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ON ASYMPTOTIC BEHAVIOUR OF PROBABILITIES OF SMALL DEVIATIONS FOR COMPOUND COX PROCESSES

We derive logarithmic asymtotics for probabilities of small deviations for compound Cox processes in the space of trajectories. We find conditions under which these asymptotics are the same as those for sums of independent identically distributed random variables and homogeneous processes with independent increments. We show that if these conditions do not hold, the asymptotics of small deviations for compound Cox processes are quite different.

Introduction

The asymptotic behaviour of probabilities of small deviations has been investigated for various classes of stochastic processes. Asymptotics of probabilities of small deviations for sums of independent random variables and homogeneous processes with independent increments were found in Mogul'skii [1], Borovkov and Mogul'skii [2] and references therein. Note that the last class of processes includes stable, Poisson and compound Poisson processes. Various results for Gaussian processes and references may be found in surveys by Ledoux [3], Li and Shao [4] and Lifshits [5]. Further results were obtained for various stochastic processes, generated by sums of random numbers of independent random variables. In this case, the number of summands is a stochastic process which is usually independent with the summands. Increments of such the processes may be dependent. These processes are called compound processes. Now we give definitions for some of them.

Let $X, X_1, X_2, \dots$ be a sequence of independent, identically distributed random variables. Put $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geqslant 1$, $S_0 = 0$.

Let $\nu(t)$ be a standard Poisson process, independent with the sequence $\{X_k\}$. Then the stochastic process $\eta(t) = S_{\nu(\lambda t)}$, $\lambda > 0$, is called a compound Poisson process.

If $\delta(t)$ is a renewal process, independent with $\{X_k\}$, then $\zeta(t) = S_{\delta(t)}$ is called a compound renewal process. When the renewal times have an exponential distribution, the compound renewal process coincides with the compound Poisson process.

Small deviations of the renewal and compound renewal processes has been studied in Frolov, Martikainen, Steinebach [6].

Now we turn to the definition of the compound Cox process. We start with the definition of a Cox process which we borrow from the paper of Embrechts and Klüppelberg [7].

Let $\Lambda(t)$, $t > 0$, be a random measure, i.e. a.s. (almost surely) $\Lambda(0) = 0$, $\Lambda(t) < \infty$ for all $t > 0$ and $\Lambda(t)$ has non-decreasing trajectories. Assume that $\Lambda(t)$ does not depend on the standard Poisson process $\nu(t)$. The point process $N(t) = \nu(\Lambda(t))$ is called a Cox process. If, in particular, the trajectories of $\Lambda(t)$ are continuous a.s., then for every

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realization $\lambda(t)$ of the measure $\Lambda(t)$ the process $N(t)$ is a non-homogeneous Poisson process with the intensity measure $\lambda(t)$.

Now we define the compound Cox process in the same way as we introduce the processes $\eta(t)$ and $\zeta(t)$ above.

Let $N(t)$ be a Cox process, independent with the sequence $\{X_k\}$. The stochastic process $S(t) = S_{N(t)}$ is called a compound Cox process.

Compound Cox processes play an important role in actuarial and financial mathematics. They describe, for example, the processes of total claims of an insurance company in a collective risk model (cf. [7]).

Note that under additional conditions (cf.,e.g. [7]), the Cox processes are renewal processes. Nevertheless, in the sequel, we assume that $\Lambda(t)$ is such that our Cox process will be the renewal process only if it is a Poisson process. So, we do not consider renewal processes here.

The logarithmic asymptotics of small deviations of the compound Cox processes has been investigated by Frolov [8]. In there, we have described the behaviour of

$$P_T = P\Big(\sup_{0\leqslant t\leqslant T}\Big|S(t) - c\Lambda(t)\Big| \leqslant x_T\Big),$$

where $\{x_T\}$ is a positive real function such that $x_T \to \infty$ and $x_T^2 = o(f(T))$ as $T \to \infty$, $f(T)$ is a function, depending on properties of the measure $\Lambda(t)$, $c = EX$ for $EX < \infty$ and $c = 0$ otherwise.

Note that if $EX < \infty$, then $S(t)$ is centered by a random function, generally speaking. Nevertheless, in the case of the homogeneous Poisson process, $\Lambda(t) = \lambda t$ and our centering coincides with $ES(t)$. The same holds true when $N(t)$ is a non-homogeneous Poisson process. Moreover, centering functions of this type are used in appropriate models of the risk process. It turns out that fluctuations of risk in such models have to be compensate by insurance premiums of random amounts since, otherwise, the ruin probability of insurance company may be separated from zero and may be non-decreasing with increasing of initial capital of insurance company (cf. [7]). Therefore we consider this centering.

Here, we present generalizations of the results in Frolov [8]. P_T is the probability that trajectories of the compound Cox process, centered at $c\Lambda(t)$ and normed by x_T, lie in a strip of width 2 around zero. We consider below more general sets instead of strips and derive logarithmic asymtotics for such probabilities. We find conditions under which these asymptotics are the same as those for sums of independent identically distributed random variables and homogeneous processes with independent increments. We show that the asymptotics of small deviations for compound Cox processes are quite different, if these conditions do not hold.

1. Results

Let $X, X_1, X_2, \dots$ be a sequence of independent, identically distributed random variables. Put $S_n = X_1 + X_2 + \cdots + X_n$ for $n \geqslant 1$, $S_0 = 0$.

Let $\nu(t)$ be a standard Poisson process, independent with the sequence $\{X_k\}$.

Denote $\xi(t) = S_{\nu(t)}$. Note that $\xi(t)$ is a compound Poisson process and, therefore, it is a homogeneous process with independent increments.

Let $\Lambda(t)$, $t > 0$, be a random measure, i.e. a.s. (almost surely) $\Lambda(0) = 0$, $\Lambda(t) < \infty$ for all $t > 0$ and $\Lambda(t)$ has non-decreasing trajectories. Assume that the trajectories of $\Lambda(t)$ are a.s. continuous and $\Lambda(\infty) = \infty$ a.s. Suppose that $\Lambda(t)$ is independent with the process $\nu(t)$ and the sequence $\{X_k\}$.

Define the Cox process $N(t)$ and the compound Cox process $S(t)$ by the relations $N(t) = \nu(\Lambda(t))$ and $S(t) = S_{N(t)}$.

Let x_T be a real function with $x_T \to \infty$ as $T \to \infty$.

Let $g_1(t)$, $g_2(t)$, $t \in [0,1]$, be continuous functions such that $g_1(0) < 0 < g_2(0)$, $g_1(t)$ is non-increasing and $g_2(t)$ is non-decreasing. Put

$$P_T = P\left(g_1\left(\frac{\Lambda(t)}{\Lambda(T)}\right) x_T \leqslant S(t) - c\Lambda(t) \leqslant g_2\left(\frac{\Lambda(t)}{\Lambda(T)}\right) x_T \text{ for all } t \in [0,T]\right).$$

The probability P_T is well defined since trajectories of $S(t)$ are jump functions.

In the sequel, we will describe the asymptotic behaviour of probabilities of small deviations P_T.

In general case, $g_i(\Lambda(t)/\Lambda(T))$ in the definition of P_T are random. We mentioned above that the random centering of $S(t)$ is well motivated. So, random normings may also be used. Nevertheless, there are important cases in which $g_i(\Lambda(t)/\Lambda(T))$ are non-random functions.

The first case is that $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$. In this case the results have been obtained in Frolov [8].

If $N(t)$ is a Poisson process, then $\Lambda(t)$ is a continuous increasing function. This is the second case.

The third case arises when $\Lambda(t) = \Lambda f(t)$, where Λ is a positive random variable and $f(t)$ is a continuous function. Then $g_i(\Lambda(t)/\Lambda(T)) = g_i(f(t)/f(T))$. If, in addition, $f(t) = t^\beta$, $\beta > 0$, we have

$$\begin{aligned} P_T &= P\left(g_3\left(\frac{t}{T}\right) x_T \leqslant S(t) - c\Lambda(t) \leqslant g_4\left(\frac{t}{T}\right) x_T \text{ for all } t \in [0,T]\right) \\ &= P\left(g_3(t) x_T \leqslant S(Tt) - c\Lambda(Tt) \leqslant g_4(t) x_T \text{ for all } t \in [0,1]\right), \end{aligned}$$

where $g_{i+2}(t) = g_i(t^\beta)$, $i = 1,2$. Note that in this last case, one can consider more general sets in the definition of P_T. To this end, consider a family of processes $\{s_T(t) = S(Tt)/x_T; 0 \leqslant t \leqslant 1, T \in (0,\infty)\}$. The trajectories $s_T(\cdot)$ belong to the Skorohod space $D[0,1]$ and we can introduce $P(s_T(\cdot) \in G)$, where $G \in \mathbf{G}$ and $\mathbf{G}$ is a class of subsets of $D[0,1]$. One can define this class in the same way as in Mogul'skii [1] with the following additional assumption: on the first step of the definition on p. 757, functions $L_1(t)$ and $L_2(t)$ ($L_1(t) > L_2(t)$) have to be non-decreasing and non-increasing correspondingly. Of course, the class $\mathbf{G}$ will be a subclass of the class in Mogul'skii [1], but it will be wide enough. It seems that the most interesting set is the set $G_0 = \{g \in D[0,1] : g(0) = 0, g_3(t) < g(t) < g_4(t), \text{ for all } t \in [0,1]\}$, where $g_3(t)$, $g_4(t)$, $t \in [0,1]$, are continuous functions such that $g_3(0) < 0 < g_4(0)$, $g_3(t)$ is non-increasing and $g_4(t)$ is non-decreasing. The asymptotics of $P(s_T(\cdot) \in G_0)$ may be derived from our results below.

Our first result is the following theorem.

Theorem 1. *Assume that there exist a positive, increasing, continuous function $f(T)$, $f(T) \to \infty$ as $T \to \infty$, and a non-negative random variables Λ such that the distributions of $\Lambda(T)/f(T)$ converge weakly to the distribution Λ as $T \to \infty$. Denote $a_T = \operatorname{ess\,inf} \Lambda(T)/f(T)$, $a = \operatorname{ess\,inf} \Lambda$. Assume that $a_T \to a$ as $T \to \infty$ and $a > 0$.*

If $EX < \infty$, then put $c = EX$. Put $c = 0$ otherwise.

Suppose that the distribution of the random variable $\xi_1 = \xi(1) - c$ belongs to a domain of attraction of a strictly stable law F_α with the index $\alpha \in (0,2]$, i.e. the distributions of $(\xi(n) - cn)/B_n$ converge weakly to F_α as $n \to \infty$, where $\{B_n\}$ is a sequence of positive constants. Assume that F_α is not concentrated on the half of line.

Then for every positive function x_T with $x_T \to \infty$ and $x_T = o(B_{[f(T)]})$ as $T \to \infty$, the following relation holds

$$\log P_T = -CH_\alpha a \frac{f(T)}{x_T^\alpha} L(x_T)\big(1 + o(1)\big) \text{ as } T \to \infty, \tag{1}$$

where $L(x) = x^{\alpha-2}\, E\xi_1^2 I\{|\xi_1| < x\}$ is a slowly varying at infinity function, C is an absolute positive constant, depending only on the distribution F_α, $H_\alpha = \int_0^1 (g_2(t)-g_1(t))^{-\alpha} dt$. If $\alpha = 2$, then $C = \pi^2/8$. Here and in the sequel, $I\{B\}$ denotes the indicator of the event B, $[x]$ denotes the integer part of x.

Theorem 1 for $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$ has been obtained in Frolov [8].

Conditions, necessary and sufficient for belonging of the distribution of ξ_1 to a domain of attraction of F_α, are well known. These conditions are usually stated in terms of asymptotic behaviours for tails or truncated moments (cf., for example, Feller [9], Chapter XVII, §5). Nevertheless, to check the conditions of Theorem 1, it is more convenient to apply Theorem 2.6.5, p. 103 from Ibragimov and Linnik [10] where such conditions are given in terms of an asymptotic behaviour of a characteristic function at zero. Since $Ee^{it\xi_1} = \exp\left\{-itc + Ee^{itX} - 1\right\}$, one can easily check these conditions.

The simplest example of $\Lambda(t)$ is $\Lambda(t) = \Lambda f(t)$, where the random variable Λ and the function $f(t)$ satisfy the conditions of Theorem 1.(Note that the random variable Λ may be degenerate and we will deal in this case with the non-homogeneous Poisson process $N(t)$ and the corresponding process $S(t)$.) We will arrive to another examples, if $\Lambda(t)$ will be a stochastic process which satisfies the law of large numbers and has appropriate trajectories.

Theorem 1 yields that if $a > 0$, then the behaviour of $\log P_T$ is the same as that for probabilities of small deviations for sums of independent identically distributed random variables and homogeneous processes with independent increments. Now we turn to the case $a = 0$. In this case the asymptotic of $\log P_T$ may be quite different.

Theorem 2. *Assume that the conditions of Theorem 1 hold and $a = 0$.*

Then for every positive function x_T with $x_T \to \infty$ and $x_T = o(B_{[f(T)]})$ as $T \to \infty$, the following relation holds

$$\log P_T \;=\; o\left(\frac{f(T)}{x_T^\alpha}\, L(x_T)\right) \text{ as } T \to \infty, \tag{2}$$

where $L(x)$ is the function from Theorem 1.

In the case $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$, Theorem 2 has been proved in Frolov [8].

Theorem 2 does not give the exact asymptotic of $\log P_T$. In the next result we find this asymptotic which depends on the behaviour of the distribution functions of $\Lambda(T)/f(T)$ and Λ at zero.

Theorem 3. *Assume that the conditions of Theorem 1 hold and $a_T = a = 0$ for all T. Put $F_T(\lambda) = P(\Lambda(T) < \lambda f(T))$ and $F(\lambda) = P(\Lambda < \lambda)$.*

For every positive function x_T with $x_T \to \infty$ and $x_T = o(B_{[f(T)]})$ as $T \to \infty$, define ε_T by the relation

$$\varepsilon_T = \sup\left\{\varepsilon > 0 : \frac{\varepsilon}{-\log F_T(\varepsilon)} \leqslant \frac{x_T^\alpha}{CH_\alpha f(T) L(x_T)}\right\},$$

where the function $L(x)$ and the constants C and H_α are from Theorem 1. Assume that ε_T is equivalent to a continuous decreasing function and $F_T(\varepsilon_T) \to 0$ as $T \to \infty$.

Suppose that for every $\tau > 0$, the following relation holds

$$\log F_T(\tau\varepsilon_T) \sim \log F_T(\varepsilon_T) \text{ as } T \to \infty. \tag{3}$$

Then

$$\log P_T = \log F_T(\varepsilon_T)\big(1+o(1)\big) = -\varepsilon_T C H_\alpha \frac{f(T)}{x_T^\alpha} L(x_T)\big(1+o(1)\big) \text{ as } T \to \infty. \tag{4}$$

Here $\varepsilon_T \to 0$ *as* $T \to \infty$.

Theorem 3 for $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$ and continuous $F_T(\lambda)$ and $F(\lambda)$ has been obtained in Frolov [8].

Note that if for all T the distribution functions $F_T(\lambda)$ and $F(\lambda)$ are continuous and positive in the non-degenerate interval $[0, \lambda_0]$, then in Theorem 3, ε_T is a continuous decreasing function and $F_T(\varepsilon_T) \to 0$ as $T \to \infty$. Indeed, $F_T(\varepsilon_T) \leqslant \Delta + F(\varepsilon_T)$, where $\Delta = \sup_{0 \leqslant x \leqslant \lambda_0} |F_T(x) - F(x)| \to 0$ as $T \to \infty$ by the weak convergence of $F_T(\lambda)$ to $F(\lambda)$ and the continuity of the limit function.

We now show that that the asymptotic of $\log P_T$ in (1) and (4) are quite different. To this goal, suppose that $F_T(x) \equiv F(x)$ for all T. If, for example, $F(x) = x^p$ for $x \in [0,1]$, where $p > 0$, then $\log F_T(\varepsilon_T) \sim -p \log(\frac{f(T)}{x_T^\alpha} L(x_T))$ as $T \to \infty$. If $F(x) = (-\log x)^{-p}$ for $x \in (0, e^{-1}]$, where $p > 0$, then $\log F_T(\varepsilon_T) \sim -p \log\log(\frac{f(T)}{x_T^\alpha} L(x_T))$ as $T \to \infty$.

It turns out that the condition (3) can not be omitted in Theorem 3. This follows from the next result.

Theorem 4. *Assume that all the conditions of Theorem 3 hold except the condition* (3). *Assume that for all* $\tau > 0$, *the following relation holds* $\log F_T(\tau\varepsilon_T) \sim \tau^p \log F_T(\varepsilon_T)$ *as* $T \to \infty$, *where* $p > 0$.

Then

$$\log P_T = o\left(\varepsilon_T \frac{f(T)}{x_T^\alpha} L(x_T)\right) \text{ as } T \to \infty. \tag{5}$$

Theorem 4 for $g_1(t) \equiv -1$ and $g_2(t) \equiv 1$ and continuous $F_T(\lambda)$ and $F(\lambda)$ has been proved in Frolov [8].

2. Proofs

Put $\bar{\xi}(t) = \xi(t) - ct$.

In what follows, we will use the following result.

Lemma 1. *If the function* $f(t)$ *satisfies the conditions of Theorem 1, then the probability* $P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T$ *for all* $t \in [0,1])$ *is non-increasing in* λ.

Proof of Lemma 1. Take $\lambda > 1$. Then

$$\begin{aligned}
&P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) \\
&= P\left(g_1\left(\frac{u}{\lambda f(T)}\right) x_T \leqslant \bar{\xi}(u) \leqslant g_2\left(\frac{u}{\lambda f(T)}\right) x_T \text{ for all } u \in [0, \lambda f(T)]\right) \\
&\leqslant P\left(g_1\left(\frac{u}{\lambda f(T)}\right) x_T \leqslant \bar{\xi}(u) \leqslant g_2\left(\frac{u}{\lambda f(T)}\right) x_T \text{ for all } u \in [0, f(T)]\right) \\
&\leqslant P\left(g_1\left(\frac{u}{f(T)}\right) x_T \leqslant \bar{\xi}(u) \leqslant g_2\left(\frac{u}{f(T)}\right) x_T \text{ for all } u \in [0, f(T)]\right) \\
&= P(g_1(t)x_T \leqslant \bar{\xi}(f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]).
\end{aligned}$$

In the last inequality we have used that $g_1(t)$ is non-increasing and $g_2(t)$ is non-decreasing.

We also need the following result on asymptotics of small deviations for compound Poisson process $\xi(t)$.

Lemma 2. *Let* $g(T)$ *be an increasing, continuous function with* $g(T) \to \infty$ *as* $T \to \infty$.

If the conditions of Theorem 1 hold, then for every positive function x_T with $x_T \to \infty$ and $x_T = o(B_{[g(T)]})$ as $T \to \infty$, the following relation holds

$$\log P\left(g_1(t)x_T \leqslant \bar{\xi}(g(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]\right) = -CH_\alpha \frac{g(T)}{x_T^\alpha} L(x_T)\big(1+o(1)\big) \tag{6}$$

as $T \to \infty$, where $L(x)$ and C, H_α are the function and the constants from Theorem 1.

Proof of Lemma 2. Take a sequence $\{T_k\}$ such that $T_k \nearrow \infty$ as $k \to \infty$.

Since $\xi(t)$ is a homogeneous process with independent increments, we have by Theorem 4 from Mogul'skii [1] and Lemma 1

$$\begin{aligned} &P\left(g_1(t)x_{T_k} \leqslant \bar{\xi}(g(T_k)t) \leqslant g_2(t)x_{T_k} \text{ for all } t \in [0,1]\right) \\ \leqslant &P\left(g_1(t)x_{T_k} \leqslant \bar{\xi}([g(T_k)]t) \leqslant g_2(t)x_{T_k} \text{ for all } t \in [0,1]\right) \\ &= \exp\left\{-CH_\alpha \frac{[g(T_k)]}{x_{T_k}^\alpha} L(x_{T_k})(1+o(1))\right\} \text{ as } k \to \infty. \end{aligned}$$

The lower bound for the probability in (6) may be derived in the same way. Taking into account that the sequence $\{T_k\}$ may be chosen arbitrarily, we get (6).

Proof of Theorem 1. Put $F_T(\lambda) = P(\Lambda(T) < \lambda f(T))$.

Taking into account Lemma 1 and the independence of $\bar{\xi}(t)$ and $\Lambda(t)$, we have

$$\begin{aligned} P_T &= P\left(g_1\left(\frac{\Lambda(t)}{\Lambda(T)}\right)x_T \leqslant \bar{\xi}(\Lambda(t)) \leqslant g_2\left(\frac{\Lambda(t)}{\Lambda(T)}\right)x_T \text{ for all } t \in [0,T]\right) \\ &= P(g_1(t)x_T \leqslant \bar{\xi}(\Lambda(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) \\ &= \int_0^\infty P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) dF_T(\lambda) \\ &= \int_{a_T}^\infty P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) dF_T(\lambda) \\ &\leqslant P(g_1(t)x_T \leqslant \bar{\xi}(a_T f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]). \end{aligned} \tag{7}$$

Take $\varepsilon \in (0,a)$. Then $a_T \geqslant a-\varepsilon$ for all sufficiently large T. By Lemma 1 it follows that

$$P_T \leqslant P(g_1(t)x_T \leqslant \bar{\xi}((a-\varepsilon)f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]). \tag{8}$$

for all sufficiently large T.

The norming constants B_n may be chosen such that $B_n = n^{1/\alpha}L_1(n)$, where $L_1(x)$ is a slowly varying at infinity function (cf., for example, [10], p. 48). It follows that the condition $x_T = o(B_{[f(T)]})$ as $T \to \infty$ is equivalent to the condition $x_T = o(B_{[bf(T)]})$ as $T \to \infty$, where b is an arbitrary fixed positive constant. By Lemma 2

$$\begin{aligned} &\log P\left(g_1(t)x_T \leqslant \bar{\xi}((a-\varepsilon)f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]\right) \\ &\qquad\qquad = -CH_\alpha(a-\varepsilon)\frac{f(T)}{x_T^\alpha} L(x_T)\big(1+o(1)\big) \end{aligned}$$

as $T \to \infty$. The latter and (8) yield that

$$\limsup_{T\to\infty} \frac{x_T^\alpha}{f(T)L(x_T)} \log P_T \leqslant -CH_\alpha(a-\varepsilon).$$

Taking in the last inequality the limit as $\varepsilon \to 0$, we get

$$\limsup_{T\to\infty} \frac{x_T^\alpha}{f(T)L(x_T)} \log P_T \leqslant -CH_\alpha a. \tag{9}$$

Take $\varepsilon > 0$. Since $a_T \to a$ as $T \to \infty$, we get from (7) and Lemma 1 that

$$\begin{aligned} P_T &\geqslant \int_{a_T}^{(1+\varepsilon)^2 a_T} P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) dF_T(\lambda) \\ &\geqslant P(g_1(t)x_T \leqslant \bar{\xi}((1+\varepsilon)^2 a_T f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) F_T((1+\varepsilon)^2 a_T) \\ &\geqslant P(g_1(t)x_T \leqslant \bar{\xi}((1+\varepsilon)^3 a f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) F_T((1+\varepsilon)a) \end{aligned}$$

for all sufficiently large T. Choose ε such that $(1+\varepsilon)a$ is a point of continuity for $F(\lambda) = P(\Lambda < \lambda)$. By the weak convergence of $F_T(\lambda)$ to $F(\lambda)$ we have

$$P_T \geqslant \frac{1}{2} P(g_1(t)x_T \leqslant \bar{\xi}((1+\varepsilon)^3 a f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) F((1+\varepsilon)a) \tag{10}$$

for all sufficiently large T. Since $x_T = o(B_{[(1+\varepsilon)^3 a f(T)]})$ as $T \to \infty$, by Lemma 2

$$\begin{aligned} &\log P\left(g_1(t)x_T \leqslant \bar{\xi}((1+\varepsilon)^3 a f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]\right) \\ &\qquad\qquad = -C(1+\varepsilon)^3 a H_\alpha \frac{f(T)}{x_T^\alpha} L(x_T)\big(1+o(1)\big) \end{aligned}$$

as $T \to \infty$. It follows from (10) that

$$\liminf_{T\to\infty} \frac{x_T^\alpha}{f(T)L(x_T)} \log P_T \geqslant -CaH_\alpha(1+\varepsilon)^3. \tag{11}$$

Taking in the last inequality the limit as $\varepsilon \to 0$, we get

$$\liminf_{T\to\infty} \frac{x_T^\alpha}{f(T)L(x_T)} \log P_T \geqslant -CaH_\alpha,$$

which together with (9) yields (1).

Proof of Theorem 2. It is clear that $\log P_T \leqslant 0$ and we need only prove the lower bound.

Take $\varepsilon > 0$. Using (7) and Lemma 1, we have

$$\begin{aligned} P_T &\geqslant \int_{\varepsilon}^{(1+\varepsilon)^2 \varepsilon} P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) dF_T(\lambda) \\ &\geqslant P(g_1(t)x_T \leqslant \bar{\xi}((1+\varepsilon)^2 \varepsilon f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) (F_T((1+\varepsilon)^2\varepsilon) - F_T(\varepsilon)) \end{aligned}$$

for all sufficiently large T. In the same way as in the proof of Theorem 1, the last implies (11) with ε instead of a. Taking the limit as $\varepsilon \to 0$, we get (2).

Proof of Theorem 3. Put $b_T = CH_\alpha \frac{f(T)}{x_T^\alpha} L(x_T)$. The condition $x_T = o(B_{[f(T)]})$ as $T \to \infty$ and formulae for the norming constants B_n, which may be chosen to satisfy

$$\frac{nL(B_n)}{B_n^\alpha} \to d \quad \text{as} \quad n \to \infty, \tag{12}$$

(cf., for example, [9], Chapter XVII, §5), imply that $b_T \to \infty$ as $T \to \infty$.

We will first prove that $\varepsilon_T \to 0$ as $T \to \infty$.

Suppose that there exists a sequence $\{T_k\}$ such that $T_k \nearrow \infty$ as $k \to \infty$ and $\varepsilon_{T_k} > \varepsilon > 0$ for all sufficiently large k, where ε is a point of continuity of $F(\lambda)$. Then $-\log F_{T_k}(\varepsilon_{T_k}) \leqslant -\log F_{T_k}(\varepsilon) \leqslant -\log(F(\varepsilon)/2) < \infty$ for all sufficiently large k in view of the weak convergence of $F_T(\lambda)$ to $F(\lambda)$. Hence $1/b_{T_k} \geqslant -\varepsilon_{T_k}/\log F_{T_k}(\varepsilon_{T_k}) > -\varepsilon/\log(F(\varepsilon)/2) > 0$ which contradicts to the relation $b_T \to \infty$ as $T \to \infty$.

By the definition of ε_T, we get $-3\varepsilon_T/\log(F_T(3\varepsilon_T)) > 1/b_T$. This and (3) imply that $-2\varepsilon_T/\log(F_T(\varepsilon_T)) \geqslant 1/b_T$ for all sufficiently large T. The latter and the relation

$F_T(\varepsilon_T) \to 0$ as $T \to \infty$ give $1/b_T = o(\varepsilon_T)$ as $T \to \infty$. The last relation, the definition of b_T and (12) with $n = [f(T)\varepsilon_T\tau]$, $\tau > 0$, yield that

$$\frac{x_T^\alpha L(B_{[f(T)\varepsilon_T\tau]})}{B^\alpha_{[f(T)\varepsilon_T\tau]} L(x_T)} \to 0 \text{ as } T \to \infty.$$

Using the well known representation of a slowly varying function (cf., for example, [9] Chapter VIII, §9), we have that

$$\frac{L(B_{[f(T)\varepsilon_T\tau]})}{L(x_T)} = \frac{l(B_{[f(T)\varepsilon_T\tau]})}{l(x_T)} \exp\left\{\int_{x_T}^{B_{[f(T)\varepsilon_T\tau]}} \frac{\varrho(u)}{u} du\right\},$$

where $l(u) \to l < \infty$ and $\varrho(u) \to 0$ as $u \to \infty$. This yields that

$$\frac{L(B_{[f(T)\varepsilon_T\tau]})}{L(x_T)} \geqslant \frac{1}{2}\left(\frac{x_T}{B_{[f(T)\varepsilon_T\tau]}}\right)^{-\alpha/2}$$

for all sufficiently large T. It follows that $x_T = o(B_{[f(T)\varepsilon_T\tau]})$ as $T \to \infty$, where $\tau > 0$.

Without loss of generality we will assume in the rest of the proof that ε_T is continuous. Otherwise, one can replace ε_T by an equivalent function in the sequel.

We have from (7) and Lemma 1

$$\begin{aligned}
P_T &= \int_0^{\varepsilon_T} P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1])dF_T(\lambda) \\
&+ \int_{\varepsilon_T}^{\infty} P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1])dF_T(\lambda) \\
&\leqslant F_T(\varepsilon_T) + P(g_1(t)x_T \leqslant \bar{\xi}(\varepsilon_T f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]) \\
&\leqslant e^{-\varepsilon_T b_T} + P(g_1(t)x_T \leqslant \bar{\xi}(\varepsilon_T f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1]).
\end{aligned}$$

Applying Lemma 2, we get the upper bound in (4).

We now turn to the lower bound. Take $\tau > 0$. By Lemmas 1 and 2

$$\begin{aligned}
P_T &\geqslant \int_0^{\tau\varepsilon_T} P(g_1(t)x_T \leqslant \bar{\xi}(\lambda f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1])dF_T(\lambda) \\
&\geqslant P(g_1(t)x_T \leqslant \bar{\xi}(\tau\varepsilon_T f(T)t) \leqslant g_2(t)x_T \text{ for all } t \in [0,1])F_T(\tau\varepsilon_T) \\
&= e^{-\tau\varepsilon_T b_T(1+o(1))} F_T(\tau\varepsilon_T)
\end{aligned}$$

as $T \to \infty$. By the definition of ε_T we get $\log F_T((1+\tau)\varepsilon_T) > -(1+\tau)\varepsilon_T b_T$. It follows from the last inequality and (3) that

$$P_T \geqslant e^{-\tau\varepsilon_T b_T(1+o(1)) + \log F_T((1+\tau)\varepsilon_T)(1+o(1))} \geqslant e^{-(1+2\tau)\varepsilon_T b_T(1+o(1))} \tag{13}$$

as $T \to \infty$. Since τ may be chosen arbitrarily small, we get the lower bound in (4).

Proof of Theorem 4. As in the proof of Theorem 2, we need only prove the lower bound.

In the same way as in proof of (13), we get

$$P_T \geqslant e^{-(\tau^p(1+\tau)^{1-p}+\tau)\varepsilon_T b_T(1+o(1))}$$

as $T \to \infty$ which yields (5).

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