequalities for continuous functions [3] and its applications. In 1983 (Ukr. Math. Journ., N 2), Borysenko considered a generalization of the idea of Demidovich [1] by using the two-parametric scale of increasing functions to investigate the properties of solutions of nonlinear impulsive differential systems in the linear approximation (linear impulsive differential systems) and used the following estimate of the Cauchy matrix of an impulsive shorted system: $\|C(t,t_0)\| \leqslant ce^{\alpha(t-t_0)}[t/t_0]^{\beta}$, where c, α, and β are some constants, and $t \geqslant t_0 \geqslant 1$.

In the monograph by Lakshmikantham, Bainov, Simeonov [4], the problem of stability by Lyapunov of solutions of the impulsive nonlinear differential system under a pulse influence at fixed time moments,

$$\frac{dx}{dt} = a(t)x + g(t,x), \qquad t \neq t_i, \qquad \Delta x|_{t=t_i} = Bx + I_i(x), \tag{1}$$

was investigated by assuming that the Cauchy matrix of the linear system $dx/dt = a(t)x$ without pulses satisfies the estimate $\|C(t,\tau)\| \leqslant \varphi(t)\psi(\tau)$ [2], and that the Cauchy matrix of the impulsive linear system

$$\frac{dx}{dt} = a(t)x, \qquad t \neq t_i, \qquad \Delta x = Bx, \qquad t \neq t_i, \tag{2}$$

satisfies the estimate $\|C(t,\tau)\| \leqslant \varphi(t)\psi(\tau) \prod\limits_{\tau<t_i<t} \gamma_i \varphi(t_i^+)\psi(t_i^+)$, where $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$; $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\|g(t,x)\| \leqslant l)(t)\|x\|^m$, $m > 1$, $l(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$, $l(t) \in C(\mathbb{R}^+)$, $\|I_i(x)\| \leqslant \gamma_i\|x\|$, $\gamma_i = \mathrm{const} > 0$.

In the monograph by Samoilenko, Borysenko, Matarazzo, Toscano, Yasinsky [5], the problems of the stability by Lyapunov and the practical stability by Chetaev of solutions of system (1) were investigated by assuming that the Cauchy matrix of the shorted system (2) satisfies the estimate $\|C(t,\tau)\| \leqslant \overline{\eta}(t)\bar{l}(\tau)$, $\overline{\eta}\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\bar{l}\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\overline{\eta} \in C(\mathbb{R}^+)$, $\bar{l} \in C(\mathbb{R}^+)$ and $\|g(t,x)\| \leqslant l(t)\|x\|^m$, $m = \mathrm{const} > 0$, $\|I_i(x)\| \leqslant \gamma_i\|x\|$.

The conditions for the stability, as well as for the practical stability, of the trivial solution of system (1) in the linear approximation (2) were found in [4, 5] by using analogies of the Gronwall–Bellman–Bihari lemmas for the discontinuous functions for integral inequalities of such a type:

$$u(t) \leqslant c + \int\limits_{t_0}^{t} \nu(\tau)u^m(\tau)\,d\tau + \sum_{t_0<t_i<t} k_i u(t_i - 0), \qquad m \geqslant 1. \tag{3}$$

In report [6], Danylo S. Borysenko found the integral inequality

$$u(t) \leqslant \varphi(t) + \int\limits_{t_0}^{t} \nu(\tau)u^m(\tau)\,d\tau + \sum_{t_0<t_i<t} k_i u^p(t_i - 0) \tag{4}$$

and obtained new analogies of the Gronwall–Bellman–Bihari lemmas for the discontinuous functions (see Remark 2).

In works [7, 8] with the use of results obtained in [6], the new estimates of solutions of impulsive nonlinear systems with nonlinearities on the right-hand side of a system not only of the Lipschitz type but also of the Hölder one were obtained.

The investigations in [9, 10] are devoted to the problems of the stability and the practical stability of solutions of impulsive nonlinear systems in the linear and nonlinear approximations under assumption that the pulse forces are characterized by functions of the Lipschitz type.

In the monograph by Borysenko, Iovane [11] and in works [12–15], the method of integral inequalities for discontinuous functions and its applications to the qualitative analysis of properties of the solutions of impulsive differential systems with nonlinearities of different kinds on the right-hand sides have obtained the further development (by including also the impulsive systems of partial differential hyperbolic equations).

In this article, we use the results of investigations performed by Bainov, Iovane, Lakshmikantham, Leela, Martynyuk, Samoilenko, Simeonov (see, e. g., [1–15]).

Preliminary considerations. Let us introduce the impulsive system of ordinary differential equations of such a form:

$$\frac{dx}{dt} = f(t,x), \qquad t \neq t_i(x), \qquad \Delta x|_{t=t_i(x)} = I_i(x). \tag{5}$$

Let the following assumptions be fulfilled:

(H1) f, I_i are defined in the domain $\Omega = \{(t,x)\colon t \in J = [t_0, T],\ T \leqslant \infty,\ t_0 \geqslant 1,\ \|x\| \leqslant h\}$ and $f(t,0) = I_i(0)$, $\forall\, t \in J$, $\forall\, i \in \mathbb{N}$;

(H2) $f(t,x) = A(t)x + r(t,x)$, $I_i(x) = B_i x + J_i(x)$, where $A(t)$, B_i are some matrices; $\det(B_i + E) \neq 0\ \forall\, i$;

(H3) $\{t_i(x)\}\colon t_1(x) < t_2(x) < \cdots$, $\lim\limits_{i\to\infty} t_i(x) = \infty$ uniformly for $x \in \Omega$;

(H4) $\exists\, \theta = \mathrm{const} > 0\colon \inf\limits_{\|x\|\leqslant h} t_i(x) - \sup\limits_{\|x\|\leqslant h} t_{i-1}(x) = \theta > 0$, $\forall\, i \in \mathbb{N}$;

(H5) if $x(t) = x(t, t_0, x_0)$ the solution of Cauchy problem for system (5) and t_i^*-moments of time: $t_i^* = x(t) \bigcap (t = t_i(x))$, then $\inf\limits_{\|x\|\leqslant h} t_i^*(x) \geqslant t_i^* \geqslant \sup\limits_{\|x\|\leqslant h} t_{i-1}(x)$, $\forall\, i \in \mathbb{N}$;

(H6) $\exists \theta_i = \mathrm{const} > 0\colon \theta_1 \tau < i(t, t+\tau) < \theta_2 \tau$; $t_0 < t_1^* < t_2^* < \cdots$, $\lim\limits_{i\to\infty} t_i^* = \infty$; where $i(a,b)$ is a number of points $\{t_i^*\} \in [a,b] \subset [t_0, T]$, $T \leqslant \infty$ (θ_i depends only on τ);

(H7) $\exists\, L = \mathrm{const} > 0\colon \left\| \dfrac{\partial t_i(x)}{\partial x} \right\| \leqslant L\ \forall\, i \in \mathbb{N},\ x \in \Omega$, $\sup\limits_{0\leqslant\sigma\leqslant 1} \left(\dfrac{\partial t_i(x + \sigma I_i(x))}{\partial x}, I_i(x) \right) \leqslant 0$;

(H8) $\exists\, \bar r(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$, $\bar r(t) \leqslant r = \mathrm{const} < \infty$, $\bar r \in C(\mathbb{R}^+)\colon \|r(t,x)\| \leqslant \bar r(t)\|x\|^{\alpha^*}$, $\alpha^* = \mathrm{const} > 0$; $\exists\, k_i = \mathrm{const} > 0\colon \|J_i(x)\| \leqslant k_i \|x\|^{\beta}$, $\beta = \mathrm{const} > 0$, $i \in \mathbb{N}$;

(H9) the Cauchy matrix $C(t, t_0)$ of the shorted system

$$\frac{dx}{dt} = A(t)x, \qquad t \neq t_i^*, \qquad \Delta x = B_i x, \qquad t = t_i^* \tag{6}$$

satisfies such an estimate: $\|C(t,t_0)\| \leqslant c\exp[(\widetilde\alpha + \theta_i \ln\alpha)(t - t_0)][t/t_0]^{\widetilde\beta}$, where the $\widetilde\alpha$ parameter is the characteristic index of a Lyapunov nontrivial solution of the system $dx/dt = A(t)x$, the $\widetilde\beta$ parameter is connected with the characteristic Lyapunov degree of this system, and

$$\alpha^2 = \max_j \lambda_j[(B_j + E)^T (B_j + E)], \qquad \theta_i \ln\alpha = \begin{cases} \theta_1 \ln\alpha, & 0 < \alpha < 1; \\ \theta_2 \ln\alpha, & \alpha > 1. \end{cases}$$

Remark 1. As for the notion of stability and practical stability of solutions of system (5), see, e. g., [5].

Remark 2. To estimate the solutions of system (5), we will use such results [6–8]:

A. Let $u(t)$ be a nonnegative piecewise continuous function at $t \geqslant t_0$, with first-kind discontinuities at the points $\{t_i\}$ satisfying the "integro-sum" inequality

$$u(t) \leqslant \varphi(t) + \int\limits_{t_0}^{t} g(\tau)u(\tau)\,d\tau + \sum_{t_0 < t_i < t} a_i u^m(t_i - 0), \tag{7}$$

where $\varphi(t) > 0$, $g(t) \geqslant 0$, $a_i = \mathrm{const} \geqslant 0$, $\{t_i\}\colon t_1 < t_2 < \cdots$, $\lim\limits_{i\to\infty} t_i = \infty$, $m = \mathrm{const} > 0$, and $\varphi(t)$ is nondecreasing on $J = [t_0, T]$, $T \leqslant \infty$.

Then

$$u(t) \leqslant \varphi(t) \prod_{t_0 < t_i < t} (1 + a_i \varphi^{m-1}(t_i)) \exp\left[\int\limits_{t_0}^{t} g(s)\,ds \right], \tag{8}$$

where $0 < m \leqslant 1$, $\forall t \geqslant t_0$;

$$u(t) \leqslant \varphi(t) \prod_{t_0<t_i<t} (1 + a_i \varphi^{m-1}(t_i)) \exp\left[m \int_{t_0}^{t} g(s)\, ds\right], \tag{9}$$

where $m \geqslant 1$, $\forall t \geqslant t_0$.

B. Let the function $u(t)$ be nonnegative piecewise continuous on J with first-kind discontinuities at the points $\{t_i\}$ $t_0 < t_1 < t_2 < \cdots$, $\lim\limits_{i\to\infty} t_i = \infty$ and satisfy the inequality

$$u(t) \leqslant \varphi(t) + \int_{t_0}^{t} g(\tau) u^m(\tau)\, d\tau + \sum_{t_0<t_i<t} a_i u^m(t_i - 0), \tag{10}$$

where $\varphi(t)$ is a positive function monotonously nondecreasing on J, $g \geqslant 0$, $a_i \geqslant 0$, $m > 0$, $m \neq 1$.

Then

$$u(t) \leqslant \varphi(t) \prod_{t_0<t_i<t} (1 + a_i \varphi^{m-1}(t_i)) \left[1 + (1-m) \int_{t_0}^{t} \varphi^{m-1}(\tau) g(\tau)\, d\tau\right]^{1/(1-m)}, \tag{11}$$

$$\forall t \geqslant t_0, \qquad 0 < m < 1;$$

$$u(t) \leqslant \varphi(t) \prod_{t_0<t_i<t} (1 + a_i m \varphi^{m-1}(t_i)) \times$$
$$\times \left[1 - (m-1) \left[\prod_{t_0<t_i<t} (1 + a_i m \varphi^{m-1}(t_i))\right]^{m-1} \int_{t_0}^{t} \varphi^{m-1}(\tau) g(\tau)\, d\tau\right]^{-1/(m-1)}; \tag{12}$$

$$\forall t \geqslant t_0\colon \int_{t_0}^{t} g(\tau) \varphi^{m-1}(\tau)\, d\tau \leqslant \frac{1}{m}, \qquad m > 1; \tag{13}$$

$$\prod_{t_0<t_i<t} (1 + a_i m \varphi^{m-1}(t_i)) < \left(1 + \frac{1}{m-1}\right)^{1/(m-1)}.$$

Main results. In this section, we obtain the new conditions of boundedness of the solutions of system (5) by using the property of boundedness of the shorted system $dx/dt = A(t)x$, $t \neq t_i(x)$, $\Delta x|_{t=t_i(x)} = B_i x$; in addition, the conditions for the stability by Lyapunov and the practical stability (uniform, attractive) by Chetaev of the trivial solution of system (5) will be found.

Theorem 1. *Let assumptions* $(H1)$–$(H9)$ *be valid for system* (5), *and let the following conditions be fulfilled:*

a^1) $\overline{\alpha} + \theta_i \ln \alpha = \widetilde{\beta} = 0$; $\alpha^* = 1$, $0 < \beta \leqslant 1$;

b^1) $\exists\, a = \mathrm{const} > 0\colon \|A(t)\| \leqslant a$;

c^1) $L < \dfrac{1}{(a+r)h}$;

d^1) $\exists\, m_1(t_0) = \mathrm{const} > 0\colon \prod\limits_{t_0<t_i^*<t} (1 + c^\beta k_i \|x_0\|^{\beta-1}) \leqslant (1 + m_1(t_0)\|x_0\|^{\beta-1})$ $\forall t \in J$;

e^1) $\exists\, m_2(t_0) = \text{const} > 0$: $\displaystyle\int\limits_{t_0}^{t} \overline{r}(\tau)\, d\tau \leqslant m_2(t_0) < \infty \ \forall\, t \in J$;

f^1) $c(1 + m_1(t_0)\lambda^{\beta-1}) \exp[cm_2(t_0)] < \Lambda/\lambda$;

g^1) $\exists\, m_3(t_0) = \text{const} > 0$: $\|x_0\|(1 + m_1(t_0)\|x_0\|^{\beta-1}) \leqslant m_3(t_0)\|x_0\|^{\beta}$;

h^1) $\lambda < \sqrt[\beta]{\Lambda(cm_3(t_0)\exp[cm_2(t_0)])^{-1}}$.

Then

I. a^2) *all solutions of system* (5) *are bounded in* Ω, *if conditions* a^1–e^1 *hold;*

II. Trivial solution (t. s.) of system (5) *is*

b^2) *practically stable (p. s.) relative to* λ, Λ, J, *if conditions* a^1–f^1 *or* a^1–e^1, g^1, h^1 *hold;*

c^2) *uniformly practically stable (u. p. s.) relative to* t_0, *if conditions* b^2 *hold and* $m_1(t_0)$ *(*$i = 1, 2, 3$*) are independent of* t_0.

Remark 3. If $t_i(x) = t_i = \text{const}$, $\beta = 1$, $A(t) = A$, the result of Theorem 1 is similar to Theorems 4.3.13 and 4.3.14 in [5], pp. 287, 288); if $\beta = 1$, it coincides with Theorem 4.1 in [11], p. 86.

Theorem 2. *Let the following conditions be fulfilled:*

a^3) *assumptions* (H1)–(H9) *hold;*

b^3) $\overline{\alpha} + \theta_i \ln\alpha = 0$, $\widetilde{\beta} < 0$, $\alpha^* = 1$, $0 < \beta \leqslant 1$;

c^3) *conditions* b^1, c^1, e^1 *of Theorem 1 take place;*

d^3) $\exists\, m_4(t_0) = \text{const} > 0$: $\displaystyle\prod_{t_0 < t_i^* < t} \left\{1 + c^{\beta}\left[\frac{t_i^*}{t_0}\right]^{\widetilde{\beta}(\beta-1)} k_i\|x_0\|^{\beta-1}\right\} \leqslant (1 + m_4(t_0)\|x_0\|^{\beta-1})$ $\forall\, t \in J = [t_0, T]$;

e^3) $c(\lambda + m_4(t_0)\lambda^{\beta}) \exp[cm_2(t_0)] < \Lambda$;

f^3) $\exists\, m_5(t_0) = \text{const} > 0$: $\|x_0\|(1 + m_4(t_0)\|x_0\|^{\beta-1}) \leqslant m_5(t_0)\|x_0\|^{\beta}$;

g^3) $\lambda < (\Lambda(cm_5(t_0)\exp[cm_2(t_0)])^{-1})^{1/\beta}$.

Then (t. s.) of system (5) *is:*

a^4) (λ, Λ, J)*-stable; moreover, attractive practically stable (a. p. s.) relative to* $(\lambda, \Lambda, \Lambda^*, J)$, *here,* $\lambda < \Lambda^* < \Lambda$; *if only* a^3–e^3 *or* a^3–d^3, f^3, g^3 *take place;*

a^5) *(u. p. s.) relative to* (λ, Λ, J), *only if* m_2, m_4, m_5 *are independent of* t_0; *moreover, attractive (u. p. s.) (a. u. p. s.) relative to* $(\lambda, \Lambda, \Lambda^*, J)$.

Remark 4. Theorem 2 gives new conditions of (p. s.) (t. s.) (uniform, attractive) of the perturbed system (5) on some hypersurfaces by using only the second scale of increasing functions (characteristic degree by Lyapunov), which was first considered by Demidovich [1] (where new conditions for the stability of solutions in the linear approximation were found). Theorem 2 generalizes the idea to consider the two-parametric scale of increasing functions including both the Lyapunov indices and the characteristic degree by Lyapunov.

Theorem 3. *Let condition* a^3 *of Theorem 2 be valid, and let the following conditions be fulfilled:*

b^4) $\overline{\alpha} + \theta_i \ln\alpha < 0$, $\widetilde{\beta} \leqslant 0$, $\alpha^* = 1$, $0 < \beta \leqslant 1$;

c^4) *condition* c^3 *of Theorem 2 is fulfilled;*

d^4) $\exists\, m_6(t_0) = \text{const} > 0$: $\displaystyle D(t_0, t) \stackrel{def}{=} \prod_{t_0 < t_I^* < t} \left(1 + c^{\beta}\left[\frac{t_i^*}{t_o}\right]^{\widetilde{\beta}(\beta-1)} \exp[(\widetilde{\alpha} + \theta_i \ln\alpha)(\beta - 1)(t_i^* - t_0)]\|x_0\|^{\beta-1}k_i\right) \leqslant (1 + m_6(t_0)\|x_0\|^{\beta-1})$, $\forall\, t \in J = [t_0, T]$;

e^4) $c(1 + m_6(t_0)\lambda^{\beta-1})\exp[cm_2(t_0)] < \Lambda/\lambda$;

f^4) $\exists\, m_7(t_0) = \text{const} > 0\colon \|x_0\|(1 + m_6(t_0)\|x_0\|^{\beta-1}) \leqslant m_7(t_0)\|x_0\|^{\beta}$;

g^4) $\lambda < (\Lambda(cm_6(t_0)\exp[cm_2(t_0)])^{-1})^{1/\beta}$.

Then (t. s.) of system (1) *is:*

a^6) (λ, Λ, J)*-stable; moreover, (a. p. s.) relative to* $(\lambda, \Lambda, \Lambda^*, J)$ *if only* (H1)–(H9), b^4–e^4 *or* b^4–d^4, f^4, g^4 *take place;*

a^7) *(u. p. s) relative to* (λ, Λ, J), *if* m_2, m_6, m_7 *are independent of* t_0*; moreover, (a. u. p. s.) relative to* $(\lambda, \Lambda, \Lambda^*, J)$.

Remark 5. Estimate (11), in which we use the two-parametric scale of increasing functions, generalizes the estimate by Demidovich [1] to the case of the impulsive differential system (5).

Remark 6. For $0 < \beta \leqslant 1$, the constants k_i must be sufficiently small, and $\lim\limits_{i\to\infty} k_i = 0$. Analogously, for Theorems 1–3 (case $0 < \beta \leqslant 1$), the next statements take place:

Theorem 4. *Let us assume that*

a^{1*}) $\widetilde{\alpha} + \theta_i \ln\alpha = \widetilde{\beta} = 0$, $\alpha^* = 1$, $\beta \geqslant 1$;

b^{1*}) *assumptions* (H1)–(H9) *hold;*

c^{1*}) b^1, c^1, e^1 *of Theorem 1 take place;*

d^{1*}) $\exists\, m_1^*(t_0) = \text{const} > 0\colon \prod\limits_{t_0 < t_i^* < t} (1 + c^{\beta}k_i\|x_0\|^{\beta-1}) \leqslant m_1^*(t_0) < \infty$, $\forall\, t \in J$;

e^{1*}) $cm_1^*(t_0)\exp[\beta cm_2(t_0)] < \Lambda/\lambda$.

Then

I. All solutions of system (5) *are bounded in* Ω, *if conditions* $a^{1*} - d^{1*}$ *hold;*

II. (t. s.) of system (5) *is:*

b^{2*}) *stable by Lyapunov, if conditions* a^{1*}–d^{1*} *hold (stable uniformly, if* m_1^*, m_2 *are independent of* t_0*);*

c^{2*}) *(u. p. s.), if* $d^{1*}), b^{2*}$) *hold and* $m_1^*(t_0)$, $m_2(t_0)$ *are independent of* t_0.

Theorem 5. *Suppose that conditions* a^3, c^3 *of Theorem 2 take place and*

a^{3*}) $\widetilde{\alpha} + \theta_i \ln\alpha = 0$, $\widetilde{\beta} < 0$, $\alpha^* = 1$, $\beta \geqslant 1$;

b^{3*}) $\exists\, m_2^*(t_0) = \text{const} > 0\colon \prod\limits_{t_0 < t_i^* < t} \left(1 + c^{\beta}\left[\dfrac{t_i^*}{t_0}\right]^{\widetilde{\beta}(\beta-1)}\|x_0\|^{\beta-1}k_i\right) \leqslant m_2^*(t_0) < \infty$, $\forall\, t \in J$;

c^{3*}) $cm_2^*(t_0)\lambda\exp[\beta cm_2(t_0)] < \Lambda/\lambda$.

Then (t. s.) of system (5) *is:*

i) asymptotically stable by Lyapunov, only if a^{3*}, b^{3*} *take place (stable uniformly, if* $m_2^*(t_0)$, $m_2(t_0)$ *are independent of* t_0*);*

ii) (p. s.) if only $a^{3*} - c^{3*}$ *take place (stable uniformly, if* $m_2^*(t_0)$, $m_2(t_0)$ *are independent of* t_0*).*

Theorem 6. *Assume that*

a^{4*}) *assumptions* (H1)–(H9), *conditions* b^1, c^1, e^1 *of Theorem 1 hold;*

b^{4*}) $\exists\, m_3^*(t_0) = \text{const} > 0\colon D(t_0, t) \leqslant m_3^*(t_0)$, $\forall\, t \geqslant t_0$;

c^{4*}) $\widetilde{\alpha} + \theta_i \ln\alpha < 0$, $\widetilde{\beta} \leqslant 0$, $\alpha^* = 1$, $\beta \geqslant 1$;

d^{4*}) $cm_3^*(t_0)\exp[\beta cm_2(t_0)] < \Lambda/\lambda$.

Then (t. s.) of system (5) *is*

iii) asymptotically stable by Lyapunov, if only a^{4*}–c^{4*} *take place (stable uniformly, if* $m_3^*(t_0)$, $m_2(t_0)$ *are independent of* t_0*);*

iv) (a. u. p. s.) (if $m_3^*(t_0) = m_3$, $m_2(t_0) = m_2$*), iii and* d^{4*} *hold.*

Remark 7. For the case $\alpha^* = \beta > 0$ ($\alpha^* \neq 1$), it is possible, by using result B of Remark 2, to make a similar qualitative analysis of the properties of the solutions of system (5).

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