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MINIMAX PREDICTION PROBLEM FOR MULTIDIMENSIONAL STATIONARY STOCHASTIC SEQUENCES

The considered problem is estimation of the unknown value of the functionals $A\vec{\xi} = \sum_{j=0}^{\infty} \vec{a}(j)\vec{\xi}(j)$ and $A_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}(j)\vec{\xi}(j)$ which depend on the unknown values of a multidimensional stationary stochastic sequence $\vec{\xi}(j)$ based on observations of the sequence $\vec{\xi}(j)$, $j < 0$, from the class Ξ of sequences which satisfy conditions $E\vec{\xi}(j) = 0$, $\|\vec{\xi}(j)\|^2 \leq P$. The maximum values of the mean-square errors of the optimal estimates of the functionals $A\vec{\xi}$ and $A_N\vec{\xi}$ are found. It is shown that these maximum values of the errors in the class Ξ give the moving average sequences which are determined by eigenvectors of compact operators constructed with the help of the sequence $\vec{a}(j)$.

1. Introduction

Traditional methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and sequences are employed under the condition that spectral densities of processes are known exactly (see, for example, selected works of A. N. Kolmogorov (1992), survey by T. Kailath (1974), Yu. A. Rozanov (1990), N. Wiener (1966); A. M. Yaglom (1987)). In practice, however, complete information on the spectral densities is impossible in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies one of the traditional estimation methods provided that the estimated or selected density is the true one. This procedure can result in a significant increasing of the value of error as K. S. Vastola and H. V. Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of the admissible spectral

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densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor (1985). The paper by Ulf Grenander (1957) should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. J. Franke (1984, 1985, 1991), J. Franke and H. V. Poor (1984) investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of densities. In the papers by M. P. Moklyachuk (1994, 1997, 1998, 2000, 2001), M. P. Moklyachuk and A. Yu. Masyutka (2005, 2006) the minimax approach to extrapolation, interpolation and filtering problems are investigated for functionals which depend on the unknown values of stationary processes and sequences.

In this article we consider the problem of estimation of the unknown value of the functionals $A\vec{\xi} = \sum_{j=0}^{\infty} \vec{a}(j)\vec{\xi}(j)$ and $A_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}(j)\vec{\xi}(j)$ which depend on the unknown values of a multidimensional stationary stochastic sequence $\vec{\xi}(j) = \{\xi_k(j)\}_{k=1}^{T}$ from the class Ξ of sequences of the rank m $(1 \leq m \leq T)$ which satisfy conditions

$$E\vec{\xi}(j) = \{E\xi_k(j)\}_{k=1}^{T} = \vec{0}, \quad \|\vec{\xi}(j)\|^2 = \sum_{k=1}^{T} E\left|\xi_k(j)\right|^2 \leq P, \tag{1}$$

based on observations of the sequence $\vec{\xi}(j)$ for $j < 0$. The maximum values of the mean-square errors of the optimal estimates of the functionals $A\vec{\xi}$, $A_N\vec{\xi}$ are found. It is shown that these maximum values of the errors in the class Ξ give the moving average sequences which are determined by eigenvectors of compact operators constructed with the help of the sequence $\vec{a}(j)$.

2. Maximum value of the error of estimation of the functional $A_N\vec{\xi}$

Let $\Delta(\xi, \hat{A}_N) = E\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2$ denotes the mean-square error of the estimate $\hat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$. Denote by Λ the class of all linear estimates of the functional $A_N\vec{\xi}$.

Theorem 1. *The function $\Delta(\xi, \hat{A}_N)$ has a saddle point on the set $\Xi \times \Lambda$ and the following equality holds true*

$$\min_{\hat{A}_N \in \Lambda} \max_{\xi \in \Xi} \Delta(\xi, \hat{A}_N) = \max_{\xi \in \Xi} \min_{\hat{A}_N \in \Lambda} \Delta(\xi, \hat{A}_N) = P\nu_N^2.$$

The least favorable multidimensional stationary stochastic sequence in the class Ξ *for the optimal estimation of the functional* $A_N\vec{\xi}$ *is a moving average sequence of the order* N *of the form*

$$\vec{\xi}(j)=\sum_{u=j-N}^{j}\Phi(j-u)\vec{\eta}(u).$$

Here ν_N^2 *is the greatest eigenvalue of the compact operator* Q_N *in the space* $\mathbb{C}^{T(N+1)}$ *determined by matrix constructed with the help of the* $T\times T$ *block-matrices*

$$Q_N=\{Q_N(p,q)\}_{p,q=0}^{N}=\sum_{u=0}^{\min(N-p,N-q)}(\vec{a}(p+u))^*\vec{a}(q+u);$$

$\vec{\eta}(u)=\{\eta_k(u)\}_{k=1}^m$ *is a multidimensional stationary stochastic sequence with orthogonal values* $E\vec{\eta}(i)(\vec{\eta}(j))^*=\delta_j^iE$, *where* E *is the identity matrix,* δ_j^i *is the Kronecker symbol;* $\Phi(u),\ u=0,1,\ldots,N$ *are* $T\times m$ *matrices, elements of which are determined by the eigenvector that corresponds to* ν_N^2, *and the condition* $\|\vec{\xi}(j)\|^2=P$.

Proof. Lower bound. Denote by Ξ_R the class of all regular multidimensional stationary stochastic sequences that satisfy condition (1). Since $\Xi_R\subset\Xi$, we have

$$\max_{\xi\in\Xi}\min_{\hat{A}_N\in\Lambda}\Delta(\xi,\hat{A}_N)\geq\max_{\xi\in\Xi_R}\min_{\hat{A}_N\in\Lambda}\Delta(\xi,\hat{A}_N). \tag{2}$$

Every regular multidimensional stationary stochastic sequence admits the canonical moving average representation [20]

$$\vec{\xi}(j)=\sum_{u=-\infty}^{j}\Phi(j-u)\vec{\eta}(u), \tag{3}$$

where $\vec{\eta}(u)=\{\eta_k(u)\}_{k=1}^m$ is a standard multidimensional stationary stochastic sequence with orthogonal values, $\Phi(u)=\{\Phi_{kl}(u)\}_{k=1}^{T}{}_{l=1}^{m}$ are coefficients of the canonical representation, m ($1\leq m\leq T$) is the rank of the sequence $\vec{\xi}(j)=\{\xi_k(j)\}_{k=1}^T$. The sequence $\vec{\xi}(j)\in\Xi_R$ is determined if there is determined $\{\Phi(u):u=0,\ 1,\ \ldots\}$ such that

$$\left\|\vec{\xi}(j)\right\|^2=\sum_{k=1}^{T}E\left|\xi_k(j)\right|^2=\sum_{k=1}^{T}E\left|\sum_{u=-\infty}^{j}\sum_{l=1}^{m}\Phi_{kl}(j-u)\eta_l(u)\right|^2=$$

$$=\sum_{k=1}^{T}\sum_{u,v=-\infty}^{j}\sum_{l,n=1}^{m}\Phi_{kl}(j-u)\overline{\Phi_{kn}}(j-v)E\eta_l(u)\overline{\eta_n}(v)=$$

$$= \sum_{k=1}^{T} \sum_{u=-\infty}^{j} \sum_{l=1}^{m} |\Phi_{kl}(u)|^2 = \sum_{u=0}^{\infty} \sum_{k=1}^{T} \sum_{l=1}^{m} |\Phi_{kl}(u)|^2 = \sum_{u=0}^{\infty} \|\Phi(u)\|^2 \leq P. \quad (4)$$

The value of the mean-square error $E\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2$ is minimal if we take an estimate $\hat{A}_N\vec{\xi}$ of the form

$$\hat{A}_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}(j)\,\hat{\vec{\xi}}(j) = \sum_{j=0}^{N} \sum_{k=1}^{T} a_k(j)\hat{\xi}_k(j),$$

where $\hat{\vec{\xi}}(j)$ is an optimal estimate of the value $\vec{\xi}(j)$ based on observations of the sequence $\vec{\xi}(p)$ for $p < 0$. From the canonical representation (3) of the regular sequence and the form of the optimal estimate of its values

$$\hat{\vec{\xi}}(j) = \sum_{u=-\infty}^{-1} \Phi(j-u)\vec{\eta}(u), \quad (5)$$

it follows that

$$\min_{\hat{A}_N \in \Lambda} E\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2 = E\left|\sum_{j=0}^{N} \vec{a}(j) \sum_{u=0}^{j} \Phi(j-u)\vec{\eta}(u)\right|^2 =$$

$$= \sum_{i,j=0}^{N} \sum_{k,l=1}^{T} a_k(i)\overline{a_l}(j) \sum_{u=0}^{i} \sum_{v=0}^{j} \sum_{n,r=1}^{m} \Phi_{kn}(i-u)\overline{\Phi_{lr}}(j-v)E\eta_n(u)\overline{\eta_r}(v) =$$

$$= \sum_{i,j=0}^{N} \sum_{k,l=1}^{T} a_k(i)\overline{a_l}(j) \sum_{u=0}^{\min(i,j)} \sum_{n=1}^{m} \Phi_{kn}(i-u)\overline{\Phi_{ln}}(j-u) =$$

$$= \sum_{i,j=0}^{N} \sum_{k,l=1}^{T} a_k(i)\overline{a_l}(j)\; R_{kl}(i,j) = \sum_{i,j=0}^{N} \vec{a}(i)R(i,j)(\vec{a}(j))^*, \quad (6)$$

where

$$R(i,j) = \{R_{kl}(i,j)\}_{k,l=1}^{T},\; R_{kl}(i,j) = \sum_{u=0}^{\min(i,j)} \sum_{n=1}^{m} \Phi_{kn}(i-u)\overline{\Phi_{ln}}(j-u).$$

By changing of variables $p = i-u,\; q = j-u$, we can represent (6) in the form

$$\min_{\hat{A}_N \in \Lambda} E\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2 = \sum_{p,q=0}^{N} \sum_{k,l=1}^{T} \sum_{n=1}^{m} \Phi_{kn}(p)\overline{\Phi_{ln}}(q)\; Q_{kl}^{N}(p,q), \quad (7)$$

where

$$Q_{kl}^{N}(p,q)=\sum_{u=0}^{\min(N-p,N-q)} a_k(p+u)\overline{a_l}(q+u). \tag{8}$$

Denote by Q_N the operator in the space $\mathbb{C}^{T(N+1)}$ determined by matrix which consists of the block-matrices $\{Q_N(p,q)\}_{p,q=0}^{N}$, $Q_N(p,q)=\left\{Q_{kl}^{N}(p,q)\right\}_{k,l=1}^{T}$. Operator Q_N is self-adjoint (its matrix is Hermitian) and compact. It can be represented in the form $Q_N=A_N\cdot A_N^{*}$, where the operator A_N is determined by the matrix $\{A_N(p,q)\}_{p,q=0}^{N}$ which consists with the blocks of vector columns:

$$A_N(p,q)=\begin{cases} \vec{a}^{*}(p+q),\ p+q\leq N,\\ \vec{0},\ p+q>N.\end{cases}$$

The operator Q_N has positive real-valued eigenvalues. It follows from (7) that $\Phi_{ij}(p)=0$ for $p>N+1$. Denote $\tilde{\Phi}(p)=\left\{P^{-1/2}\Phi_{ij}(p)\right\}_{i=1}^{T}{}_{j=1}^{m}$, $\tilde{\Phi}=\left\{\tilde{\Phi}(p)\right\}_{p=0}^{N}$. Then condition (4) has the form

$$\|\tilde{\Phi}\|^2=\sum_{p=0}^{N}\|\Phi(p)\|^2\leq 1, \tag{9}$$

where $\|\tilde{\Phi}\|$ is the norm in the space $\mathbb{C}^{Tm(N+1)}$.
From (7) and (9) it follows that

$$\min_{\hat{A}_N\in\Lambda}\Delta(\xi,\hat{A}_N)=P\left\langle Q_N\tilde{\Phi},\ \tilde{\Phi}\right\rangle,$$

where $\langle\cdot,\ \cdot\rangle$ is the scalar product in the space $\mathbb{C}^{Tm(N+1)}$ and $Q_N\tilde{\Phi}$ is the matrix constructed with the help of block-matrices

$$Q_N\tilde{\Phi}=\left\{Q_N\tilde{\Phi}(p,q)\right\}_{p,q=0}^{N}=\left\{Q_N(p,q)\cdot\tilde{\Phi}(q)\right\}_{p,q=0}^{N}.$$

Taking into account (2), we will have the following lower bound for maximum of the mean-square error

$$\max_{\xi\in\Xi}\min_{\hat{A}_N\in\Lambda}\Delta(\xi,\hat{A}_N)\geq P\max_{\|\tilde{\Phi}\|\leq 1}\left\langle Q_N\tilde{\Phi},\tilde{\Phi}\right\rangle=P\nu_N^2. \tag{10}$$

Here ν_N^2 is the greatest eigenvalue of the operator Q_N.

Upper bound. We will use the following inequality to find an upper bound of the mean-square error

$$\min_{\hat{A}_N\in\Lambda}\max_{\xi\in\Xi}\Delta(\xi,\hat{A}_N)\leq\min_{\hat{A}_N\in\Lambda_1}\max_{\xi\in\Xi}\Delta(\xi,\hat{A}_N). \tag{11}$$

Here Λ_1 is the class of all linear estimates of the functional $A_N\vec{\xi}$, which have the form

$$\hat{A}_N\vec{\xi} = \sum_{j=-\infty}^{-1} \vec{c}(j)\vec{\xi}(j). \tag{12}$$

where $\vec{c}(j) = \{c_k(j)\}_{k=1}^{T}$ are complex vectors such that $\sum_{j=-\infty}^{-1} \|\vec{c}(j)\|^2 < \infty$. The spectral representations of the multidimensional stationary stochastic sequence and its correlation functions gives us a possibility right the relation

$$\Delta(\xi, \hat{A}_N) = E\left|A_N\vec{\xi} - \hat{A}_N\vec{\xi}\right|^2 = E\left|\sum_{j=0}^{N} \vec{a}(j)\vec{\xi}(j) - \sum_{j=-\infty}^{-1} \vec{c}(j)\vec{\xi}(j)\right|^2 =$$

$$= E\left|\int_{-\pi}^{\pi} \left(\sum_{j=0}^{N} \vec{a}(j)e^{ij\lambda} - \sum_{j=-\infty}^{-1} \vec{c}(j)e^{ij\lambda}\right) Z(d\lambda)\right|^2 =$$

$$= \int_{-\pi}^{\pi} \left(A_N(e^{i\lambda}) - C(e^{i\lambda})\right) F(d\lambda) \left(A_N(e^{i\lambda}) - C(e^{i\lambda})\right)^*,$$

$$A_N(e^{i\lambda}) = \sum_{j=0}^{N} \vec{a}(j)e^{ij\lambda},\ C(e^{i\lambda}) = \sum_{j=-\infty}^{-1} \vec{c}(j)e^{ij\lambda}.$$

Here $Z(d\lambda) = \{Z_k(d\lambda)\}_{k=1}^{T}$ is the spectral random measure and $F(d\lambda) = \{F_{kl}(d\lambda)\}_{k,l=1}^{T}$ is the spectral matrix-valued measure of the multidimensional stationary sequence. Elements $F_{kl}(d\lambda)$ of the spectral matrix-valued measure are complex measures with bounded variation which satisfy the following conditions [20]

$$F_{kk}(d\lambda) \geq 0, \quad |F_{kl}(d\lambda)|^2 \leq F_{kk}(d\lambda)F_{ll}(d\lambda). \tag{13}$$

Conditions (1) mean that

$$\int_{-\pi}^{\pi} Tr\, F(d\lambda) \leq P. \tag{14}$$

From these reasons

$$\max_{\xi\in\Xi} \Delta(\xi, \hat{A}_N) =$$

$$= \max_{F} \int_{-\pi}^{\pi} \left(A_N(e^{i\lambda}) - C(e^{i\lambda})\right) F(d\lambda) \left(A_N(e^{i\lambda}) - C(e^{i\lambda})\right)^* \leq$$

$$\leq \max_{F} \int_{-\pi}^{\pi} \left\|A_N(e^{i\lambda}) - C(e^{i\lambda})\right\|^2 \|F(d\lambda)\| \leq$$

$$\leq \max_{\lambda\in[-\pi,\pi]} \left\|A_N(e^{i\lambda}) - C(e^{i\lambda})\right\|^2 \int_{-\pi}^{\pi} \|F(d\lambda)\|.$$

It follows from (13) and (14) that the spectral matrix-valued measure satisfy condition

$$\int_{-\pi}^{\pi} \|F(d\lambda)\| = \int_{-\pi}^{\pi} \left(\sum_{k,l=1}^{T} \|F_{kl}(d\lambda)\|^2 \right)^{1/2} \leq \int_{-\pi}^{\pi} \left(\sum_{k,l=1}^{T} F_{kk}(d\lambda) F_{ll}(d\lambda) \right)^{1/2} =$$

$$= \int_{-\pi}^{\pi} \sum_{k=1}^{T} F_{kk}(d\lambda) = \int_{-\pi}^{\pi} Tr\, F(d\lambda) \leq P.$$

From these reasons we have

$$\max_{\xi\in\Xi} \Delta(\xi, \hat{A}_N) \leq P \max_{\lambda\in[-\pi,\pi]} \left\| A_N(e^{i\lambda}) - C(e^{i\lambda}) \right\|^2.$$

To estimate

$$\max_{\lambda\in[-\pi,\pi]} \left\| A_N(e^{i\lambda}) - C(e^{i\lambda}) \right\|^2$$

we consider the class of vector-valued power series $\vec{f}(z) = \sum_{n=0}^{\infty} \vec{\alpha}(n) z^n$, which are regular in the unit disk $|z| < 1$ and start from given summands $\sum_{j=0}^{N} \vec{d}(j) z^j$. Denote by ρ_N^2 the greatest eigenvalue of the matrix $H = \{H(p,q)\}_{p,q=0}^{N}$ constructed with the help of block-matrices

$$H(p,q) = \sum_{j=0}^{\min(p,q)} \vec{d}^*(p-j) \vec{d}(q-j),\ p,q = \overline{0,N}.$$

Then we have the following relation [2]

$$\min_{\{\vec{\alpha}(n): n\geq N+1\}} \max_{|z|=1} \left\| \vec{f}(z) \right\|^2 = \rho_N^2.$$

Since in our case $\vec{d}(p) = \vec{a}(N-p),\ p = \overline{0,N}$, we have to find the greatest eigenvalue of the matrix constructed with the help of block-matrices

$$G_N = \{G_N(p,q)\}_{p,q=0}^{N},\ G_N(p,q) = \sum_{u=0}^{\min(p,q)} (\vec{a}(N-p+u))^* \vec{a}(N-q+u).$$

Denote this eigenvalue by ω_N^2. The we will have

$$\min_{\hat{A}_N\in\Lambda_1} \max_{\xi\in\Xi} \Delta(\xi, \hat{A}_N) \leq P\omega_N^2.$$

It follows from (11) that

$$\min_{\hat{A}_N\in\Lambda} \max_{\xi\in\Xi} \Delta(\xi, \hat{A}_N) \leq P\omega_N^2. \qquad (15)$$

Now note that $G_N(N-p, N-q) = Q_N(p,q)$. Therefore $\omega_N^2 = \nu_N^2$. Relations (10) and (15) give us the inequality

$$\min_{\hat{A}_N \in \Lambda} \max_{\xi \in \Xi} \Delta(\xi, \hat{A}_N) \leq \max_{\xi \in \Xi} \min_{\hat{A}_N \in \Lambda} \Delta(\xi, \hat{A}_N). \qquad (16)$$

Since the opposite inequality holds true, we have equality in (16). The proof is complete.

From the proof of the theorem we have a construction of the optimal minimax estimate of the functional $A_N\vec{\xi}$.

Corollary 1. *The optimal minimax estimate $\hat{A}_N\vec{\xi}$ of the functional $A_N\vec{\xi}$ is of the form*

$$\hat{A}_N\vec{\xi} = \sum_{j=0}^{N} \vec{a}(j) \left(\sum_{u=j-N}^{-1} \Phi(j-u)\vec{\eta}(u) \right),$$

where $\vec{\eta}(u) = \{\eta_k(u)\}_{k=1}^m$ is a standard vector-valued stationary stochastic sequence with orthogonal values, $\Phi(u) = \{\Phi_{ij}(u)\}_{i=1}^{T}{}_{j=1}^{m}$ is uniquely determined by eigenvector of the operator Q_N which corresponds to the greatest eigenvalue ν_N^2, and condition $\|\vec{\xi}(j)\|^2 = P$. In particular case of stationary sequence of the minimal rank ($m = 1$), the vector $\Phi = \{\Phi(u)\}_{u=0}^N$, that consists of the block-vectors $\Phi(u) = \{\Phi_k(u)\}_{k=1}^T$, is eigenvector of the operator Q_N, which corresponds to the greatest eigenvalue ν_N^2.

Example 1. Let $= 1$ and let $A_1\vec{\xi} = \xi_1(0) + \xi_2(0) + \xi_1(1) + \xi_2(1)$. The eigenvalues of the operator Q_1 are equal to $\lambda_{1,2} = 2 \pm \sqrt{2}$. Therefore $\nu_1^2 = 2 + \sqrt{2}$. Eigenvector of the operator which corresponds to the greatest eigenvalue $\nu_1^2 = 2 + \sqrt{2}$ is of the form $\Phi = \{\Phi(0), \Phi(1)\}$, where

$$\Phi(0) = \begin{pmatrix} \sqrt{2}/2, & 1/2 \end{pmatrix}, \Phi(1) = \begin{pmatrix} 1/2, & 0 \end{pmatrix}.$$

In the case of the minimal rank ($m = 1$), the least favorable sequence $\vec{\xi}(j)$ is of the form

$$\vec{\xi}(j) = \Phi(0)\eta(j) + \Phi(1)\eta(j-1) = 1/2 \cdot \begin{pmatrix} \sqrt{2}\eta(j) + \eta(j-1), & \eta(j) \end{pmatrix}.$$

The optimal linear minimax estimate $\hat{A}_1\vec{\xi}$ of the functional $A_1\vec{\xi}$ is of the form

$$\hat{A}_1\vec{\xi} = \vec{a}(0)\Phi(1)\eta(-1) = \eta(-1)/2.$$

In the case of the maximal rank ($m = 2$), the least favorable sequence $\vec{\xi}(j)$ is of the form

$$\vec{\xi}(j) = \Psi(0)\vec{\eta}(j) + \Psi(1)\vec{\eta}(j-1),$$

where $\Psi(0)$, $\Psi(1)$ are 2×2 matrices constructed with the help of the vector-columns

$$\Psi(0) = 1/\sqrt{2} \cdot \{\Phi(0), \Phi(0)\}, \Psi(1) = 1/\sqrt{2} \cdot \{\Phi(1), \Phi(1)\}.$$

Therefore

$$\vec{\xi}(j) = \frac{1}{2\sqrt{2}} \left(\sqrt{2}(\eta_1(j) + \eta_2(j)) + \eta_1(j-1) + \eta_2(j-1), \quad \eta_1(j) + \eta_2(j) \right).$$

The optimal linear minimax estimate $\hat{A}_1\vec{\xi}$ of the functional $A_1\vec{\xi}$ is of the form

$$\hat{A}_1\vec{\xi} = \vec{a}(0)\Psi(1)\vec{\eta}(-1) = \frac{1}{2\sqrt{2}} (\eta_1(-1) + \eta_2(-1)).$$

The mean-square errors in both cases are not greater than $2 + \sqrt{2}$.

EXAMPLE 2. Let $= 1$ and let $A_1\vec{\xi} = \xi_1(0) + \xi_2(0) + \xi_1(1) + \xi_2(1)$. eigenvalues of the operator Q_1 are equal to $3 \pm \sqrt{5}$. Therefore $\nu_1^2 = 3 + \sqrt{5}$. Eigenvector of the operator which corresponds to the greatest eigenvalue ν_1^2 is of the form $\Phi = \{\Phi(0), \Phi(1)\}$, where

$$\Phi_1(0) = \Phi_2(0) = \sqrt{(5+\sqrt{5})/20}, \Phi_1(1) = \Phi_2(1) = \sqrt{(5-\sqrt{5})/20}.$$

In the case of the minimal rank ($m = 1$), the least favorable sequence $\vec{\xi}(j)$ is of the form

$$\vec{\xi}(j) = \Phi(0)\eta(j) + \Phi(1)\eta(j-1) =$$
$$= \sqrt{(5+\sqrt{5})/20} \left(\eta(j), \quad \eta(j) \right) + \sqrt{(5-\sqrt{5})/20} \left(\eta(j-1), \quad \eta(j-1) \right).$$

The optimal linear minimax estimate is of the form

$$\hat{A}_1\vec{\xi} = \vec{a}(0)\Phi(1)\eta(-1) = \sqrt{(5-\sqrt{5})/5}\eta(-1).$$

In the case of the maximal rank ($m = 2$), the least favorable sequence $\vec{\xi}(j)$ is of the form

$$\vec{\xi}(j) = \Psi(0)\vec{\eta}(j) + \Psi(1)\vec{\eta}(j-1) = \sqrt{(5+\sqrt{5})/40} \cdot I \cdot \vec{\eta}(j) + \sqrt{(5-\sqrt{5})/40} \cdot I \cdot \vec{\eta}(j-1),$$

where I is a square matrix elements of which are units.
The optimal linear minimax estimate is of the form

$$\hat{A}_1\vec{\xi} = \vec{a}(0)\Psi(1)\vec{\eta}(-1) = \sqrt{(5-\sqrt{5})/10} (\eta_1(-1) + \eta_2(-1)).$$

The mean-square errors in both cases are not greater than $3 + \sqrt{5}$.

3. MAXIMUM VALUE OF THE ERROR OF ESTIMATION OF THE FUNCTIONAL $A\vec{\xi}$

Theorem 2. *Let the sequence of vectors $\vec{a}(j), j = 0, 1, \ldots$ satisfies conditions*

$$\sum_{k=1}^{T} \sum_{j=0}^{\infty} |a_k(j)| < \infty, \quad \sum_{j=0}^{\infty} (j+1) \|\vec{a}(j)\|^2 < \infty, \tag{17}$$

The function $\Delta(\xi, \hat{A})$ has a saddle point on the set $\Xi \times \Lambda$ and the following equality holds true

$$\min_{\hat{A}\in\Lambda}\max_{\xi\in\Xi}\Delta(\xi,\hat{A}) = \max_{\xi\in\Xi}\min_{\hat{A}\in\Lambda}\Delta(\xi,\hat{A}) = P\nu^2.$$

The least favorable multidimensional stochastic sequence in the class Ξ for the optimal estimate of the functional $A\vec{\xi}$ is a moving average sequence of the form

$$\vec{\xi}(j) = \sum_{u=-\infty}^{j}\Phi(j-u)\vec{\eta}(u).$$

Here ν^2 is the greatest eigenvalue and $\Phi = \{\Phi(u)\}_{u=0}^{\infty}$ is the corresponding eigenvector of the compact operator Q in the space ℓ_2 determined by matrix constructed with the help of the block-matrices

$$Q = \{Q(p,q)\}_{p,q=0}^{\infty}, \quad Q(p,q) = \sum_{u=0}^{\infty}\vec{a}^*(p+u)\vec{a}(q+u),$$

$\vec{\eta}(u) = \{\eta_k(u)\}_{k=1}^{m}$ is a multidimensional stationary stochastic sequence with orthogonal values; $\Phi = \{\Phi(u)\}_{u=0}^{\infty}$ are matrices, elements of which are determined by the eigenvector of the operator Q that corresponds to ν^2, and the condition $\|\vec{\xi}(j)\|^2 = P$.

Proof. Lower bound. Let $\xi \in \Xi_R$. Then we have the inequality

$$\max_{\xi\in\Xi}\min_{\hat{A}\in\Lambda}\Delta(\xi,\hat{A}) \geq \max_{\xi\in\Xi_R}\min_{\hat{A}\in\Lambda}\Delta(\xi,\hat{A}). \tag{18}$$

Making use the canonical representation of the regular stationary sequence (3) and the form (5) of the optimal estimate, we get

$$\min_{\hat{A}\in\Lambda}\Delta(\xi,\hat{A}) = \min_{\hat{A}\in\Lambda} E\left|A\vec{\xi} - \hat{A}\vec{\xi}\right|^2 = E\left|\sum_{j=0}^{\infty}\vec{a}(j)\sum_{u=0}^{j}\Phi(j-u)\vec{\eta}(u)\right|^2 =$$

$$= \sum_{p,q=0}^{\infty}\sum_{k,l=1}^{T}\sum_{n=1}^{m}\Phi_{kn}(p)\overline{\Phi_{ln}(q)}Q_{kl}(p,q), \tag{19}$$

where

$$Q(p,q) = \{Q_{kl}(p,q)\}_{k,l=1}^{T}, \quad Q_{kl}(p,q) = \sum_{u=0}^{\infty}a_k(p+u)\overline{a_l}(q+u). \tag{20}$$

Denote by Q the operator in the Hilbert space ℓ_2 determined by the matrix constructed with the help of block-matrices $Q = \{Q(p,q)\}_{p,q=0}^{\infty}$. Since there are satisfied conditions (17) and

$$\sum_{p,q=0}^{\infty}\|Q(p,q)\|^2 = \sum_{p,q=0}^{\infty}\sum_{k,l=1}^{T}|Q_{kl}(p,q)|^2 =$$

$$= \sum_{p,q=0}^{\infty} \sum_{k,l=1}^{T} \left| \sum_{u=0}^{\infty} a_k(p+u)\overline{a_l}(q+u) \right|^2 \leq$$

$$\leq \sum_{p,q=0}^{\infty} \sum_{k,l=1}^{T} \left(\sum_{u=0}^{\infty} |a_k(p+u)|^2 \cdot \sum_{u=0}^{\infty} |a_l(q+u)|^2 \right) =$$

$$= \left(\sum_{p=0}^{\infty} \sum_{k=1}^{T} \sum_{u=0}^{\infty} |a_k(p+u)|^2 \right)^2 =$$

$$= \left(\sum_{p=0}^{\infty} \sum_{u=0}^{\infty} \|\vec{a}(p+u)\|^2 \right)^2 = \left(\sum_{p=0}^{\infty} (p+1) \|\vec{a}(p)\|^2 \right)^2,$$

we have

$$\|Q\| \leq N(Q) \leq \sum_{p=0}^{\infty} (p+1) \|\vec{a}(p)\|^2 < \infty,$$

where $N(Q)$ is the Hilbert-Schmidt norm of the operator Q. The operator Q is a self-adjoint Hilbert-Schmidt operator 1. It can be represented in the form $Q = A \cdot A^*$, where the operator A is determined by matrix constructed with the help of block-columns $A = \{A(p,q)\}_{p,q=0}^{\infty} = \{\vec{a}(p+q)\}_{p,q=0}^{\infty}$. For these reasons the operator Q has real-valued positive eigenvalues. The operator A is an Hilbert-Schmidt operator and his Hilbert-Schmidt norm is equal to

$$N(A) = \left(\sum_{p=0}^{\infty} (p+1) \|\vec{a}(p)\|^2 \right)^{1/2}.$$

Since $\tilde{\Phi}(p) = \left\{ P^{-1/2} \Phi_{ij}(p) \right\}_{i=1}^{T} {}_{j=1}^{m}$, then (19) can be represented in the form

$$\min_{\hat{A} \in \Lambda} \Delta(\xi, \hat{A}) = P \left\langle Q\tilde{\Phi}, \tilde{\Phi} \right\rangle.$$

Taking into account restrictions (4), we will have

$$\max_{\xi \in \Xi_R} \min_{\hat{A} \in \Lambda} \Delta(\xi, \hat{A}) = P \max_{\|\tilde{\Phi}\|=1} \left\langle Q\tilde{\Phi}, \tilde{\Phi} \right\rangle = P\nu^2,$$

where ν^2 is the greatest eigenvalue of the operator Q, $\langle \cdot, \cdot \rangle$ is the scalar product in the space ℓ_2. It follows from (18) that we can estimate the maximin value of the error

$$\max_{\xi \in \Xi} \min_{\hat{A} \in \Lambda} \Delta(\xi, \hat{A}) \geq P\nu^2. \tag{21}$$

Upper bound. Consider the sequence of operators Q_N determined by matrices (8) and the operator Q determined by matrices (20). Since conditions (17), we have

$$N(Q - Q_N) = \sum_{p=N+1}^{\infty} (p+1) \left\|\vec{a}(p)\right\|^2 \to 0,$$

for $N \to \infty$. Taking into account that

$$\|Q - Q_N\| \leq N(Q - Q_N),$$

we will get

$$\lim_{N\to\infty} \|Q - Q_N\| = 0.$$

So the sequence of operators Q_N converges to the operator Q. For this reason [1, 3] $\lim\limits_{N\to\infty} \nu_N^2 = \nu^2$, where ν_N^2 is the greatest eigenvalue of the operator Q_N, and ν^2 is the greatest eigenvalue of the operator Q. From theorem 1 it follows that

$$\min_{\hat{A}\in\Lambda} \max_{\xi\in\Xi} \Delta(\xi, \hat{A}) = \lim_{N\to\infty} \min_{\hat{A}_N\in\Lambda} \max_{\xi\in\Xi} \Delta(\xi, \hat{A}_N) = \lim_{N\to\infty} P\nu_N^2 = P\nu^2. \quad (22)$$

From relations (22) and (21) we have

$$\min_{\hat{A}\in\Lambda} \max_{\xi\in\Xi} \Delta(\xi, \hat{A}) = P\nu^2 \leq \max_{\xi\in\Xi} \min_{\hat{A}\in\Lambda} \Delta(\xi, \hat{A}),$$

where only equality is possible. Theorem is proved.

Corollary 2. *The optimal minimax estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ is of the form*

$$\hat{A}\vec{\xi} = \sum_{j=0}^{\infty} \vec{a}(j) \left[\sum_{u=-\infty}^{-1} \Phi(j-u)\vec{\eta}(u) \right],$$

where $\vec{\eta}(u) = \{\eta_k(u)\}_{k=1}^m$ is a standard vector-valued stationary stochastic sequence with orthogonal values, $\Phi(u) = \{\Phi_{ij}(u)\}_{i=1}^{T}{}_{j=1}^{m}$, $u = 0, 1, \ldots$ is uniquely determined by eigenvector of the operator Q which corresponds to the greatest eigenvalue ν^2, and condition $\|\vec{\xi}(j)\|^2 = P$. In particular case of stationary sequence of the minimal rank ($m = 1$) the vector which consists of the block-vectors $\Phi = \{\Phi(u)\}_{u=0}^{\infty}$, is eigenvector of the operator Q, which corresponds to the greatest eigenvalue ν^2.

Example 3. Let $= 1$ and let $A\vec{\xi} = \sum_{k=1}^{T} \sum_{j=0}^{\infty} e^{-\lambda j} \xi_k(j)$, $\lambda > 0$. Elements of the block-matrices of the operator Q are of the form

$$Q_{kl}(p, q) = \sum_{u=0}^{\infty} a_k(p+u)\overline{a_l(q+u)} = e^{-\lambda(p+q)}(1 - e^{-2\lambda})^{-1}. \quad (23)$$

The eigenvalues of the operator Q are determined by the system of equations

$$\mu\Phi_k(p) = \sum_{l=1}^{T}\sum_{s=0}^{\infty} e^{-\lambda(p+s)}(1-e^{-2\lambda})^{-1}\Phi_l(s),\ k=\overline{1,T},\ p=0,\ 1,\ \dots . \tag{24}$$

From (24) we will get that $\Phi_k(p)$ is of the form $\Phi_k(p) = Ce^{-\lambda p},\ k=\overline{1,T}$. The constant C is determined by the normalizing condition. In the case of the minimal rank $(m=1)$ we have

$$C = (1-e^{-2\lambda})^{1/2}T^{-1/2}, \quad \Phi_k(p) = T^{-1/2}(1-e^{-2\lambda})^{1/2}e^{-\lambda p},\ k=\overline{1,T}.$$

Substitution of theses expressions into (24) gives us $\mu = T(1-e^{-2\lambda})^{-2}$. In the case of the minimal rank $(m=1)$, The least favorable in the class Ξ vector-valued stationary sequence $\vec{\xi}(j)$ is a moving average sequence of the form

$$\vec{\xi}(j) = T^{-1/2}(1-e^{-2\lambda})^{1/2}e^{-\lambda j}\sum_{u=-\infty}^{j} e^{\lambda u}\eta(u)I,$$

where I is a square matrix elements of which are units, $\vec{\eta}(u) = \{\eta_k(u)\}_{k=1}^{T}$ is a standard vector sequence with orthogonal values.
The optimal linear minimax estimate $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ is of the form

$$\hat{A}\vec{\xi} = T^{1/2}(1-e^{-2\lambda})^{1/2}\sum_{j=0}^{\infty} e^{-2\lambda j}\left[\sum_{u=-\infty}^{-1} e^{\lambda u}\eta(u)\right].$$

In the case of maximal rank $m=T$ we will have

$$\vec{\xi}(j) = T^{-1}(1-e^{-2\lambda})^{1/2}e^{-\lambda j}\sum_{u=-\infty}^{j} e^{\lambda u}\, I\,\vec{\eta}(u),$$

$$\hat{A}\vec{\xi} = (1-e^{-2\lambda})^{1/2}\sum_{j=0}^{\infty} e^{-2\lambda j}\left[\sum_{k=1}^{T}\sum_{u=-\infty}^{-1} e^{\lambda u}\eta_k(u)\right],$$

where I is a square matrix elements of which are units, $\vec{\eta}(u) = \{\eta_k(u)\}_{k=1}^{T}$ is a standard vector sequence with orthogonal values.
The mean-square errors in both cases are not greater than $T(1-e^{-2\lambda})^{-2}$.

4. Conclusions

We propose formulas for calculation the mean square errors and the spectral characteristic of the optimal linear estimate of the unknown value of the functionals $A\vec{\xi} = \sum_{j=0}^{\infty}\vec{a}(j)\vec{\xi}(j)$ and $A_N\vec{\xi} = \sum_{j=0}^{N}\vec{a}(j)\vec{\xi}(j)$ which depend on the unknown values of a multidimensional stationary stochastic sequence $\vec{\xi}(j)$ based on observations of the sequence $\vec{\xi}(j)$ from the class Ξ of sequences which satisfy conditions $E\vec{\xi}(j) = 0,\ \|\vec{\xi}(j)\|^2 \le P$, for $j<0$.

Formulas are proposed that determine the maximum values of the mean-square errors of the optimal estimates of the functionals $A\vec{\xi}$ and $A_N\vec{\xi}$ in the class Ξ. It is shown that these maximum values of the errors in the class Ξ give the moving average sequences determined by eigenvectors of compact operators constructed with the help of the sequence $\vec{a}(j)$.

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