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A FAMILY OF MARTINGALES GENERATED BY A PROCESS WITH INDEPENDENT INCREMENTS

An explicit procedure to construct a family of martingales generated by a process with independent increments is presented. The main tools are the polynomials that give the relationship between the moments and cumulants, and a set of martingales related to the jumps of the process called Teugels martingales.

1. Introduction

In this work, we present an explicit procedure to generate a family of martingales from a process $X=\{X_t,\, t\geq 0\}$ with independent increments and continuous in probability. We extend our results exposed in [8], where we dealt with Lévy processes (independent and stationary increments); in that case, the martingales obtained were of the form $M_t = P(X_t, t)$, where $P(x,t)$ is a polynomial in x and t, and then they are *time–space* harmonic polynomials relative to X. Here, the martingales constructed are polynomials on X_t but, in general, not in t. Part of the paper is devoted to define the Teugels martingales of a process with independent increments; such martingales, introduced by Nualart and Schoutens [5] for Lévy processes, are a building block of the stochastic calculus with that type of processes.

2. Independent increment processes and their Teugels martingales

Let $X=\{X_t,\, t\geq 0\}$ be a process with independent increments, $X_0=0$, continuous in probability and cadlag; such processes are also called additive processes, and we will indistinctly use both names. Moreover, assume that X_t is centered and has moments of all orders. It is well known that the law of X_t is infinitely divisible for all $t\geq 0$. Let σ_t^2 be the variance of the Gaussian part of X_t, and let ν_t be its Lévy measure; for all these notions, we refer to Sato [6] or Skorohod [7].

Denote, by $\widetilde{\nu}$, the (unique) measure on $\mathcal{B}((0,\infty)\times\mathbb{R}_0)$ defined by

$$\widetilde{\nu}((0,t]\times B)=\nu_t(B),\ \ B\in\mathcal{B}(\mathbb{R}_0),$$

where $\mathbb{R}_0=\mathbb{R}-\{0\}$. By the standard approximation argument, we have that, for a measurable function $f:\mathbb{R}_0\to\mathbb{R}$ and for every $t>0$,

$$\iint_{(0,t]\times\mathbb{R}_0}|f(x)|\,\widetilde{\nu}(ds,dx)<\infty \quad\Longleftrightarrow\quad \int_{\mathbb{R}_0}|f(x)|\,\nu_t(dx)<\infty,$$

and, in this case,

$$\iint_{(0,t]\times\mathbb{R}_0}f(x)\,\widetilde{\nu}(ds,dx)=\int_{\mathbb{R}_0}f(x)\,\nu_t(dx).$$

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Note that since, for every $t \geq 0$, ν_t is a Lévy measure, $\widetilde{\nu}$ is σ–finite. To prove this, we observe that $\nu_t(\{|x| > 1\}) < \infty$, $\nu_t((1/(n+1), 1/n]) < \infty$, and $\nu_t([-1/n, -1/(n+1))) < \infty$, $n \geq 1$. So there is a numerable partition of $\mathbb{R}_0$ with sets of finite ν_t measure, $\forall t > 0$. Then, we can construct a numerable partition of $(0,\infty) \times \mathbb{R}_0$, each set being with finite $\widetilde{\nu}$-measure.

Write

$$N(C) = \#\{t:\, (t, \Delta X_t) \in C\}, \quad C \in \mathcal{B}((0,\infty) \times \mathbb{R}_0),$$

the jump measure of the process, where $\Delta X_t = X_t - X_{t-}$. It is a Poisson random measure on $(0,\infty) \times \mathbb{R}_0$ with intensity measure $\widetilde{\nu}$ (Sato [6, Theorem 19.2]). Define the compensated jump measure

$$d\widetilde{N}(t,x) = dN(t,x) - d\widetilde{\nu}(t,x).$$

The process admits the Lévy–Itô representation

$$X_t = G_t + \iint_{(0,t]\times\mathbb{R}_0} x\, d\widetilde{N}(t,x), \tag{2}$$

where $\{G_t,\, t \geq 0\}$ is a centered continuous Gaussian process with independent increments and variance $\mathbb{E}[G_t^2] = \sigma_t^2$.

The relationship between the moments of an infinitely divisible law and the *moments* of its Lévy measure is also well known (see Sato [6,Theorem 25.4]). In our case, as the process has moments of all orders, for all $t \geq 0$,

$$\int_{\{|x|>1\}} |x|\, \nu_t(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}_0} |x|^n\, \nu_t(dx) < \infty,\ \forall n \geq 2.$$

Write

$$F_2(t) = \sigma_t^2 + \int_{\mathbb{R}_0} x^2\, \nu_t(dx) \quad \text{and} \quad F_n(t) = \int_{\mathbb{R}_0} x^n\, \nu_t(dx),\ n \geq 3. \tag{1}$$

Since $\int_{\{|x|>1\}} |x|\, \nu_t(dx) < \infty$ and $\mathbb{E}[X_t] = 0$, the characteristic function of X_t can be written as

$$\phi_t(u) = \exp\big\{ -\frac{1}{2}\, \sigma_t^2\, u^2 + \int_{\mathbb{R}_0} \big(e^{iux} - 1 - iux\big)\nu_t(dx)\big\}.$$

It is deduced that, for $n \geq 2$, $F_n(t)$ is the cumulant of order n of X_t (for $n = 2$, $\mathbb{E}[X_t^2] = F_2(t)$). Also σ_t^2 is continuous and increasing (Sato [6, Theorem 9.8]).

Proposition 1. *The functions $F_n(t)$, $n \geq 2$, are continuous and have finite variation on finite intervals, and, for n even, they are increasing.*

Proof.

Consider $0 < u < t < v$, and write $U = [u,v]$. From the continuity in probability of X,

$$\lim_{s \to t,\, s \in U} X_s^n = X_t^n, \text{ in probability.}$$

Moreover, $\forall s \in U,\ |X_s| \leq \sup_{r \in U} |X_r|$, and since X is a martingale, by Doob's inequality,

$$\mathbb{E}\big[\sup_{r\in U} |X_r|^n\big] \leq C \sup_{r\in U} \mathbb{E}[|X_r|^n] \leq C\mathbb{E}[|X_v|^n] < \infty.$$

So it follows by dominated convergence that the function $t \mapsto \mathbb{E}[X_t^n]$ is continuous. Since the cumulants are polynomials of the moments, the continuity of all functions $F_n(t)$ is deduced.

To prove that $F_n(t)$ has finite variation on finite intervals, consider a partition of $[0,t]$: $0 < t_0 < \cdots < t_k = t$. Then

$$\sum_{j=1}^{k} \big|F_n(t_j) - F_n(t_{j-1})\big| = \sum_{j=1}^{k} \Big| \iint_{(t_{j-1},t_j]\times\mathbb{R}_0} x^n \widetilde{\nu}(ds,dx)\Big|$$
$$\le \sum_{j=1}^{k} \iint_{(t_{j-1},t_j]\times\mathbb{R}_0} |x|^n \, \widetilde{\nu}(ds,dx) = \iint_{(0,t]\times\mathbb{R}_0} |x|^n \, \widetilde{\nu}(ds,dx) < \infty. \quad \blacksquare$$

Consider the *variations* of the process X (see Meyer [4]):

$$X_t^{(1)} = X_t,$$
$$X_t^{(2)} = [X,X]_t = \sigma_t^2 + \sum_{0<s\le t} \big(\Delta X_s\big)^2$$
$$X_t^{(n)} = \sum_{0<s\le t} \big(\Delta X_s\big)^n, \ n \ge 3.$$

By Kyprianou [2, Theorem 2.7], for $n \ge 3$ (the case $n = 2$ is similar), the characteristic function of $X^{(n)}$ is

$$\exp\Big\{ \iint_{(0,t]\times\mathbb{R}_0} \big(e^{iux^n} - 1\big) \, \widetilde{\nu}(ds,dx) \Big\} = \exp\Big\{ \int_{\mathbb{R}_0} \big(e^{iux} - 1\big) \, \nu_t^{(n)}(dx) \Big\},$$

where $\nu_t^{(n)}$ is the measure image of ν_t by the function $x \mapsto x^n$ which is a Lévy measure. So $X^{(n)}$ has independent increments. Also by Kyprianou [2, Theorem 2.7], for $n \ge 2$,

$$\mathbb{E}[X_t^{(n)}] = F_n(t) \quad \text{and} \quad \mathbb{E}\big[\big(X_t^{(n)}\big)^2\big] = F_{2n}(t) + \big(F_n(t)\big)^2.$$

Therefore, combining the independence of the increments and the continuity of $F_n(t)$, it is deduced that $X^{(n)}$ is continuous in probability.

By Proposition 1, $F_n(t)$ has finite variation on finite intervals. Hence, the process

$$X_t^{(n)} = F_n(t) + \big(X_t^{(n)} - F_n(t)\big)$$

is a semimartingale.

The Teugels martingales introduced by Nualart and Schoutens [5] for Lévy processes can be extended to additive processes. In the same way as in [5], these martingales are obtained centering the processes $X^{(n)}$:

$$Y_t^{(1)} = X_t,$$
$$Y_t^{(n)} = X^{(n)} - F_n(t), \ n \ge 2,$$

They are square integrable martingales with optional quadratic covariation

$$[Y^{(n)}, Y^{(m)}]_t = X^{(n+m)},$$

and, since $F_{2n}(t)$ is increasing, the predictable quadratic variation of $Y^{(n)}$ is

$$\langle Y^{(n)} \rangle_t = F_{2n}(t).$$

3. The polynomials of cumulants

The formal expression

$$\exp\left\{\sum_{n=1}^{\infty}\kappa_n\frac{u^n}{n!}\right\}=\sum_{n=0}^{\infty}\mu_n\frac{u^n}{n!}. \tag{3}$$

relates the sequences of numbers $\{\kappa_n,\ n\geq 1\}$ and $\{\mu_n,\ n\geq 0\}$. When we consider a random variable Z with moment generating function in some open interval containing 0, then both series converge in a neighborhood of 0, and (3) is the relationship between the moment generating function, $\psi(u)=\mathbb{E}[e^{uZ}]$, and the cumulant generating function, $\log\psi(u)$. Moreover, μ_n (respectively, κ_n) is the moment (respectively, the cumulant) of order n of Z, and the well-known relations between moments and cumulants can be deduced from (3). The first three ones are

$$\begin{aligned}\mu_1&=\kappa_1,\\ \mu_2&=\kappa_1^2+\kappa_2,\\ \mu_3&=\kappa_1^3+3\kappa_1\kappa_2+\kappa_3,\dots\end{aligned}$$

If the random variable Z has only finite moments up to order n, the corresponding relationship is true up to this order.

There is a general explicit expression of the moments in terms of cumulants in Kendall and Stuart [1], or formulas involving the partitions of a set, see McCullagh [3]. In general, μ_n is a polynomial of $\kappa_1,\dots,\kappa_n$, called Kendall polynomial. Denote, by $\Gamma_n(x_1,\dots,x_n)$, $n\geq 1$, this polynomial, that is, we have

$$\mu_n=\Gamma_n(\kappa_1,\dots,\kappa_n).$$

Also write $\Gamma_0=1$. These polynomials enjoy very interesting properties, as the recurrence formula that follows from Stanley [9, Proposition 5.1.7]:

$$\Gamma_{n+1}(x_1,\dots,x_{n+1})=\sum_{j=0}^{n}\binom{n}{j}\Gamma_j(x_1,\dots,x_j)\,x_{n+1-j}. \tag{4}$$

We also have

$$\frac{\partial\Gamma_n(x_1,\dots,x_n)}{\partial x_j}=\binom{n}{j}\Gamma_{n-j}(x_1,\dots,x_{n-j}),\quad j=1,\dots,n. \tag{5}$$

Computing the Taylor expansion of $\Gamma_n(x_1+y,x_2,\dots,x_n)$ at $y=0$, we get the following expression we will need later:

$$\Gamma_n(x_1+y,x_2,\dots,x_n)=\sum_{j=0}^{n}\binom{n}{j}\Gamma_{n-j}(x_1,\dots,x_{n-j})\,y^j. \tag{6}$$

Interchanging the roles of x_1 and y and evaluating the function at 0, we obtain

$$\Gamma_n(x_1,x_2,\dots,x_n)=\sum_{j=0}^{n}\binom{n}{j}\Gamma_{n-j}(0,x_2,\dots,x_{n-j})\,x_1^j. \tag{7}$$

4. A family of martingales relative to the additive process

The main result of the paper is the following Theorem:

Theorem 1. *Let X be a centered additive process with finite moments of all orders. Then the process*

$$M_t^{(n)}=\Gamma_n\Big(X_t,-F_2(t),\dots,-F_n(t)\Big)$$

is a martingale.

Proof.

Let $n \geq 2$. We apply the multidimensional Itô formula to the semimartingales $X_t,\ F_2(t),\dots,F_n(t)$. By Proposition 1, the functions $F_2(t),\dots,F_n(t)$ and σ_t^2 are continuous and of finite variation. From (5) and the fact that $[X,X]_t^c = \sigma_t^2$ and $[F_j,F_j]_t^c = 0$, we have

$$\begin{aligned} M_t^{(n)} = n\int_0^t M_{s-}^{(n-1)}\,dX_s - \sum_{j=2}^n \binom{n}{j}\int_0^t M_s^{(n-j)} dF_j(s) \\ + \frac{1}{2}n(n-1)\int_0^t M_s^{(n-2)}\,d(\sigma_s^2) \\ + \sum_{0<s\leq t}\Big(\Gamma_n\big(X_{s-}+\Delta X_s, -F_2(s),\dots,-F_n(s)\big) - \Gamma_n\big(X_{s-},-F_2(s),\dots,-F_n(s)\big) \\ - n\Delta X_s\,\Gamma_{n-1}\big(X_{s-},-F_2(s),\dots,-F_n(s)\big)\Big). \end{aligned}$$

Applying (6),

$$\Gamma_n\big(X_{s-}+\Delta X_s,-F_2(s),\dots,-F_n(s)\big) = \sum_{j=0}^n\binom{n}{j}M_{s-}^{(n-j)}\big(\Delta X_s\big)^j.$$

Then, the jumps part given in the expression of $M_t^{(n)}$ is

$$\begin{aligned} \sum_{0<s\leq t}\sum_{j=2}^n\binom{n}{j}M_{s-}^{(n-j)}\big(\Delta X_s\big)^j &= \sum_{j=2}^n\binom{n}{j}\int_0^t M_{s-}^{(n-j)}\,dX_s^{(j)} - \binom{n}{2}\int_0^t M_s^{(n-2)}\,d(\sigma_s^2) \\ &= \sum_{j=2}^n\binom{n}{j}\int_0^t M_{s-}^{(n-j)}\,d\big(Y_s^{(j)}+F_j(s)\big) - \binom{n}{2}\int_0^t M_s^{(n-2)}\,d(\sigma_s^2). \end{aligned}$$

Therefore,

$$M_t^{(n)} = \sum_{j=1}^n\binom{n}{j}\int_0^t M_{s-}^{(n-j)}\,dY_s^{(j)}. \tag{8}$$

Moreover, $\big(M_t^{(k)}\big)^2$ is a polynomial in $X_t,\ F_2(t),\dots,F_k(t)$. Taking expectations and using the relations between moments and cumulants, as well as the fact that the cumulants of X_t are $F_n(t)$, $n\geq 2$, we obtain that

$$E\big[\big(M_t^{(k)}\big)^2\big] = P(F_2(t),\dots,F_{2k}(t)),$$

for a suitable polynomial P. Then, for every $t\geq 0$, we have

$$\begin{aligned} \mathbb{E}\big[\int_0^t\big(M_{s-}^{(k)}\big)^2\,d\langle Y^{(j)}\rangle_s\big] &= \int_0^t\mathbb{E}\Big[\big(M_{s-}^{(k)}\big)^2\Big]dF_{2j}(s) \\ &= \int_0^t P(F_2(s),\dots,F_{2k}(s))\,dF_{2j}(s) < \infty, \end{aligned}$$

So all the stochastic integrals on the right-hand side of (8) are martingales. ■

Remark 1. *It is worth to note that the preceding Theorem implies that the function*

$$g_n(x,t) = \Gamma_n\Big(x,-F_2(t),\dots,-F_n(t)\Big)$$

is a time–space harmonic function with respect to X_t. By (7),

$$g_n(x,t) = \sum_{j=0}^n\Gamma_{n-j}(0,-F_2(t),\dots,F_n(t))x^j.$$

In general, $g_n(x,t)$ is a polynomial in x. If $F_n(t)$, $n \geq 2$, are polynomials in t, then $g_n(x,t)$ is a time–space harmonic polynomial; this happens for all Lévy processes with moments of all orders and for some additive process; see the example below.

Example. Let $\Lambda(t) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous increasing function, and let J be a Poisson random measure on $\mathbb{R}_+$ with intensity measure $\mu(A) = \int_A \Lambda(dt)$, $A \in \mathcal{B}(\mathbb{R}_+)$. Then the process $X = \{X_t,\, t \geq 0\}$ defined pathwise,

$$X_t(\omega) = \int_0^t J(ds, \omega) - \Lambda(t)$$

, is an additive process; it is a Cox process with deterministic hazard function $\Lambda(t)$. From the characteristic function of X_t, we deduce that the Lévy measure is

$$\nu_t(dx) = \Lambda(t)\delta_1(dx),$$

where δ_1 is a Dirac delta measure concentrated in the point 1. Hence,

$$F_n(t) = \Lambda(t),\ n \geq 2.$$

Note that the conditions we have assumed on Λ are necessary to obtain an additive process, but it is not necessary (though not very restrictive) to assume that Λ is absolutely continuous with respect to the Lebesgue measure.

The function defined in Remark 1 is

$$g_n(x,t) = \Gamma_n\big(x, -\Lambda(t), \ldots, -\Lambda(t)\big).$$

Hence, when $\Lambda(t)$ is a polynomial, $g_n(x,t)$ is a time–space harmonic polynomial.

Denote, by $\overline{C}_n(x,t)$, the Charlier polynomial with leading coefficient equal to 1. Then (see [8])

$$g_n(x,t) = \sum_{j=1}^{n} \lambda_j^{(n)} \overline{C}_j(x, \Lambda(t)),$$

where $\lambda_1^{(n)} = 1$ and

$$\lambda_{k+1}^{(n)} = \sum_{j=k}^{n-1} \binom{n}{j} \lambda_k^{(j)}, \qquad k = 1, \ldots, n-1.$$

Bibliography

1. M.Kendall and A.Stuart, *The Advanced Theory of Statistics, Vol. 1, 4th edition*, MacMillan, New York, 1977.
2. A.E.Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, Berlin, 2006.
3. P.McCullagh, *Tensor Methods in Statistics*, Chapman and Hall, London, 1987.
4. P.A.Meyer, *Un cours sur les integrales stochastiques*, Séminaire de Probabilités X, Springer, New York, 1976, pp. 245–400. (French)
5. D.Nualart and W.Schoutens, *Chaotic and predictable representation for Lévy processes*, Stochastic Process. Appl. **90** (2000), 109–122.
6. K.Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.
7. A.V.Skorohod, *Random Processes with Independent Increments*, Kluwer Academic Publ., Dordrecht, Boston, London, 1986.
8. J.L.Solé and F.Utzet, *Time-space harmonic polynomials relative to a Lévy process*, Bernoulli (2007).
9. R.P.Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

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