Theory of Stochastic Processes
Vol.14 (30), no.3-4, 2008, pp.17-26

OLEKSANDR D. BORYSENKO AND OLGA V. BORYSENKO

LIMIT BEHAVIOR OF NON-AUTONOMOUS RANDOM OSCILLATING SYSTEM OF THIRD ORDER UNDER RANDOM PERIODIC EXTERNAL DISTURBANCES IN RESONANCE CASE

The asymptotic behavior of the general type third order non-autonomous oscillating system under the action of small non-linear random periodic perturbations of "white"and "Poisson" types in resonance case is investigated.

1. Introduction

The asymptotic behavior of the general type third order non-autonomous oscillating system under the action of small non-linear random periodic perturbations of "white" and "Poisson" types in the non-resonance case is investigated in O.D.Borysenko, O.V.Borysenko [1]. The overview of papers devoted to the averaging method, proposed by N.M.Krylov, N.N.Bogolyubov [2], and its applications to random oscillatory systems of different types is presented in O.D.Borysenko, O.V.Borysenko [3] with corresponding references.

In this paper we will investigate the behaviour, as $\varepsilon \to 0$, of the general type third order non-autonomous oscillating system driven by stochastic differential equation

$$\begin{aligned} &x'''(t) + ax''(t) + b^2x'(t) + ab^2x(t) = \\ &= \varepsilon^{k_0} f_0(\mu_0 t, x(t), x'(t), x''(t)) + f_\varepsilon(t, x(t), x'(t), x''(t)) \end{aligned} \tag{1}$$

2000 *Mathematics Subject Classifications.* 60H10
Key words and phrases. Asymptotic behavior, third order autonomous oscillating system, small non-linear random perturbations, resonance case

with non-random initial conditions $x(0) = x_0, x'(0) = x'_0, x''(0) = x''_0$, where $\varepsilon > 0$ is a small parameter, $f_\varepsilon(t, x, x', x'')$ is a random function such that

$$\int_0^t f_\varepsilon(s, x(s), x'(s), x''(s))\, ds =$$
$$= \sum_{j=1}^m \varepsilon^{k_j} \int_0^t f_j(\mu_j s, x(s), x'(s), x''(s))\, dw_j(s) +$$
$$+ \varepsilon^{k_{m+1}} \int_0^t \int_{\mathrm{R}} f_{m+1}(\mu_{m+1} s, x(s), x'(s), x''(s), z)\, \tilde{\nu}(ds, dz),$$

$k_j > 0, j = \overline{0, m+1}$; $a > 0, b > 0$; $f_j, j = \overline{0, m+1}$ are non-random functions, periodic on $\mu_j t, j = \overline{0, m+1}$ with period 2π; $\{w_j(t), j = \overline{1, m}\}$ are independent one-dimensional Wiener processes; $\tilde{\nu}(dt, dy) = \nu(dt, dy) - \Pi(dy)dt$, $E\nu(dt, dy) = \Pi(dy)dt$, $\nu(dt, dy)$ is the Poisson measure independent on $w_j(t), j = \overline{1, m}$; $\Pi(A)$ is a finite measure on Borel sets in R.

We will consider the equation (1) as a system of stochastic differential equations

$$\begin{aligned}
dx(t) &= x'(t)dt \\
dx'(t) &= x''(t)dt \\
dx''(t) &= [-ax''(t) - b^2 x'(t) - ab^2 x(t) + \\
&\quad + \varepsilon^{k_0} f_0(\mu_0 t, x(t), x'(t), x''(t))]dt + \\
&\quad + \sum_{j=1}^m \varepsilon^{k_j} f_j(\mu_j t, x(t), x'(t), x''(t)) dw_j(t) + \\
&\quad + \varepsilon^{k_{m+1}} \int_{\mathrm{R}} f_{m+1}(\mu_{m+1} t, x(t), x'(t), x''(t), z) \tilde{\nu}(dt, dz), \\
&\quad x(0) = x_0,\ x'(0) = x'_0,\ x''(0) = x''_0.
\end{aligned} \tag{2}$$

In what follows we will use the constant $K > 0$ for the notation of different constants, which are not depend on ε.

From Borysenko O. and Malyshev I. [4], using the obvious modifications we obtain following results

Lemma. *Let for each $x \in \mathrm{R}^d$ there exists*

$$\lim_{T\to\infty} \frac{1}{T} \int_A^{T+A} f(t, x)\, dt = \bar{f}(x)$$

uniformly with respect to A, the function $\bar{f}(x)$ is bounded, continuous, function $f(t, x)$ is bounded and continuous in x uniformly with respect to (t, x) in any region $t \in [0, \infty), |x| \le K$, and stochastic processes $\xi(t) \in \mathrm{R}^d$, $\eta(t) \in \mathrm{R}$ are continuous, then

$$\lim_{\varepsilon\to 0} \int_0^t f\left(\frac{s}{\varepsilon} + \eta(s), \xi(s)\right) ds = \int_0^t \bar{f}(\xi(s))\, ds$$

almost surely for all arbitrary $t \in [0, T]$.

Remark. Let $f(t, x, z)$ is bounded and uniformly continuous in x with respect to $t \in [0, \infty)$ and $z \in \mathrm{R}$ in every compact set $|x| \le K, x \in \mathrm{R}^d$. Let $\Pi(\cdot)$ be a finite measure on the σ-algebra of Borel sets in R and let

$$\lim_{T\to\infty} \frac{1}{T} \int_A^{T+A} f(t, x, z)\, dt = \bar{f}(x, z),$$

uniformly with respect to A for each $x \in \mathrm{R}^d, z \in \mathrm{R}$, where $\bar{f}(x, z)$ is bounded, uniformly continuous in x with respect to $z \in \mathrm{R}$ in every compact set $|x| \le K$. Then for any continuous processes $\xi(t) \in \mathrm{R}^d$ and $\eta(t) \in \mathrm{R}$ we have

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\mathrm{R}} f\left(\frac{s}{\varepsilon} + \eta(s), \xi(s), z\right) \Pi(dz)ds = \int_0^t \int_{\mathrm{R}} \bar{f}(\xi(s), z)\, \Pi(dz)ds.$$

2. Main result

We will study the resonance case: $\mu_j = \dfrac{p_j}{q_j} \cdot b$ for some $j = \overline{0, m+1}$, where p_j and q_j are relatively prime integers. Let us consider the following representation of processes $x(t), x'(t), x''(t)$:

$$x(t) = C(t) \exp\{-at\} + A_1(t) \cos(bt) + A_2(t) \sin(bt),$$

$$x'(t) = -aC(t) \exp\{-at\} - bA_1(t) \sin(bt) + bA_2(t) \cos(bt),$$
$$x''(t) = a^2 C(t) \exp\{-at\} - b^2 A_1(t) \cos(bt) - b^2 A_2(t) \sin(bt),$$
$$N(t) = C(t) \exp\{-at\}.$$

Then

$$N(t) = \frac{b^2 x(t) + x''(t)}{a^2 + b^2},$$
$$A_1(t) = \cos\alpha \cos(bt + \alpha) x(t) - \frac{\sin bt}{b} x'(t) - \frac{\sin\alpha \sin(bt + \alpha)}{b^2} x''(t),$$
$$A_2(t) = \cos\alpha \sin(bt + \alpha) x(t) + \frac{\cos bt}{b} x'(t) + \frac{\sin\alpha \cos(bt + \alpha)}{b^2} x''(t),$$

where $\alpha = \mathrm{arctg}\,(b/a)$. We can apply Ito formula [5] to stochastic process $\xi(t) = (N(t), A_1(t), A_2(t))$ and obtain for the process $\xi(t)$ the system of stochastic differential equations

$$dN(t) = -aN(t)\, dt + \frac{1}{a^2 + b^2} dM(t),$$
$$dA_1(t) = -\frac{\sin\alpha \sin(bt + \alpha)}{b^2} dM(t), \quad dA_2(t) = \frac{\sin\alpha \cos(bt + \alpha)}{b^2} dM(t),$$

$$dM(t)=\varepsilon^{k_0}\tilde f_0(\mu_0 t,N(t),A_1(t),A_2(t),t)dt+ \tag{3}$$

$$+\sum_{j=1}^{m}\varepsilon^{k_j}\tilde f_j(\mu_j t,N(t),A_1(t),A_2(t),t)dw_j(t)+$$

$$+\varepsilon^{k_{m+1}}\int_{\mathrm{R}}\tilde f_{m+1}(\mu_{m+1}t,N(t),A_1(t),A_2(t),t,z)\tilde\nu(dt,dz)],$$

$$N(0)=\frac{b^2x_0+x_0''}{a^2+b^2},A_1(0)=\frac{a^2x_0-x_0''}{a^2+b^2},A_2(0)=\frac{ax_0''+(a^2+b^2)x_0'+ab^2x_0}{b(a^2+b^2)},$$

where $\tilde f_j(\mu_j t,N,A_1,A_2,t)=$ $f_j(\mu_j t,N+A_1\cos bt+A_2\sin bt,-aN-bA_1\sin bt+bA_2\cos bt,a^2N-b^2A_1\cos bt-b^2A_2\sin bt)$, $j=\overline{0,m}$, $\tilde f_{m+1}(\mu_{m+1}t,N,A_1,A_2,t,z)=$ $f_{m+1}(\mu_{m+1}t,N+A_1\cos bt+A_2\sin bt,-aN-bA_1\sin bt+bA_2\cos bt,a^2N-b^2A_1\cos bt-b^2A_2\sin bt,z)$.

Theorem. *Let $\Pi(\mathrm{R})<\infty$, $t\in[0,t_0]$, $k=\min(k_0,2k_j,j=\overline{1,m+1})$. Let us suppose, that functions $f_j,j=\overline{0,m+1}$ bounded and satisfy Lipschitz condition on x,x',x''. If given below matrix $\bar\sigma^2(A_1,A_2)$ is positive definite, then:*

1. Let $\mu_j=\frac{p_j}{q_j}\cdot b$, for $j=\overline{0,m+1}$, where p_j and q_j some relatively prime integers. If $k_0=2k_j$, $j=\overline{1,m+1}$, then the stochastic process $\xi_\varepsilon(t)=\xi(t/\varepsilon^k)$ weakly converges, as $\varepsilon\to 0$, to the stochastic process $\bar\xi(t)=(0,\bar A_1(t),\bar A_2(t))$, where $\bar A(t)=(\bar A_1(t),\bar A_2(t))$ is the solution of the system of stochastic differential equations

$$d\bar A(t)=\bar\alpha(\bar A(t))dt+\bar\sigma(\bar A(t))d\bar w(t), \tag{4}$$

$$\bar A(0)=(A_1(0),A_2(0)),$$

where $\bar\alpha(\bar A)=(\bar\alpha^{(1)}(A_1,A_2),\bar\alpha^{(2)}(A_1,A_2))$,

$$\bar\alpha^{(1)}(A_1,A_2)=-\frac{1}{4\pi^2b(a^2+b^2)}\times$$
$$\sum_{p_0n+q_0l=0}\int_0^{2\pi}\int_0^{2\pi}\hat f_0(\psi,A_1,A_2,t)(a\sin\psi+b\cos\psi)e^{-i(n\psi+lt)}\,dt\,d\psi,$$

$$\bar\alpha^{(2)}(A_1,A_2)=\frac{1}{4\pi^2b(a^2+b^2)}\times$$
$$\sum_{p_0n+q_0l=0}\int_0^{2\pi}\int_0^{2\pi}\hat f_0(\psi,A_1,A_2,t)(a\cos\psi-b\sin\psi)e^{-i(n\psi+lt)}\,dt\,d\psi,$$

$$\bar\sigma(A_1,A_2)=\left\{\bar B(A_1,A_2)\right\}^{\frac12}=\left\{\frac{1}{4\pi^2b^2(a^2+b^2)^2}\times\right.$$

$$\left[\sum_{j=1}^{m}\sum_{p_jn+q_jl=0}\int_0^{2\pi}\int_0^{2\pi}\hat f_j^2(\psi,A_1,A_2,t)B(\psi)e^{-i(n\psi+lt)}\,dt\,d\psi+\right.$$

$$\left.\left.\sum_{p_{m+1}n+q_{m+1}l=0}\int_0^{2\pi}\int_0^{2\pi}\int_R\hat f_{m+1}^2(\psi,A_1,A_2,t,z)B(\psi)e^{-i(n\psi+lt)}\,\Pi(dz)\,dt\,d\psi\right]\right\}^{\frac12},$$

$$B(\psi)=(B_{ij}(\psi),i,j=1,2),\quad B_{11}(\psi)=(a\sin\psi+b\cos\psi)^2,$$

$$B_{12}(\psi)=B_{21}(\psi)=-(a\sin\psi+b\cos\psi)(a\cos\psi-b\sin\psi),$$

$$B_{22}(\psi)=(a\cos\psi-b\sin\psi)^2,$$

$$\hat f_j(\psi,A_1,A_2,t)=\tilde f_j(\psi,0,A_1,A_2,t),\ \ j=\overline{0,m}$$

$$\hat f_{m+1}(\psi,A_1,A_2,t,z)=\tilde f_{m+1}(\psi,0,A_1,A_2,t,z),$$

$\bar w(t)=(\bar w_j(t),j=1,2)$, $\bar w_j(t),j=1,2$ – *independent one-dimensional Wiener processes.*

2. If $k<k_0$ then in the averaging equation (4) we must put $\hat f_0\equiv 0$; if $k<2k_j$ for some $1\le j\le m+1$, then in the averaging equation (4) we must put $\hat f_j\equiv 0$ for all such j.

3. If $\mu_j\ne\frac{p_j}{q_j}\cdot b$ for some $j=\overline{0,m+1}$ and arbitrary relatively prime integers p_j and q_j, then in averaging coefficients in (4) we must put $l=n=0$ in corresponding sums containing $\hat f_j$.

Proof. Let us make a change of variable $t\to t/\varepsilon^k$ in equation (3) and obtain for the process $\xi_\varepsilon(t)=(N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t))=(N(t/\varepsilon^k),A_1(t/\varepsilon^k),A_2(t/\varepsilon^k))$ the system of stochastic differential equations

$$dN_\varepsilon(t)=\left[-\frac{a}{\varepsilon^k}N_\varepsilon(t)+\frac{\varepsilon^{k_0-k}}{a^2+b^2}\tilde f_0(\mu_0t/\varepsilon^k,N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t),t/\varepsilon^k)\right]dt+$$

$$+\sum_{j=1}^{m}\frac{\varepsilon^{k_j-k/2}}{a^2+b^2}\tilde f_j(\mu_jt/\varepsilon^k,N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t),t/\varepsilon^k)dw_j^\varepsilon(t)+$$

$$+\frac{\varepsilon^{k_{m+1}}}{a^2+b^2}\int_{\mathrm R}\tilde f_{m+1}(\mu_{m+1}t/\varepsilon^k,N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t),t/\varepsilon^k,z)\tilde\nu_\varepsilon(dt,dz),$$

$$dA_1^\varepsilon(t)=-\frac{\sin\alpha\sin(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_0-k}\tilde f_0(\mu_0t/\varepsilon^k,N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t))dt+\quad(5)$$

$$+\sum_{j=1}^{m}\varepsilon^{k_j-k/2}\tilde f_j(\mu_jt/\varepsilon^k,N_\varepsilon(t),A_1^\varepsilon(t),A_2^\varepsilon(t))dw_j^\varepsilon(t)+$$

$$+\varepsilon^{k_{m+1}}\int_{\mathrm{R}}\tilde{f}_{m+1}(\mu_{m+1}t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt,dz)],$$

$$dA_2^\varepsilon(t)=\frac{\sin\alpha\cos(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_0-k}\tilde{f}_0(\mu_0 t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k)dt+$$

$$+\sum_{j=1}^m \varepsilon^{k_j-k/2}\tilde{f}_j(\mu_j t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k)dw_j^\varepsilon(t)+$$

$$+\varepsilon^{k_{m+1}}\int_{\mathrm{R}}\tilde{f}_{m+1}(\mu_{m+1}t/\varepsilon^k, N_\varepsilon(t), A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k, z)\tilde{\nu}_\varepsilon(dt,dz)],$$

where $w_j^\varepsilon(t)=\varepsilon^{k/2}w_j(t/\varepsilon^k)$, $\tilde{\nu}_\varepsilon(t,A)=\nu(t/\varepsilon^k,A)-\Pi(A)t/\varepsilon^k$, here A is Borel set in R. For any $\varepsilon>0$ the processes $w_j^\varepsilon(t),\ j=\overline{1,m}$ are the independent Wiener processes and $\tilde{\nu}_\varepsilon(t,A)$ is the centered Poisson measure independent on $w_j^\varepsilon(t),\ j=\overline{1,m}$.

Since we have relationship $N_\varepsilon(t)=\exp\{-at/\varepsilon^k\}C(t/\varepsilon^k)$ and process $C_\varepsilon(t)=C(t/\varepsilon^k)$ satisfies the stochastic equation

$$C_\varepsilon(t)=C(0)+\varepsilon^{k_0-k}\int_0^t\frac{e^{as/\varepsilon^k}}{a^2+b^2}\tilde{f}_0(\mu_0 s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s), s/\varepsilon^k)\,ds+$$

$$+\sum_{j=1}^m\varepsilon^{k_j-k/2}\int_0^t\frac{e^{as/\varepsilon^k}}{a^2+b^2}\tilde{f}_j(\mu_j s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s), s/\varepsilon^k)\,dw_j^\varepsilon(s)+$$

$$+\varepsilon^{k_{m+1}}\int_0^t\int_{\mathrm{R}}\frac{e^{as/\varepsilon^k}}{a^2+b^2}\tilde{f}_{m+1}(\mu_{m+1}s/\varepsilon^k, N_\varepsilon(s), A_1^\varepsilon(s), A_2^\varepsilon(s), s/\varepsilon^k, z)\,\tilde{\nu}_\varepsilon(dt,dz),$$

where $C(0)=\frac{b^2x_0+x_0''}{a^2+b^2}$, we can obtain estimate

$$\mathrm{E}|N_\varepsilon(t)|^2\le K[e^{-2at/\varepsilon^k}+\varepsilon^k(1-e^{-2at/\varepsilon^k})(t\varepsilon^{2(k_0-k)}+\sum_{j=1}^{m+1}\varepsilon^{2k_j-k})].$$

Therefore $\lim_{\varepsilon\to0}\mathrm{E}|N_\varepsilon(t)|^2=0$ and it is sufficient to study the behaviour, as $\varepsilon\to0$, of solution to the system of stochastic differential equations

$$dA_1^\varepsilon(t)=-\frac{\sin\alpha\sin(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_0-k}\hat{f}_0(\mu_0 t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dt+$$

$$+\sum_{j=1}^m\varepsilon^{k_j-k/2}\hat{f}_j(\mu_j t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t))dw_j^\varepsilon(t)+$$

$$+\varepsilon^{k_{m+1}}\int_{\mathrm{R}}\hat{f}_{m+1}(\mu_{m+1}t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), z)\tilde{\nu}_\varepsilon(dt,dz)],$$

$$dA_2^\varepsilon(t)=\frac{\sin\alpha\cos(bt/\varepsilon^k+\alpha)}{b^2}[\varepsilon^{k_0-k}\hat{f}_0(\mu_0 t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k)dt+\qquad(6)$$

$$+\sum_{j=1}^{m}\varepsilon^{k_j-k/2}\hat{f}_j(\mu_j t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k)dw_j^\varepsilon(t)+$$

$$+\varepsilon^{k_{m+1}}\int_{\mathrm{R}}\hat{f}_{m+1}(\mu_{m+1}t/\varepsilon^k, A_1^\varepsilon(t), A_2^\varepsilon(t), t/\varepsilon^k, z)\tilde{\nu}_\varepsilon(dt,dz)],$$

with initial conditions $A_1^\varepsilon(0)=A_1(0)$, $A_2^\varepsilon(0)=A_2(0)$.

Let us denote $A_\varepsilon(t)=(A_1^\varepsilon(t),A_2^\varepsilon(t))$. Using conditions on coefficients of equation (6) and properties of stochastic integrals we obtain estimates

$$\mathrm{E}||A_\varepsilon(t)||^2\le K\left(1+t^2\varepsilon^{2(k_0-k)}+t\sum_{j=1}^{m+1}\varepsilon^{2k_j-k}\right),$$

$$\mathrm{E}||A_\varepsilon(t)-A_\varepsilon(s)||^2\le K\left(|t-s|^2\varepsilon^{2(k_0-k)}+|t-s|\sum_{j=1}^{m+1}\varepsilon^{2k_j-k}\right).$$

Similarly for the process $\zeta_\varepsilon(t)=(\zeta_1^\varepsilon(t),\zeta_2^\varepsilon(t))$, where

$$\zeta_1^\varepsilon(t)=-\sum_{j=1}^{m}\varepsilon^{k_j-k/2}\int_0^t\frac{\sin\alpha\sin(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat{f}_j(\frac{\mu_j s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), \frac{s}{\varepsilon^k})dw_j^\varepsilon(s)-$$

$$-\varepsilon^{k_{m+1}}\int_0^t\int_{\mathrm{R}}\frac{\sin\alpha\sin(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat{f}_{m+1}(\frac{\mu_{m+1}s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), \frac{s}{\varepsilon^k}, z)\tilde{\nu}_\varepsilon(ds,dz)],$$

$$\zeta_2^\varepsilon(t)=\sum_{j=1}^{m}\varepsilon^{k_j-k/2}\int_0^t\frac{\sin\alpha\cos(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat{f}_j(\frac{\mu_j s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), \frac{s}{\varepsilon^k})dw_j^\varepsilon(s)+$$

$$+\varepsilon^{k_{m+1}}\int_0^t\int_{\mathrm{R}}\frac{\sin\alpha\cos(\frac{bs}{\varepsilon^k}+\alpha)}{b^2}\hat{f}_{m+1}(\frac{\mu_{m+1}s}{\varepsilon^k}, A_1^\varepsilon(s), A_2^\varepsilon(s), \frac{s}{\varepsilon^k}, z)\tilde{\nu}_\varepsilon(ds,dz)]$$

we derive estimates

$$\mathrm{E}||\zeta_\varepsilon(t)||^2\le Kt\sum_{j=1}^{m+1}\varepsilon^{2k_j-k},\ \mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^2\le K|t-s|\sum_{j=1}^{m+1}\varepsilon^{2k_j-k}.$$

Therefore for stochastic process $\eta_\varepsilon(t)=(A_\varepsilon(t),\zeta_\varepsilon(t))$ conditions of weak compactness [6] are fulfilled

$$\lim_{h\downarrow 0}\overline{\lim_{\varepsilon\to 0}}\sup_{|t-s|<h}\mathrm{P}\{|\eta_\varepsilon(t)-\eta_\varepsilon(s)|>\delta\}=0 \text{ for any } \delta>0,\ t,s\in[0,T],$$

$$\lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}}\sup_{t\in[0,T]}\mathrm{P}\{|\eta_\varepsilon(t)|>N\}=0,$$

and for any sequence $\varepsilon_n\to 0, n=1,2,\dots$ there exists a subsequence $\varepsilon_m=\varepsilon_{n(m)}\to 0, m=1,2,\dots$, probability space, stochastic processes

$\bar{A}_{\varepsilon_m}(t) = (\bar{A}_1^{\varepsilon_m}(t), \bar{A}_2^{\varepsilon_m}(t))$, $\bar{\zeta}_{\varepsilon_m}(t)$, $\bar{A}(t) = (\bar{A}_1(t), \bar{A}_2(t))$, $\bar{\zeta}(t)$ defined on this space, such that $\bar{A}_{\varepsilon_m}(t) \to \bar{A}(t), \bar{\zeta}_{\varepsilon_m}(t) \to \bar{\zeta}(t)$ in probability, as $\varepsilon_m \to 0$, and finite-dimensional distributions of $\bar{A}_{\varepsilon_m}(t), \bar{\zeta}_{\varepsilon_m}(t)$ are coincide with finite-dimensional distributions of $A_{\varepsilon_m}(t), \zeta_{\varepsilon_m}(t)$. Since we interesting in limit behaviour of distributions, we can consider processes $A_{\varepsilon_m}(t)$, and $\zeta_{\varepsilon_m}(t)$ instead of $\bar{A}_{\varepsilon_m}(t), \bar{\zeta}_{\varepsilon_m}(t)$. From (6) we obtain equation

$$A_{\varepsilon_m}(t) = A(0) + \int_0^t \alpha_{\varepsilon_m}(s, A_{\varepsilon_m}(s))\, ds + \zeta_{\varepsilon_m}(t), \quad A_0 = (A_1(0), A_2(0)), \tag{7}$$

where $\alpha_\varepsilon(t, A) = (\alpha_\varepsilon^{(1)}(t, A_1, A_2), \alpha_\varepsilon^{(2)}(t, A_1, A_2))$,

$$\alpha_\varepsilon^{(1)}(t, A_1, A_2) = -\varepsilon^{k_0 - k} \frac{\sin\alpha \sin(bt/\varepsilon^k + \alpha)}{b^2} \hat{f}_0(\mu_0 t/\varepsilon^k, A_1, A_2, t/\varepsilon^k),$$

$$\alpha_\varepsilon^{(2)}(t, A_1, A_2) = \varepsilon^{k_0 - k} \frac{\sin\alpha \cos(bt/\varepsilon^k + \alpha)}{b^2} \hat{f}_0(\mu_0 t/\varepsilon^k, A_1, A_2, t/\varepsilon^k).$$

It should be noted that process $\zeta_\varepsilon(t)$ is the vector-valued square integrable martingale with matrix characteristic

$$\langle \zeta_\varepsilon^{(l)}, \zeta_\varepsilon^{(n)} \rangle(t) = \sum_{j=1}^m \int_0^t \sigma_\varepsilon^{(l,j)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s)) \sigma_\varepsilon^{(n,j)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s))\, ds +$$

$$+ \frac{1}{\varepsilon^k} \int_0^t \int_{\mathrm{R}} \gamma_\varepsilon^{(l)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z) \gamma_\varepsilon^{(n)}(s, A_1^\varepsilon(s), A_2^\varepsilon(s), z)\, \Pi(dz) ds, \ l, n = 1, 2,$$

where

$$\sigma_\varepsilon^{(1,j)}(s, A_1, A_2) = -\varepsilon^{k_j - k/2} \frac{\sin\alpha \sin(\frac{bs}{\varepsilon^k} + \alpha)}{b^2} \hat{f}_j(\frac{\mu_j s}{\varepsilon^k}, A_1, A_2, \frac{s}{\varepsilon^k}),$$

$$\sigma_\varepsilon^{(2,j)}(s, A_1, A_2) = \varepsilon^{k_j - k/2} \frac{\sin\alpha \cos(\frac{bs}{\varepsilon^k} + \alpha)}{b^2} \hat{f}_j(\frac{\mu_j s}{\varepsilon^k}, A_1, A_2, \frac{s}{\varepsilon^k}),$$

$$\gamma_\varepsilon^{(1)}(s, A_1, A_2, z) = -\varepsilon^{k_{m+1}} \frac{\sin\alpha \sin(\frac{bs}{\varepsilon^k} + \alpha)}{b^2} \hat{f}_{m+1}(\frac{\mu_{m+1} s}{\varepsilon^k}, A_1, A_2, \frac{s}{\varepsilon^k}, z),$$

$$\gamma_\varepsilon^{(2)}(s, A_1, A_2, z) = \varepsilon^{k_{m+1}} \frac{\sin\alpha \cos(\frac{bs}{\varepsilon^k} + \alpha)}{b^2} \hat{f}_{m+1}(\frac{\mu_{m+1} s}{\varepsilon^k}, A_1, A_2, \frac{s}{\varepsilon^k}, z).$$

For processes $A_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ following estimates hold

$$\mathrm{E}||A_\varepsilon(t) - A_\varepsilon(s)||^4 \le K \left[\varepsilon^{4(k_0 - k)} |t - s|^4 + \mathrm{E}||\zeta_\varepsilon(t) - \zeta_\varepsilon(s)||^4 \right], \tag{8}$$

$$\mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^4 \le K\left[\sum_{j=1}^{m+1}\varepsilon^{4k_j-2k}|t-s|^2+\right.$$

$$\left.+\varepsilon^{4k_{m+1}-3k/2}|t-s|^{3/2}+\varepsilon^{4k_{m+1}-k}|t-s|\right], \tag{9}$$

$$\mathrm{E}||A_\varepsilon(t)-A_\varepsilon(s)||^8 \le K, \quad \mathrm{E}||\zeta_\varepsilon(t)-\zeta_\varepsilon(s)||^8 \le K. \tag{10}$$

Since $A_{\varepsilon_m}(t)\to\bar{A}(t),\zeta_{\varepsilon_m}(t)\to\bar{\zeta}(t)$ in probability, as $\varepsilon_m\to 0$, then, using (10), from (8) and (9) we obtain estimates

$$\mathrm{E}||\bar{A}(t)-\bar{A}(s)||^4\le K(|t-s|^4+|t-s|^2), \quad \mathrm{E}||\bar{\zeta}(t)-\bar{\zeta}(s)||^4\le C|t-s|^2.$$

Therefore processes $\bar{A}(t)$ and $\bar{\zeta}(t)$ satisfy the Kolmogorov's continuity condition [7].

Let us consider the case $k_0=2k_j,\ j=\overline{1,m+1}$. Under these conditions we have for $l,n=1,2$

$$\lim_{\varepsilon\to 0}\frac{1}{t}\int_0^t \alpha_\varepsilon^{(l)}(s,A_1,A_2)ds=\bar{\alpha}^{(l)}(A_1,A_2),$$

$$\lim_{\varepsilon\to 0}\frac{1}{t}\int_0^t\left[\sum_{j=1}^{m}\sigma_\varepsilon^{(l,j)}(s,A_1,A_2)\sigma_\varepsilon^{(n,j)}(s,A_1,A_2)+\right. \tag{11}$$

$$\left.+\frac{1}{\varepsilon^k}\int_R\gamma_\varepsilon^{(l)}(s,A_1,A_2,z)\gamma_\varepsilon^{(n)}(s,A_1,A_2,z)\Pi(dz)\right]ds=\bar{B}_{ln}(A_1,A_2),$$

where functions $\bar{\alpha}^{(l)}(A_1,A_2)$ and $\bar{B}(A_1,A_2)=\{\bar{B}_{ln}(A_1,A_2),\ l,n=1,2\}$ are defined in the condition of theorem. Since processes $\bar{A}(t),\bar{\zeta}(t)$ are continuous, then from Lemma and relationships (7), (11) it follows

$$\bar{A}(t)=A(0)+\int_0^t\bar{\alpha}(\bar{A}_1(s),\bar{A}_2(s))ds+\bar{\zeta}(t), \quad A(0)=(A_1(0),A_2(0)), \tag{12}$$

where $\bar{\zeta}(t)$ is continuous vector-valued martingale with matrix characteristic

$$\langle\bar{\zeta}^{(l)},\bar{\zeta}^{(n)}\rangle(t)=\int_0^t\bar{B}_{ln}(\bar{A}_1(s),\bar{A}_2(s))ds, \quad l,n=1,2.$$

Hence [8] there exists Wiener process $\bar{w}(t)=(\bar{w}_j(t),j=1,2)$, such that

$$\bar{\zeta}(t)=\int_0^t\bar{\sigma}(\bar{A}_1(s),\bar{A}_2(s))\,d\bar{w}(s),\ \bar{\sigma}(A_1,A_2)=\left\{\bar{B}(A_1,A_2)\right\}^{1/2}. \tag{13}$$

Relationships (12), (13) mean, that process $\bar{A}(t)$ satisfies equation (4). Under conditions of theorem the equation (4) has unique solution. Therefore process $\bar{A}(t)$ does not depend on choosing of sub-sequence $\varepsilon_m \to 0$, and finite-dimensional distributions of process $A_{\varepsilon_m}(t)$ converge to finite-dimensional distributions of process $\bar{A}(t)$. Since processes $A_{\varepsilon_m}(t)$ and $\bar{A}(t)$ are Markov processes, then using the conditions for weak convergence of Markov processes [7], we complete the proof of statement 1 of theorem.

Let us consider the case $k < k_0$. Then coefficients $\alpha_\varepsilon^{(i)}(t, A_1, A_2)$, $i = 1, 2$ of equation (7) tend to zero, as $\varepsilon \to 0$. Repeating with obvious modifications the proof of statement 1) of theorem we obtain proof of the first statement of 2).

In the case $k < 2k_j,\ j = \overline{1, m}$ in (11) we have

$$\sigma_\varepsilon^{(l,j)}(t, A_1, A_2)\sigma_\varepsilon^{(n,j)}(t, A_1, A_2) = O(\varepsilon^{2k_j - k}),\ l, n = 1, 2.$$

Then we can complete the proof in this case as above. In the same way we consider the case $k < 2k_{m+1}$. The statement 3) follows from result of [1].□

References

1. Borysenko O.D., Borysenko O.V. *Limit behavior of non-autonomous random oscillating system of third order under random periodic external disturbances*, Journ. Numer. and Applied Math., **1(96)** (2008), 87 – 95.
2. Krylov N.M., Bogolyubov N.N., *Introduction to non-linear mechanics*, Kyiv: Publ.Acad.Scien. UkrSSR, 1937.
3. Borysenko O.D., Borysenko O.V. *Limit behavior of autonomous random oscillating system of third order*, Theory of Stoch. Proc., **13(29)** (2007), no.4, 19 – 28.
4. Borysenko O.D., Malyshev I.G., *The limit behaviour of integral functional of the solution of stochastic differential equation depending on small parameter*, Theory of Stoch. Proc., **7(23)** (2001), no.1-2, 30 – 36.
5. Gikhman, I.I., Skorokhod, A.V., *Stochastic Differential Equations*, Springer-Verlag, Berlin, 1972.
6. Skorokhod, A.V., *Studies in the Theory of Random Processes,* Addison-Wesley, 1965.
7. Gikhman, I.I., Skorokhod, A.V., *The Theory of Stochastic Processes,* v.I, Springer-Verlag, Berlin, 1974.
8. Ikeda, N. and Watanabe, S., *Stochastic Differential Equations and Diffusion Processes,* North Holland, Amsterdam and Kadansha, Tokyo, 1981.

Department of Probability Theory and Mathematical Statistics, Kyiv National Taras Shevchenko University, Kyiv, Ukraine
E-mail: odb@univ.kiev.ua

Department of Mathematical Physics, National Technical University "KPI", Kyiv, Ukraine