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Nonlinear Finite Element Analysis of Rubber Composite Shells

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Нелинейный конечноэлементный расчет композитных оболочек с каучуковой матрицей

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Представлено конечноэлементное решение для композитных оболочек с каучуковой матрицей. Рассматриваются также многослойные конструкции с каучуковой основой. Для учета несжимаемости каучуковой матрицы и неоднородности структуры многослойных оболочек необходимо разработать на основе численных методов хорошо обоснованную модель, описывающую поведение указанных строительных материалов. Предложенная модель применялась авторами к вырожденному элементу оболочки в рамках деформационных теорий сдвига первого и третьего порядка. Модель позволяет достаточно точно прогнозировать нелинейное поведение многослойных оболочек с тонкими прослойками из композитного материала и слоями каучука, а также композитных оболочек с каучуковой матрицей.

Ключевые слова: композитная оболочка, каучуковая матрица, деформационные теории сдвига, конечноэлементное решение.

Introduction. In this paper, a finite element formulation for the solution of composite shells with rubber matrix is presented. Also sandwich structures with rubber cores are considered. The incompressible nature of rubber and the complexity of laminated shells bring forth the need for a sound numerical model for the prediction of such important industrial materials.

A finite element formulation based on a degenerated shell and first- and third-order shear deformation theories has been developed.

Two separate nonlinear material models are considered: one for rubber and another for the composite layers. Rubber is considered to be a hyperelastic material, for which incompressibility is formulated on a single field, where in each interpolation point the Lagrange multiplier for incompressibility condition is eliminated.

The total Lagrangian large displacement formulation is taken into account.

The mathematical formulation of rubber shells is fundamental in many industrial processes. Previous work on the subject has been performed in [1–6]. In this work, a new approach is considered by analyzing sandwich structures with a rubber core and with a bi-material formulation [7]. Also a third-order formulation is involved in such analysis for the first time.

1. **Large Deformation Analysis**.

1.1. **Generalities**. In rubber composites or composite/rubber sandwiches, continuous variation of geometry has to be considered.

The Green–Lagrange strain vector is defined as [8]

$$\gamma_{ij} = \frac{1}{2}(g_{ij} - \delta_{ij}) = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}), \tag{1}$$

where u_i are displacements and γ_{ij} is the Green strain tensor. Three strain invariants are usually used for the establishment of a constitutive law for rubber. They can be obtained as a function of the Green–Lagrange strain components as [9]

$$I_1 = 3 + 2\gamma_{ii}, \tag{2}$$

$$I_2 = 3 + 4\gamma_{ii} + 2(\gamma_{ii}\gamma_{ij} - \gamma_{ij}\gamma_{ji}), \tag{3}$$

$$I_3 = 1 + 2\gamma_{ii} + 2(\gamma_{ii}\gamma_{ij} - \gamma_{ij}\gamma_{ji}) + \frac{4}{3}\varepsilon^{ijk}\varepsilon^{rst}\gamma_{ir}\gamma_{js}\gamma_{kt}, \tag{4}$$

where ε^{ijk} and ε^{rst} are the permutation symbols.

Considering dV_0 as an elementary volume in the reference configuration and V the volume of this element in actual configuration, it can be shown [9] that

$$\frac{dV}{dV_0} = \sqrt{I_3}, \tag{5}$$

so that for an incompressible material it can be written

$$I_3 = 1. \tag{6}$$

The Piola–Kirchhoff stresses S_{ij} are energetically conjugate with the Green–Lagrange strains in the reference configuration [10]. In the deformed configuration they can be related with real (or Cauchy) stresses σ_{ij} as

$$S_{ij} = \frac{\rho_0}{\rho}\frac{\partial X_i}{\partial x_r}\sigma_{rs}\frac{\partial X_j}{\partial x_s}, \tag{7}$$

where ρ_0/ρ is the ratio between densities of both configurations, X_i and x_i are the coordinates in the reference and deformed configurations, respectively.

1.2. **Incremental Equilibrium Equations**. An incremental form of the virtual work principle is needed to take into account nonlinear material and geometrical behavior. This principle can be expressed for conservative loading as follows [11]:

$$\int_{V_0} (\Delta S_0 \delta\gamma_{ij} + S_{ij}\delta\Delta\gamma_{ij})dV_0 = \int_{S_0} \Delta t_i^0 \delta u_i dS_0 + \int_{V_0} \Delta b_i^0 \delta u_i dV_0, \qquad (8)$$

$$\Delta S_{ij} = \frac{\partial S_{ij}}{\partial \gamma_{kl}} \Delta\gamma_{kl} = D_{klij}\Delta\gamma_{kl}, \qquad (9)$$

$$\Delta\gamma_{ij} = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i} + u_{k,i}\Delta u_{k,j} + u_{k,j}\Delta u_{k,i} + \Delta u_{k,i}\Delta u_{k,j}), \qquad (10)$$

$$\delta\Delta\gamma_{ij} = \frac{1}{2}(\delta u_{k,i}\Delta u_{k,j} + \delta u_{k,j}\Delta u_{k,i}) = \Delta\delta\gamma_{ij}, \qquad (11)$$

where Δt_i^0 and Δb_i^0 are the pseudo tractions and pseudo body forces, respectively, defined in the reference configuration as

$$\Delta t_i^0 dS_0 = \Delta t_i dS, \qquad (12)$$

$$\Delta b_i^0 dV_0 = \Delta b_i dV. \qquad (13)$$

The symbols Δt_i and Δb_i are the real forces in the deformed configuration.

2. **Rubber Theory**. Synthetic rubbers are polymeric materials that withstand large elastic deformations with full instantaneous recovery after unloading. Such properties are due to their molecular structure [12, 13]. A typical behavior of rubber is nonlinear with elongations under tension up to 600%.

2.1. **Mathematical Theory**. Rubber can be considered as an incompressible hyperelastic material. In hyperelasticity, the reversibility of elastic deformation and the independence of the deformation history are ensured through the definition of stresses as gradients of a potential function, i.e., the strain energy [5, 9].

The definition of the strain energy follows the research of Mooney [14] and Rivlin [15, 16]. For an isotropic material, the strain energy per unit undeformed volume is taken as a function of strain invariants as

$$W = W(I_1, I_2, I_3). \qquad (14)$$

The strain energy must be independent of I_3 [9]. This equation can now be rewritten as

$$W = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} C_{lm}(I_1 - 3)^l (I_2 - 3)^m, \quad C_{00} = 0, \qquad (15)$$

where C_{ij} are experimental parameters. The most common form for rubber is known as the Mooney–Rivlin law [15, 16]:

$$W = C_1(I_1 - 3) + C_2(I_2 - 3). \tag{16}$$

This law can be considered acceptable for elongations up to 450–500%.

2.2. **Stress Evaluation**. Stresses are defined as derivatives of the strain energy. In rubber, deformations occur without volume variation. A new constrained functional can be defined [9] as

$$\overline{W} = W(\gamma_{ij}) - \lambda(\sqrt{I_3} - 1), \tag{17}$$

where λ imposes the incompressibility condition and can be identified as hydrostatic pressure [6, 9]. In each point, the condition $I_3 = 1$ is imposed. The stresses are now calculated as

$$S_{ij} = \frac{\partial \overline{W}}{\partial \gamma_{ij}} = \frac{\partial W}{\partial \gamma_{ij}} - \frac{1}{2}\lambda \frac{\partial I_3}{\partial \gamma_{ij}}. \tag{18}$$

2.3. **Stresses in Shells**. In shell structures, it is assumed that the stress normal to midplane is zero: $\sigma_{33} = 0$. Therefore, some of the equations can be simplified. The elimination of the Lagrange multiplier can be made at the element level explicitly avoiding a mixed or penalty formulation. The constrained functional and stresses can be written in a more convenient way [6]. According to continuum mechanics, it is assumed that $S_{ij} = S_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$. Strain invariants can now be written as

$$I_1 = 3 + 2J_1, \tag{19}$$

$$I_2 = 3 + 4J_1 + 4J_2, \tag{20}$$

$$I_3 = 1 + 2J_1 + 4J_2 + 3J_3, \tag{21}$$

$$J_1 = \gamma_{11} + \gamma_{22} + \gamma_{33}, \tag{22}$$

$$J_2 = \gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{22} + \gamma_{11}\gamma_{33} - \frac{1}{4}(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2), \tag{23}$$

$$J_3 = \gamma_{11}\gamma_{22}\gamma_{22} + \frac{1}{4}\gamma_{12}\gamma_{13}\gamma_{23} - \frac{1}{4}(\gamma_{11}\gamma_{23}^2 + \gamma_{22}\gamma_{13}^2 + \gamma_{33}\gamma_{12}^2). \tag{24}$$

Therefore, the constrained functional (17) can be expressed as

$$\overline{W} = \overline{C}_1 J_1 + \overline{C}_2 J_2 - \overline{\lambda} \overline{I}, \tag{25}$$

where

$$\overline{C}_1 = 2C_1 + 4C_2, \tag{26}$$

$$\overline{C}_2 = 4C_2, \tag{27}$$

$$\bar{\lambda} = 2\lambda, \tag{28}$$

$$\bar{I} = J_1 + 2J_2 + 4J_3. \tag{29}$$

The stresses are now obtained as

$$S_{ij} = \overline{C}_1 \frac{\partial J_1}{\partial \gamma_{ij}} + \overline{C}_2 \frac{\partial J_2}{\partial \gamma_{ij}} - \bar{\lambda} \frac{\partial \bar{I}}{\partial \gamma_{ij}} \tag{30}$$

and the material constitutive tensor (9) for the incremental solution is obtained by the derivatives of the stresses relative to strains:

$$\frac{\partial S_{ij}}{\partial \gamma_{kl}} = D_{klij} = (\overline{C}_1 - \bar{\lambda}) \frac{\partial^2 J_1}{\partial \gamma_{kl} \partial \gamma_{ij}} + (\overline{C}_2 - \bar{\lambda}) \frac{\partial J_2}{\partial \gamma_{kl}} - 4\bar{\lambda} \frac{\partial^2 J_3}{\partial \gamma_{kl} \partial \gamma_{ij}}. \tag{31}$$

If S_{33} is zero as it is supposed in shells, it can be eliminated from Eq. (8). The value of γ_{33}, not necessarily zero, is dependent on other strains. The contribution of γ_{33} cannot be excluded. From the compressibility condition the explicit value of γ_{33} can be evaluated. Considering

$$I_3 - 1 = R + S\gamma_{33} = 0, \tag{32}$$

where

$$R = \gamma_{11} + \gamma_{22} + 2\gamma_{11}\gamma_{22} - \frac{1}{2}(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2) +$$

$$+ \gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 \tag{33}$$

and

$$S = 1 + 2(\gamma_{11} + \gamma_{22}) + 4\gamma_{11}\gamma_{22} - \gamma_{12}^2. \tag{34}$$

Finally, we get

$$\gamma_{33} = -\frac{R}{S}. \tag{35}$$

The assumption $S_{33} = 0$ allows the direct calculation of the Lagrange multiplier in each point. From (30), we obtain

$$\bar{\lambda} = \frac{\overline{C}_1 \dfrac{\partial J_1}{\partial \gamma_{33}} + \overline{C}_2 \dfrac{\partial J_2}{\partial \gamma_{33}}}{\dfrac{\partial J_1}{\partial \gamma_{33}} + 2\dfrac{\partial J_2}{\partial \gamma_{33}} + 4\dfrac{\partial J_3}{\partial \gamma_{33}}}. \tag{36}$$

An iterative process using a single-filed formulation, where an automatic update of the displacement field takes place, is established. This new process, although avoiding mixed formulations, results in a highly nonlinear process.

2.4. **Reinforced Rubber**. In many applications, fiber-reinforced rubber is used by combining the strength of fibers with large rubber deformability [13, 17–20]. Examples can be found in car tyres, boats and hovercrafts. Reinforcement layers can be oriented in chosen directions [21]. This can be considered as a composite material. Sandwich laminates with stiff composite skins and rubber cores also have interesting applications where a combination of stiffness and damping is needed, for example, in trains.

In this paper, we used the first- and third-order shear deformation theories in order to capture various elastic material properties.

A bi-phase model is proposed for composite-reinforced rubber materials. The material properties are summed up by separate contribution of isotropic rubber and composite layers. Both hyperelastic and elastic material models for rubber and composites are considered.

3. **Finite Element Solution of Large Deformations**. When applying the finite element method to large deformation problems, two approaches are typical: a total Lagrangian formulation and an updated Lagrangian formulation [10, 20]. In the total Lagrangian formulation, the reference configuration is the initial one, while in the updated Lagrangian formulation, the reference configuration is continuously updated in each iteration and thus the element basis moves with changes in geometry. The choice between both configurations depends on the constitutive law and on the applications [10, 22–31]. In this paper, we adopted the total Lagrangian formulation.

3.1. **Total Lagrangian Formulation**. Equations to be solved are derived from (8). Displacements are obtained from

$$\underline{u} = \underline{N}\,\underline{d}, \tag{37}$$

$$\Delta \underline{u} = \underline{N}\Delta \underline{d}. \tag{38}$$

The Green–Lagrange strain $\underline{\gamma}$ is expressed as the sum of two vectors corresponding to the linear and nonlinear contributions:

$$\underline{\gamma} = \underline{\varepsilon}_L + \underline{\varepsilon}_{NL} \tag{39}$$

or in terms of the nodal displacements

$$\underline{\gamma} = \underline{B}\,\underline{d} = \left(\underline{B}_L + \frac{1}{2}\underline{B}_{NL}\right)\underline{d}, \tag{40}$$

where

$$\underline{\varepsilon}_L = \underline{B}_L \underline{d}, \tag{41}$$

$$\underline{\varepsilon}_{NL} = \frac{1}{2}\underline{B}_{NL}\underline{d}, \tag{42}$$

and $\underline{B}_L$ is the traditional $\underline{B}$ matrix for small deformations [20–22]. The matrix $\underline{B}_{NL}$ can be written as

$$\underline{\varepsilon}_{NL}=\frac{1}{2}\underline{A}\underline{H}=\frac{1}{2}\begin{bmatrix} \frac{\partial u}{\partial X} & 0 & 0 & \frac{\partial v}{\partial X} & 0 & 0 & \frac{\partial w}{\partial X} & 0 & 0 \\ 0 & \frac{\partial v}{\partial Y} & 0 & 0 & \frac{\partial v}{\partial Y} & 0 & 0 & \frac{\partial w}{\partial Y} & 0 \\ 0 & 0 & \frac{\partial u}{\partial Z} & 0 & 0 & \frac{\partial v}{\partial Z} & 0 & 0 & \frac{\partial w}{\partial Z} \\ \frac{\partial u}{\partial Y} & \frac{\partial u}{\partial X} & 0 & \frac{\partial v}{\partial Y} & \frac{\partial v}{\partial X} & 0 & \frac{\partial w}{\partial Y} & \frac{\partial w}{\partial X} & 0 \\ \frac{\partial u}{\partial Z} & 0 & \frac{\partial u}{\partial X} & \frac{\partial v}{\partial Z} & 0 & \frac{\partial v}{\partial X} & \frac{\partial w}{\partial Z} & 0 & \frac{\partial w}{\partial X} \\ 0 & \frac{\partial u}{\partial Z} & \frac{\partial u}{\partial Y} & 0 & \frac{\partial v}{\partial Z} & \frac{\partial v}{\partial Y} & 0 & \frac{\partial w}{\partial Z} & \frac{\partial w}{\partial Y} \end{bmatrix}\begin{bmatrix} \frac{\partial u}{\partial X} \\ \frac{\partial u}{\partial Y} \\ \frac{\partial u}{\partial Z} \\ \frac{\partial v}{\partial X} \\ \frac{\partial v}{\partial Y} \\ \frac{\partial v}{\partial Z} \\ \frac{\partial w}{\partial X} \\ \frac{\partial w}{\partial Y} \\ \frac{\partial w}{\partial Z} \end{bmatrix}, \qquad (43)$$

where $X=X_1$, $Y=X_2$, $Z=X_3$, $u=u_1$, $v=u_2$, and $w=u_3$. Equation (42) can also be written in the form

$$\underline{\varepsilon}_{NL}=\frac{1}{2}\underline{A}\underline{G}\underline{d}, \qquad (44)$$

where $\underline{G}$ is the matrix of spatial derivatives of the shape functions. Therefore,

$$\underline{B}_{NL}=\underline{A}\underline{G}. \qquad (45)$$

The Green–Lagrange strain increment can be obtained from (10) as

$$\Delta\underline{\gamma}=\Delta\underline{\varepsilon}_L+\Delta\underline{\varepsilon}_{NL}. \qquad (46)$$

It can be represented as

$$\Delta\underline{\varepsilon}_L=\underline{B}_{NL}\Delta\underline{d} \qquad (47)$$

and

$$\Delta\underline{\varepsilon}_{NL}=\Delta\underline{\varepsilon}^1_{NL}+\Delta\underline{\varepsilon}^2_{NL}=\underline{B}_{NL}\Delta\underline{d}+\frac{1}{2}\Delta\underline{B}_{NL}\Delta\underline{d}, \qquad (48)$$

where

$$\Delta\underline{B}_{NL}=\Delta\underline{A}\underline{G}. \qquad (49)$$

Here $\Delta\underline{A}$ is the matrix identical to $\underline{A}$ with spatial derivatives of increments instead of total displacements.

The variation of the Green–Lagrange strains can be written as

$$\delta\underline{\gamma} = \delta\underline{\varepsilon}_L + \delta\underline{\varepsilon}_{NL} = \left(\underline{B}_L + \frac{1}{2}\underline{B}_{NL}\right)\delta\underline{d}. \tag{50}$$

The variation of the Green–Lagrange increments is obtained from (11) as

$$\delta\underline{\gamma} = \delta\underline{A}\underline{G}\delta\underline{d}. \tag{51}$$

The stress increments can be obtained from (9) as

$$\delta\underline{S} = \underline{D}\delta\underline{\gamma}, \tag{52}$$

where $\underline{D}$ is the material constitutive matrix. Using the approximation

$$\Delta\underline{\gamma} = \Delta\underline{\varepsilon}_L + \Delta\underline{\varepsilon}^1_{NL} = (\underline{B}_L + \underline{B}_{NL})\Delta\underline{d}, \tag{53}$$

we can finally obtain the following equation of motion:

$$\int_{V_0} (\delta\underline{d}^T \underline{B}^T \underline{D}\underline{B}\Delta\underline{d} + \underline{S}^T \delta\underline{A}\underline{G}\Delta\underline{d}) dV_0 =$$

$$= \int_{S_0} \delta\underline{d}^T \underline{N}^T \Delta\underline{t}^0 dS_0 + \int_{V_0} \delta\underline{d}^T \underline{N}^T \Delta\underline{b}^0 dV_0. \tag{54}$$

The product $\delta\underline{A}^T \underline{S}$ can be written as

$$\delta\underline{A}^T \underline{S} = \underline{t}\underline{G}\delta\underline{d}, \tag{55}$$

where

$$\underline{\tau} = \begin{bmatrix} \underline{S} & 0 & 0 \\ 0 & \underline{S} & 0 \\ 0 & 0 & \underline{S} \end{bmatrix} \tag{56}$$

and

$$\underline{S} = \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix}. \tag{57}$$

As in (54), virtual displacements are arbitrary, this equation can now be written as

$$\underline{K}_T \Delta\underline{d} = \Delta\underline{f}, \tag{58}$$

where $\underline{K}_T$ is the tangential matrix in the form

$$\underline{K}_T = \underline{K}_L + \underline{K}_{NL} + \underline{K}_S. \tag{59}$$

In the previous equation, $\underline{K}_L$ is the tangential matrix for small displacements, $\underline{K}_{NL}$ is the tangential matrix for large displacements, and $\underline{K}_S$ is the geometric matrix. They can be expressed as

$$\underline{K}_L = \int_{V_0} \underline{B}_L^T \underline{D}\underline{B}_L dV_0, \tag{60}$$

$$\underline{K}_{NL} = \int_{V_0} (\underline{B}_{NL}^T \underline{D}\underline{B}_L + \underline{B}_L^T \underline{D}\underline{B}_{NL}) dV_0, \tag{61}$$

$$\underline{K}_S = \int_{V_0} \underline{G}^T \underline{\tau}\underline{G} dV_0. \tag{62}$$

3.2. **Evaluation of Strains and Stresses**. After the calculation of $\Delta\underline{d}$, the total nodal displacements can be updated as

$$\underline{\bar{d}} = \underline{d} + \Delta\underline{d} \tag{63}$$

and the total strain can be calculated as

$$\underline{\bar{\gamma}} = \left(\underline{B}_L + \frac{1}{2}\underline{B}_{NL}\right)\underline{\bar{d}} \tag{64}$$

or by

$$\Delta\underline{\bar{\gamma}} = \left(\underline{B}_L + \underline{B}_{NL} + \frac{1}{2}\Delta\underline{B}_{NL}\right)\underline{d}. \tag{65}$$

Eventually, we determine the total strain as

$$\underline{\bar{\gamma}} = \underline{\gamma} + \Delta\underline{\gamma}. \tag{66}$$

The evaluation of the stress depends on the material constitutive laws. In hyperelastic materials, the total stress is directly calculated from the total strain:

$$\bar{S}_{ij} = S_{ij}(\gamma_{ij}). \tag{67}$$

A degenerate shell element is used [28–30, 32] with a first- and third-order shear deformation theory [29, 31]. For further information, the reader should consult the above references.

4. **Results and Discussion**. Two numerical examples are considered, a three-point bending beam and an inflating rubber membrane.

4.1. **Beam under Flexure Loads**. A rubber beam, a fiber-reinforced rubber beam, and a sandwich beam with composite skins and a rubber core are considered. In Fig. 1, the general dimensions and cross sections are considered.

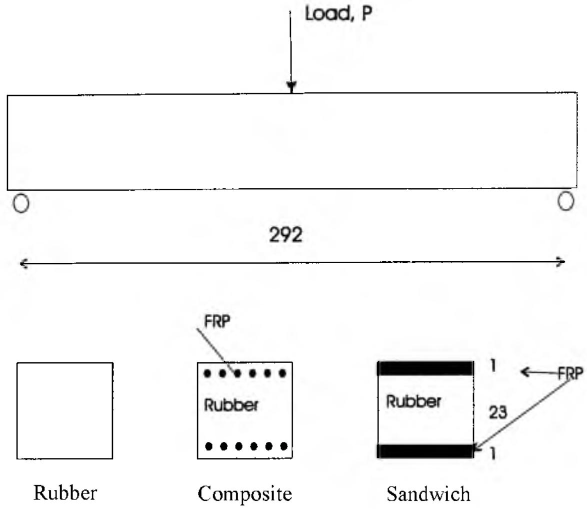

Fig. 1. Beam under flexural load.

In this paper, the following material properties are used: for rubber: $C_1 = 0.55$ and $C_2 = 0.138$; for composite material: $E_1 = E_2 = 10{,}000$, $\nu = 0.3$, and $G_{12} = G_{13} = G_{23} = 3900$. Nonlinear evolution of the load vs central displacement relation is presented in Fig. 2 for the case of a rubber beam. Calculations with the account of the first- and third-order shear deformations provide very similar results.

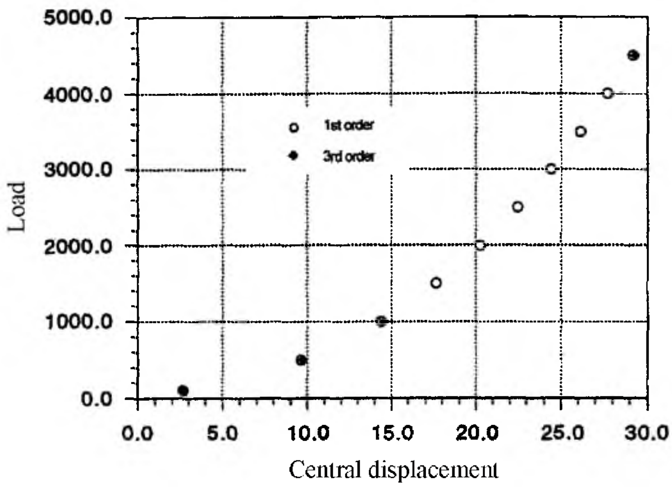

Fig. 2. Load vs central displacement for a rubber beam.

In Fig. 3, the evolution for the case of a fiber-reinforced rubber beam is presented. Some variation occurs in the nonlinear behavior between both deformation theories. Figure 4 presents the evolution of the load vs displacement for the case of a sandwich beam. Both theories present similar results.

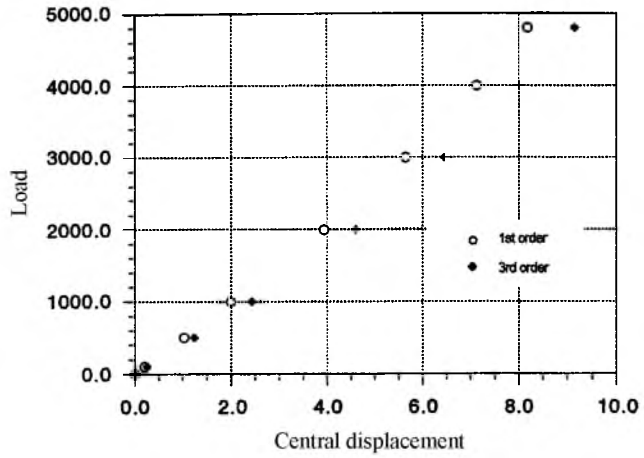

Fig. 3. Load vs central displacement for a composite section beam.

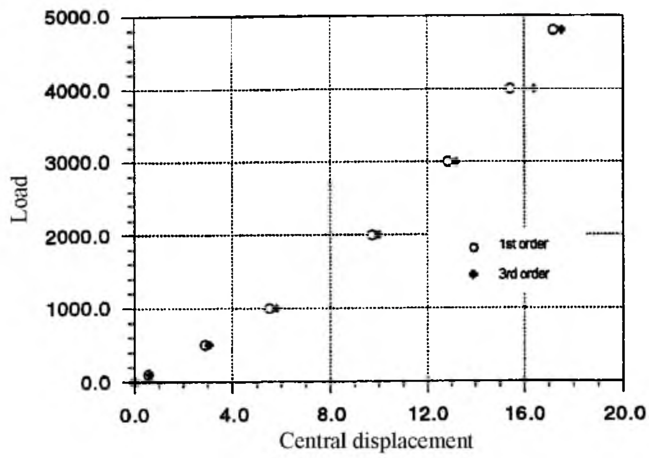

Fig. 4. Load vs central displacement for a sandwich beam.

4.2. **Inflating Membrane**. This example refers to a simply supported rubber plate under external growing pressure as seen in Fig. 5. The applied load is non-conservative as the loaded element surfaces stretch and change their orientation. The latter phenomenon involves large deformations and rotations and presents a challenge to the present model. On a large part of the load path the membrane action is predominant and the behavior of this plate in bending is similar to that of an inflating membrane [11, 33–37]. Numerical solutions for the governing differential equations were given by Adkins and Rivlin [35] and Yang and Feng [36]. Finite element solutions were presented by Oden and Key [37], Tang [38], and Mattiasson [11]. In this paper, a 13-element mesh was used as illustrated in Fig. 5. Only a 22.5° sector was used due to symmetry in the material,

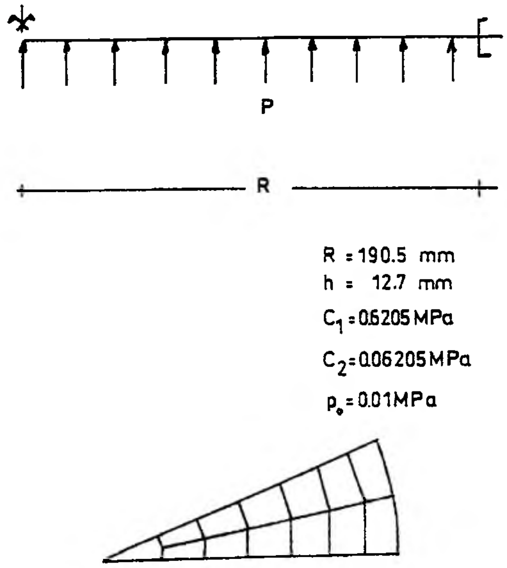

Fig. 5. Rubber plate (insufflating membrane).

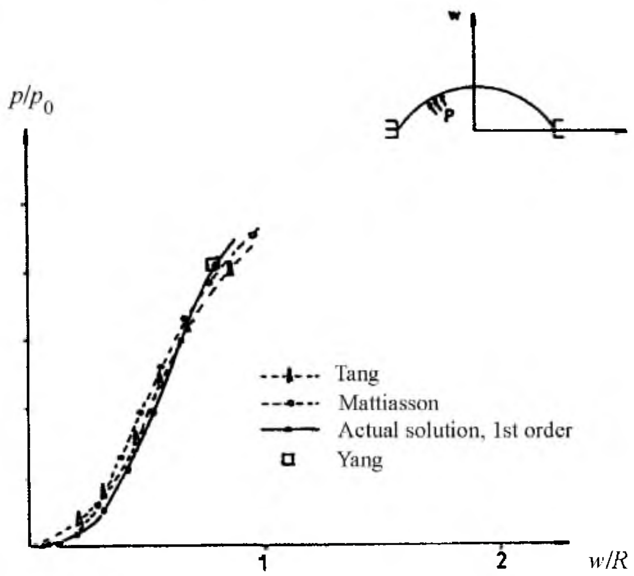

Fig. 6. Load vs central displacement for an insufflating membrane.

loading and boundary conditions. The results are compared with those presented in [11, 36, 38] and are in close agreement (Fig. 6). After a load factor of 17.3, the process was stopped due to very slow convergence.

Conclusions. In this paper, rubber, rubber composite, and sandwich shells were analyzed by a new finite element formulation that incorporates a single-field formulation for the rubber material model. A layered formulation of a degenerate shell element was used with first- and third-order shear deformation theories. A biphase material model was used for fiber-reinforced rubber shells as a better approach to homogenized shells. The nonlinear rubber material model considers a

single-field formulation, where at each integration point the Lagrange multiplier is eliminated. Geometric nonlinearities are accounted for. A beam and a plate under bending loads were treated numerical experiments. Both present interesting results and prove the accuracy of the model.

Резюме

Наведено скінченноелементний розв'язок для композитних оболонок із каучуковою матрицею. Розглядаються також багатошарові конструкції з каучуковою основою. Для урахування нестисливості каучукової матриці і неоднорідності структури багатошарових оболонок необхідно розробити на основі числових методів добре обгрунтовану модель, що дозволить описати поведінку указаних будівельних матеріалів. Запропонована модель застосовувалась авторами до виродженого елемента оболонки в рамках деформаційних теорій зсуву першого і третього порядку. Модель дозволяє достатньо точно прогнозувати нелінійну поведінку багатошарових оболонок із тонкими прошарками з композитного матеріалу і шарами каучука, а також композитних оболонок із каучуковою матрицею.

1. M. Tuomala, D. R. J. Owen, O. C. Zienkiewicz, and S. Nakazawa, "A penalty function finite element method in nonlinear elasticity," in: *Numerical Methods For Coupled Problems*, Pineridge Press, Swansea (1981).
2. E. Jankovich, F. Leblanc, M. Durand, and M. Bercovier, "A finite element method for the analysis of rubber parts. Experimental and analytical assessment," *Comp. & Struct.*, **14**, Nos. 5-6, 385–391 (1981).
3. B. Haggblad and J. A. Sunderberg, "Large strain solutions of rubber components," *Comp. & Struct.*, **17**, Nos. 5-6, 835–843 (1983).
4. H. J. Anderson, *Isotropic and Reinforced Rubber Analysis*, Ph. D. Thesis, C/Ph/82/85, University College of Swansea (1985).
5. J. M. A. Cesar De Sá, *Numerical Modelling of Incompressible Problems in Glass Forming and Rubber Technology*, Ph. D. Thesis, C/Ph/91/86, University College of Swansea (1986).
6. J. M. A. Cesar De Sá and D. R. J. Owen, "The finite element analysis of reinforced rubber shells," in: R. Taylor, D. R. J. Owen, E. Hinton (Eds.), *Computational Methods for Nonlinear Problems*, Pineridge Press, Swansea (1987).
7. A. J. M. Ferreira, J. M. A. Cesar De Sá, and A. T. Marques, "A degenerated shell element for the static linear analysis of sandwich structures," in: Proc. ICCM 9, Madrid (1993).
8. Y. C. Fung, *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs (1965).
9. J. T. Oden, *Finite Elements of Nonlinear Continua*, McGraw-Hill Book Company Inc., USA (1972).
10. K.-J. Bathe, *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs (1982).

11. K. Mattiasson, *On the Co-Rotational Finite Element Formulation for Large Deformation Problems*, Publication 83:1, Chalmers University of Technology, Dept. Structural Mechanics, Goteborg (1983).
12. L. R. G. Treloar, *The Physics of Rubber Elasticity*, 2nd edition, Oxford University Press (1958).
13. P. K. Freakley and A. R. Payne, *Theory and Practice of Engineering Rubber*, Applied Science Publ. Ltd. (1978).
14. M. Mooney, "A theory of large elastic deformation," *J. Appl. Phys.*, **11**, 582 (1940).
15. R. S. Rivlin and D. N. Saunders, "Large elastic deformation of isotropic material. Experiments on the deformation of rubber," *Phil. Trans. Roy. Soc.*, **A243**, 251–298 (1951).
16. R. S. Rivlin, *Rheology: Theory and Applications*, Vol. I, (Ed. F. K. Eitich), Ch. 10, Academic Press, New York (1956).
17. J. M. Klosner and A. Segal, *Mechanical Characterization of a Natural Rubber*, Rep. 69-42, Polytechnic Institute of Brooklyn, New York (1969).
18. L. J. Hart-Smith, "Elasticity parameters for finite deformations of rubber-like materials," *Z. Agnew. Math. Phys.*, **17**, 608–625 (1966).
19. W. D. Hutchinson, G. W. Becker, and R. F. Lander, "Determination of the stored energy function of rubber-like materials," in: Bull. 4th Meeting Interagency Chemical Rocket Propulsion Group – Working Group Mechanical Behavior, CPIA Publ. 94, Vol. L, 141–152 (1965).
20. H. P. Patel, J. L. Turner and J. D Walter, "Radial tyre cord-rubber composites," in: *Rubber Chemistry and Technology*, 4a (1977).
21. R. M. Jones, *Mechanics of Composite Materials*, McGraw-Hill Book Co., USA (1975).
22. D. R. J. Owen and J. A. Figueiras, "Anisotropic elastoplastic finite element analysis of thick and thin plates and shells," *Int. J. Num. Meth. Eng.*, **19**, 541–566 (1983).
23. O. C. Zienkiewicz, *The Finite Element Method*, McGraw-Hill, London (1977).
24. E. Hinton and D. R. J. Owen, *Finite Element Software for Plates and Shells*, Pineridge Press, Swansea (1984).
25. J. A. Figueiras, *Ultimate Load Analysis of Anisotropic and Reinforced Concrete Plates and Shells*, Ph. D. Thesis, C/Ph/72/83, University College of Swansea (1983).
26. R. H. Gallagher, "Shell elements," in: Proc. First World Congress on *Finite Element Methods in Structural Mechanics*, Bournemouth, England (1975).
27. T. J. R. Hughes and W. K. Liu, "Nonlinear finite element analysis of shells: Part 1. Three-dimensional shells," *Comp. Meth. Appl. Mech. Eng.*, **26**, 331–362 (1981).
28. A. J. M. Ferreira, J. T. Barbosa, A. T. Marques, and J. C. Sá, "Nonlinear analysis of sandwich shells: the effect of core plasticity," *Comp. & Struct.*, **76**, 337–346 (2000).

29. A. J. M. Ferreira and J. T. Barbosa, "Buckling behavior of composite shells," in: *Composite Structures*, Vol. 50, Elsevier (2000), pp. 93–98.
30. A. J. M. Ferreira, A. T. Marques, and J. C. Sá, "Analysis of reinforced concrete with external composite strengthening," *Composites* (Part B: Engineering), Elsevier, **31**, issue 6-7, 527–534 (2000).
31. A. J. M. Ferreira, P. P. Camanho, A. T. Marques, and A. A. Fernandes, "Modeling of concrete beams reinforced with frp re-bars," in: *Composite Structures*, Vol. 53 (1), Elsevier (2001), pp. 107–116.
32. S. Ahmad, B. Irons, and O.C. Zienkiewicz, "Analysis of thick and thin shell structures by curved finite elements," *Int. J. Num. Meth. Eng.*, **2**, 419–451 (1970).
33. I. Cormeau, "Elastoplastic thick shell analysis by viscoplastic solid finite elements," *Int. J. Num. Meth. Eng.*, **12**, 203–227 (1978).
34. L. J. Hart-Smith and J. D. C. Crisp, "Large elastic deformations of thin rubber membranes," *Int. J. Eng. Sci.*, **5**, 1–24 (1967).
35. J. E. Adkins and R. S. Rivlin, "Large elastic deformations of isotropic materials," *Phyl. Trans. Royal Soc.*, *Series A*, **244**, 505 (1952).
36. W. H. Yang and W. W. Feng, "On axisymmetrical deformations of nonlinear membranes," *J. Appl. Mech.*, **37**, 1002–1011 (1970).
37. J. T. Oden and J. E. Key, "Analysis of finite deformations of elastic solids by the finite element method," in: Proc. IUTAM Colloq. "*High Speed Computing Elastic Structures*," Liege (1971).
38. S. C. Tang, "Large strain analysis of an inflating membrane," *Comp. & Struct.*, **15**, No. 1, 71–78 (1982).

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