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CONVERGENCE IN SKOROKHOD \mathbf{J} -TOPOLOGY FOR COMPOSITIONS OF STOCHASTIC PROCESSES

A survey on functional limit theorems for compositions of stochastic processes is presented. Applications to stochastic processes with random scaling of time, random sums, extremes with random sample size, generalised exceeding processes, sum- and max-processes with renewal stopping, and shock processes are discussed.

1. INTRODUCTION

Let us begin from short historical remarks concerned research studies on functional limit theorems for stochastic processes.

The classical works by Khintchine (1933), Lévy (1937, 1948), Gnedenko and Kolmogorov (1949), and Doob (1953) created a general framework for the development of the theory.

As far as functional limit theorems are concerned, the papers by Kolmogorov (1931, 1933), Erdős and Kac (1946a, 1946b), Doob (1949), Donsker (1951, 1952), Gikhman (1953), Prokhorov (1953), and Kolmogorov and Prokhorov (1954) are considered to be the precursors of the theory. In particular, Donsker (1951) gave the first universal functional limit theorem, called by him an invariance principle. This theorem establishes weak convergence of functionals (continuous in uniform topology) defined on sum-processes constructed from i.i.d.(independent identically distributed) random variables to the same functionals defined on the limiting Wiener process. Kolmogorov and Prokhorov (1954) connected the invariance principle with theorems about weak convergence of measures in the functional metric space of continuous functions.

Prokhorov (1956) completed the general theory of weak convergence of measures in metric spaces and gave general conditions for convergence of continuous stochastic processes in the uniform \mathbf{U} -topology.

Skorokhod (1955a, 1955b, 1956) invented the main topology in the space \mathbf{D} of càdlàg functions, the \mathbf{J} -topology, and gave general conditions for \mathbf{J} -convergence of càdlàg processes. In the papers Skorokhod (1957, 1958), general conditions for \mathbf{J} -convergence of processes with independent increments and Markov processes were also given.

Skorokhod's original approach was based on his representation theorem reducing the weak convergence in space \mathbf{D} to \mathbf{J} -convergence of càdlàg functions. Kolmogorov (1956) has shown that the space \mathbf{D} can be equipped with an appropriate metric that makes the \mathbf{J} -convergence equivalent to the convergence in this metric. The metric, which makes \mathbf{D} a Polish space, was constructed by Billingsley (1968). These results permitted the consideration of limit theorems for càdlàg processes in the framework of the general theory of weak convergence of measures in metric spaces. This approach was used in the book by Billingsley (1968). One can find historical remarks concerning the early period of the development of the theory in the paper by Billingsley and Wishura (2000).

In the paper by Skorokhod (1956), some other topologies, including \mathbf{M} -topology, were also introduced. These topologies are not so widely used since, in most cases, càdlàg

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processes converge in the \mathbf{J} -topology. Nevertheless, they are useful in some special cases. For example, the \mathbf{M} -topology is often applied to extremal processes. The book by Whitt (2002) gives a detailed account of the corresponding results.

The original theory of functional theorems was developed in the case where stochastic processes are defined on a finite interval. An extension of functional limit theorems to stochastic processes that are defined on the interval $[0, \infty)$ is needed in limit theorems for randomly stopped stochastic processes. This is due to the possibility for the random stopping moments to be stochastically unbounded random variables. This extension of the theory was given by Stone (1963) and Lindvall (1973).

To complete the picture, we would also like to mention some other directions of development of the general theory of functional limit theorems. For example, Le Cam (1957), Varadarajan (1961), and Dudley (1966) generalised the main results on weak convergence from metric spaces to spaces of a more general type. Borovkov (1972, 1976) developed a version of the theory based on his methods of individual functionals. Stroock and Varadhan (1969, 1979) developed martingale methods that cover large classes of martingale-type stochastic processes. The general theory originated from this method is given in Liptser and Shiryaev (1986) and Jacod and Shiryaev (1987).

A complete theory can be found in the books by Skorokhod (1961, 1964), Gikhman and Skorokhod (1965, 1971), Parthasarathy (1967), Billingsley (1968), Pollard (1984), Ethier and Kurtz (1986), Liptser and Shiryaev (1986), Jacod and Shiryaev (1987), Davidson (1994), Borovkov, Mogul'skij, and Sakhanenko (1995), Whitt (2002), and Silvestrov (2004). These books also contain bibliographies of works in the area as well as the survey paper by Mishura (2000).

One model that has attracted the attention of many researchers in this area is that of limit theorems for randomly stopped stochastic processes and for compositions of stochastic processes.

This model can appear in a natural way, for example: when studying limit theorems for additive or extremal functionals of stochastic processes; in models connected with a random change of time, change point problems and problems related to optimal stopping of stochastic processes; and in different renewal models, particularly those which appear in applications for risk processes, queuing systems, etc.

The model also appears in statistical applications connected with studies of samples with a random sample size. Such sample models play an important role in sequential analysis. They also appear in sample survey models, or in statistical models where sample variables are associated with stochastic flows. The latter models are typical for insurance, queuing and reliability applications, as well as many others.

The present paper is a survey of results on general conditions of \mathbf{J} -convergence for compositions of stochastic processes.

The first book on this subject was published by me in 1974. Since that time many new results and applications have been developed in this area. These realities have stimulated me to begin work on a new book on this subject. This book was published in 2004. The present survey is mainly based on the results presented in this book.

2. \mathbf{J} -CONVERGENCE OF CÀDLÀG PROCESSES

Let \mathbf{D} be the space of real-valued càdlàg functions, i.e., functions that are continuous from the right and have finite left limits at every point of interval $[0, \infty)$. For every $\varepsilon \geq 0$, let $\zeta_\varepsilon(t), t \geq 0$ be a stochastic càdlàg process, trajectories of which belong with probability 1 to the space \mathbf{D} .

We refer to Gikhman and Skorokhod (1965, 1971), Billingsley (1968), Whitt (2002), and Silvestrov (2004) for the corresponding definitions and general facts about \mathbf{J} -convergence of càdlàg stochastic processes. The symbol $\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0$ as $\varepsilon \rightarrow 0$ is used to show that processes $\zeta_\varepsilon(t)$ converge in topology \mathbf{J} to a process $\zeta_0(t)$ on any interval

$[0, T]$, such that point $0 < T < \infty$ is a point of stochastic continuity of the process $\zeta_0(t)$. This definition is a direct translation of the original definition by Skorokhod (1956) from finite intervals to the interval $[0, \infty)$, and a slight modification of the definitions given by Stone (1963) and Lindvall (1973).

Let us formulate the general conditions for **J**-convergence of càdlàg processes on interval $[0, \infty)$, which directly follow from the conditions of **J**-convergence on finite intervals given by Skorokhod (1956):

- A:** $\zeta_\varepsilon(t), t \in S \Rightarrow \zeta_0(t), t \in S$ as $\varepsilon \rightarrow 0$, where S is a subset of $[0, \infty)$ that is everywhere dense in this interval and contains the point 0;
- B:** $\lim_{c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} P\{\Delta_J(\zeta_\varepsilon(\cdot), c, T) > \delta\} = 0$, $\delta, T > 0$,

where

$$\Delta_J(x(\cdot), c, T) = \sup_{0 \vee (t-c) \leq t' \leq t \leq t'' \leq (t+c) \wedge T} \min(|x(t') - x(t)|, |x(t'') - x(t)|).$$

Note that under conditions **A** and **B**, the set S in **A** can be extended to the set $S \cup S_0$, where S_0 is the set of points of stochastic continuity of the process $\zeta_0(t), t \geq 0$. Note that $S \cup S_0$ is $[0, \infty)$ except for at most a countable set.

Theorem 1. *Let conditions **A** and **B** hold. Then,*

$$\zeta_\varepsilon(t), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t), t \geq 0 \text{ as } \varepsilon \rightarrow 0.$$

It should be noted that conditions **A** and **B** are not only sufficient but also necessary conditions for **J**-convergence of càdlàg processes $\zeta_\varepsilon(t), t \geq 0$, which in its turn implies that condition **A** holds with the set S of points of stochastic continuity of the limiting process, and also that the condition of **J**-compactness **B** holds.

When speaking about conditions for **J**-convergence of compositions of càdlàg stochastic processes, one means sufficient conditions (expressed in terms of external processes and internal stopping processes constructing a composition) which provide Skorokhod's general conditions for **J**-convergence. Components of a composition have usually simpler structure than their composition. This hopefully will lead to conditions for **J**-convergence, expressed in terms of components, which can be simpler than the general conditions of **J**-convergence. Let us formulate the problem more precisely.

Let \mathbf{D}_+ be the subspace of non-negative and non-decreasing functions from \mathbf{D} . For every $\varepsilon \geq 0$, let $\xi_\varepsilon(t), t \geq 0$ and $\nu_\varepsilon(t), t \geq 0$ be stochastic càdlàg processes, trajectories of which belong with probability 1, respectively, to the spaces \mathbf{D} and \mathbf{D}_+ . We are interested in their composition $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$. This process is also a càdlàg process.

In the case, where the process $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$ is a composition, it is natural to try to find conditions sufficient for **A** and **B** expressed in terms of components of composition.

Natural candidates that are expected to provide **J**-convergence of the compositions $\zeta_\varepsilon(t), t \geq 0$ are the following three conditions:

- C:** $(\nu_\varepsilon(s), \xi_\varepsilon(t)), (s, t) \in V \times U \Rightarrow (\nu_0(s), \xi_0(t)), (s, t) \in V \times U$ as $\varepsilon \rightarrow 0$, where V, U are some everywhere dense subsets in $[0, \infty)$ containing 0;
- D:** $\lim_{c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} P\{\Delta_J(\xi_\varepsilon(\cdot), c, T) > \delta\} = 0$, $\delta, T > 0$;
- E:** $\lim_{c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} P\{\Delta_J(\nu_\varepsilon(\cdot), c, T) > \delta\} = 0$, $\delta, T > 0$.

As above, it can be noted that, under conditions **C** and **D** the sets V and U can be extended to the sets $V \cup V_0$ and $U \cup U_0$, where V_0 and U_0 are the sets of points of stochastic continuity of the processes $\nu_0(t), t \geq 0$ and $\xi_0(t), t \geq 0$, respectively. Since processes $\nu_\varepsilon(t)$ are monotonic, it is not necessary to involve the condition **E**. Note that $V \cup V_0$ and $U \cup U_0$ are $[0, \infty)$ except for at most some countable sets.

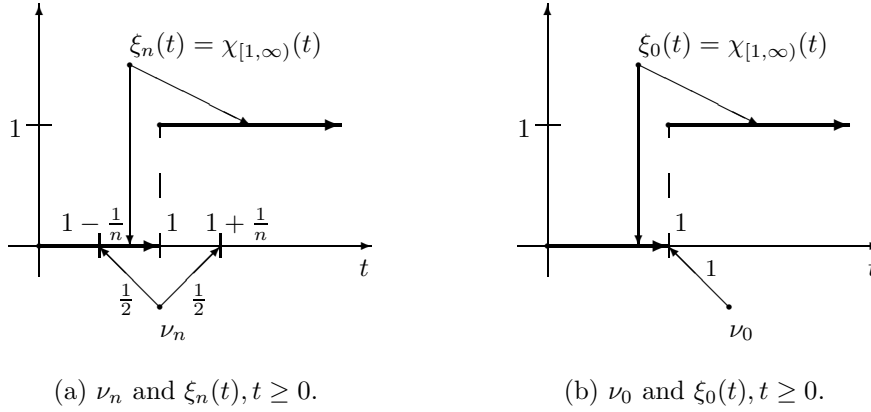


FIGURE 1. First-type continuity condition.

Conditions **C** and **D** provide \mathbf{J} -convergence of processes $\xi_\varepsilon(t), t \geq 0$. Conditions **C** and **E** provide the same for processes $\nu_\varepsilon(t), t \geq 0$. But together, all three conditions **C**, **D**, and **E** do not provide \mathbf{J} -convergence, neither for the vector processes $(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ nor for the compositions $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$. Some conditions additional to **C**, **D**, and **E** must be assumed for the processes $\xi_\varepsilon(t)$ and $\nu_\varepsilon(t)$, to provide desirable \mathbf{J} -convergence of compositions.

As follows from Theorem 1, the problem can be split in two subproblems: the first one to give sufficient conditions for the weak convergence of compositions (condition **A**), and the second to give sufficient conditions for the \mathbf{J} -compactness of compositions (condition **B**).

3. WEAK CONVERGENCE FOR COMPOSITIONS OF CÀDLÀG PROCESSES

In all examples presented below, we use a natural number n as a parameter, instead of ε , to index the corresponding càdlàg stochastic processes. Actually, we can always assume that $\varepsilon = n^{-1}$ for $n \geq 1$ and $\varepsilon = 0$ for $n = 0$.

Condition **C** can be expected to provide weak convergence of compositions on some set W dense in $[0, \infty)$.

The following example shows that this hypothesis is not true. Let the stopping process $\nu_n(t) = \nu_n, t \geq 0$ be in fact a random variable (stopping moment) that takes values $1 - n^{-1}$ and $1 + n^{-1}$ with probability $\frac{1}{2}$ and $\xi_n(t) = \chi_{[1, \infty)}(t), t \geq 0$, for $n \geq 1$. In this case, condition **C** obviously holds. The limiting stopping moment $\nu_0 = 1$ with probability 1 and the limiting process $\xi_0(t) = \chi_{[1, \infty)}(t), t \geq 0$. However, $\xi_n(\nu_n)$ is a random variable that takes values 0 and 1 with probability $\frac{1}{2}$, while $\xi_0(\nu_0) = 1$ with probability 1. Therefore, the random variables $\xi_n(\nu_n)$ do not weakly converge. Figure 1 illustrates this example.

In this example, the limiting stopping moment is a discontinuity point for the corresponding limiting external process. The oscillation of stopping moments in a neighbourhood of this discontinuity point causes that compositions $\xi_n(\nu_n)$ do not weakly converge.

Let us denote $R[x(\cdot)]$ the set of discontinuity points for a càdlàg function $x(t), t \geq 0$.

The example considered above leads to the following hypothesis. In order to provide desirable weak convergence of compositions on some set W it is enough to add to **C** the condition that the limiting process $\xi_0(t), t \geq 0$ is continuous at a random point $\nu_0(w)$ with probability 1 for every w from set W , i.e., to assume that the following condition holds:

$$\mathbf{F}_W: P\{\nu_0(w) \in R[\xi_0(\cdot)]\} = 0 \text{ for } w \in W.$$

The following more sophisticated example shows that this hypothesis is also not true. Conditions **C** and \mathbf{F}_W together are not sufficient to provide weak convergence of compositions on set W without some additional assumptions.

Let ξ_k , $k = 0, 1, \dots$ be a sequence of non-negative i.i.d. random variables with a continuous distribution function $F(x)$ and $\zeta_n = \max_{0 \leq k \leq n} \xi_k$ be the maximum of the first $n + 1$ random variables of this sequence. Let us introduce the random variables $\mu_n = \min(r : \xi_r = \zeta_n)$. By the definition, $\zeta_n = \xi_{\mu_n}$. The last representation can be rewritten in the following form: $\zeta_n/n = \xi_n(\nu_n)$, where $\xi_n(t) = \xi_{[tn]}/n$, $t \geq 0$ and $\nu_n(t) = \nu_n = \mu_n/n$, $t \geq 0$.

It is easy to see that the random variable μ_n takes values $0, \dots, n$ with probability $\frac{1}{n+1}$ and, hence, $\nu_n \Rightarrow \nu_0$ as $n \rightarrow \infty$, where ν_0 is a random variable uniformly distributed in $[0, 1]$. Since the random variables ξ_k , $k = 0, 1, \dots$ are i.i.d. random variables, $\xi_n(t) \xrightarrow{P} 0$ as $n \rightarrow \infty$, for $t \geq 0$. From Slutsky theorem and the above convergence relations, it follows that $(\nu_n, \xi_n(t)), t \geq 0 \Rightarrow (\nu_0, 0), t \geq 0$ as $n \rightarrow \infty$.

So, condition **C** holds. Condition \mathbf{F}_W also holds, with the set $W = [0, \infty)$, since the limiting process $\xi_0(t) = 0$, $t \geq 0$ is continuous.

In this case, $\xi_0(\nu_0) = 0$. However, the random variables $\zeta_n/n = \xi_n(\nu_n)$ may not converge weakly to 0 as $n \rightarrow \infty$. For example, let $F(x) = \chi_{[1, \infty)}(x)(1 - 1/x)$. Then $P\{\zeta_n/n < x\} = F(xn)^n \rightarrow \exp\{-x^{-1}\}$ as $n \rightarrow \infty$, for $x > 0$. This means that the random variables $\zeta_n/n \Rightarrow \zeta$ as $n \rightarrow \infty$, where ζ is a non-negative random variable which has the distribution function $P\{\zeta \leq x\} = \chi_{[0, \infty)}(x) \exp\{-x^{-1}\}$.

An explanation of the example above is that the processes $\xi_n(t)$, $t \geq 0$ weakly converge to the zero-process $\xi_0(t) = 0$, $t \geq 0$, but these processes do not converge in the topology **J** or even in the weaker topology **M**. They can possess too large oscillations in small intervals. This effect has the result that the max-processes $\xi_n^+(t) = \sup_{s \leq t} \xi_n(s)$, $t \geq 0$ do not converge weakly to the corresponding zero max-process $\xi_0^+(t) = \sup_{s \leq t} \xi_0(s) = 0$, $t \geq 0$. At the same time, the stopping moments ν_n and the external processes $\xi_n(t)$, $t \geq 0$ are connected so that $\xi_n(\nu_n) = \xi_n^+(\nu_n)$.

The above example shows that, in order to provide weak convergence of compositions on set W , one should add, to the condition of joint weak convergence **C** and the continuity condition \mathbf{F}_W , an additional compactness condition, for example, this can be the **J**-compactness condition **D**.

Let us use notation \mathbf{F}_w for the condition \mathbf{F}_W in the case where set $W = \{w\}$ contains only the point w . Let also denote by W_0 the set of all points w such that condition \mathbf{F}_w holds.

Let also V_0 be the set of all points of stochastic continuity of the process $\nu_0(t)$, $t \geq 0$. Note that, due to monotonicity of the processes $\nu_\varepsilon(t)$, $t \geq 0$, and the assumption that the set V is dense in $[0, \infty)$, the set V can be extended to the set $V \cup V_0$ in condition **C**. The set $V \cup V_0$ coincides with $[0, \infty)$ except for at most a countable set.

The following result was obtained in Silvestrov (1971a, 1972a).

Theorem 2. *Let conditions **C** and **D** hold. Then, for the set $S = (V \cup V_0) \cap W_0$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \rightarrow 0.$$

The set W_0 can be empty, but, under some natural assumptions, this set is dense in $[0, \infty)$, more over, it coincides with $[0, \infty)$ except for at most a countable set.

The problem has an additional new aspect if one does not prescribe a set of weak convergence but would like to guarantee only the weak convergence of compositions $\xi_\varepsilon(\nu_\varepsilon(t))$ on some subset S dense in the interval $[0, \infty)$. This is an important problem since this is required in the basic condition **A**.

Let us introduce the following condition:

$$\mathbf{G}: P\{\nu_0(t') = \nu_0(t'') \in R[\xi_0(\cdot)]\} = 0 \text{ for } 0 \leq t' < t'' < \infty.$$

The following useful lemma from Silvestrov and Teugels (2004) gives the necessary and sufficient condition for the set W_0 to be dense in $[0, \infty)$.

Lemma 1. *The condition \mathbf{G} is necessary for \mathbf{F}_W to hold for some set W , which is dense in $[0, \infty)$, and sufficient for \mathbf{F}_W to hold for some set W , which is $[0, \infty)$ except for at most a countable set.*

Condition \mathbf{G} holds, for example, if the process $\nu_0(t), t \geq 0$ is strictly monotonic with probability 1.

This condition also holds if the process $\xi_0(t) = \xi'_0(t) + \xi''_0(t), t \geq 0$ can be decomposed in a sum of two càdlàg processes such that the process $\xi'_0(t), t \geq 0$ is continuous with probability 1 while the process $\xi''_0(t), t \geq 0$ is stochastically continuous and independent of the process $\nu_0(t), t \geq 0$.

Condition \mathbf{G} guarantees that the set of weak convergence $S = (V \cup V_0) \cap W_0$, which appears in Theorem 2, coincides with $[0, \infty)$ except for at most a countable set.

It should be noted that condition \mathbf{G} does not guarantee that set S contains a pre-assigned point $w \in [0, \infty)$. In order to include a point w in the set of convergence, one should assume that condition \mathbf{F}_w holds and also to require that $w \in V \cup V'_0$. Since the point $0 \in V$, in this case, one should require condition \mathbf{F}_0 to hold, in order to include this point in the set of weak convergence.

The following variant of Theorem 2 gives conditions that guarantee that the basic condition of weak convergence \mathbf{A} holds for compositions of càdlàg processes.

Theorem 3. *Let conditions \mathbf{C} , \mathbf{D} , \mathbf{G} , and \mathbf{F}_0 hold. Then the set $S = (V \cup V_0) \cap W_0$ is $[0, \infty)$ except for at most a countable set, contains the point 0, and,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \rightarrow 0.$$

Theorem 3 gives conditions that, together, imply the desirable weak convergence of randomly stopped càdlàg processes. These conditions are: the condition \mathbf{C} of joint weak convergence of random stopping moments and external stochastic processes; the condition \mathbf{D} of \mathbf{J} -compactness of external stochastic processes; and the conditions of continuity \mathbf{G} and \mathbf{F}_0 , which guarantee that the limiting process $\nu_0(s)$ stops the limiting external process $\xi_0(t)$ at its points of continuity for every s from some set dense in $[0, \infty)$.

This combination makes a good balance between the conditions imposed on the pre-limiting external and stopping processes, on the one hand, and the limiting external and stopping processes, on the other. Pre-limiting joint distributions of stopping and external processes usually have a complicated structure. However, these distributions are involved only in the simplest and most natural way via the condition of their joint weak convergence. The second \mathbf{J} -compactness condition involves only the external processes themselves and not the stopping processes. This condition is a standard one. It was thoroughly studied for various classes of càdlàg stochastic processes. The continuity conditions involve joint distributions of the limiting stopping process and the limiting external process. These limiting joint distributions are usually simpler than the corresponding pre-limiting joint distributions. This permits one to check the continuity conditions in various practically important cases. Due to a balance between the conditions imposed on the pre-limiting and limiting external processes and internal stopping processes Theorems 2 and 3 are effective tools for use in limit theorems for randomly stopped stochastic processes.

The continuity condition \mathbf{G} is effective in the cases where the process $\nu_0(s)$ mostly stops the process $\xi_0(t)$ at points of continuity of this process. However, there are cases, where the process $\nu_0(s)$ stops the $\xi_0(t)$ at its discontinuity point for all s from some interval $[s', s'']$. This may happen, for example, for compositions with renewal type stopping. In such cases condition \mathbf{G} does not hold and should be replaced by some weaker condition.

Let us denote by $\alpha_{\varepsilon k}^{(\delta)}$ the successive moments of jumps of the process $\xi_\varepsilon(t)$, $t \geq 0$, with absolute values of jumps greater than or equal to δ .

Let also Z_0 be the set of all $\delta > 0$ such that the process $\xi_0(t)$, $t \geq 0$ has no jumps with the absolute value equal to δ with probability 1.

Let us introduce the following condition:

H_W: There exist a sequence $\delta_l \in Z_0$, $\delta_l \rightarrow 0$ as $l \rightarrow \infty$, and a sequence $0 < T_r \rightarrow \infty$ as $r \rightarrow \infty$, such that, for every $l, k, r \geq 1$, $\lim_{0 < c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \mathbb{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(w) < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0$ for $w \in W$.

Lemma 2. *Condition **F_W** implies condition **H_W**.*

Let us use notation **H_w** for the condition **H_W** in the case where set $W = \{w\}$ contain only the point w . Let also denote by W'_0 the set of all points w such that condition **H_w** holds.

Note that, according Lemma 2, the set $W_0 \subseteq W'_0$.

The following theorem from Silvestrov (2004) improves the result given in Theorem 2.

Theorem 4. *Let conditions **C** and **D** hold. Then for the set $S = (V \cup V_0) \cap W'_0$,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \rightarrow 0.$$

Let us now introduce the following condition:

I: There exist a sequence $\delta_l \in Z_0$, $\delta_l \rightarrow 0$ as $l \rightarrow \infty$, and a sequence $0 < T_r \rightarrow \infty$ as $r \rightarrow \infty$, such that, for every $l, k, r \geq 1$, $\lim_{0 < c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \mathbb{P}\{\alpha_{\varepsilon k}^{(\delta_l)} - c \leq \nu_\varepsilon(t'), \nu_\varepsilon(t'') < \alpha_{\varepsilon k}^{(\delta_l)}, \alpha_{\varepsilon k}^{(\delta_l)} < T_r\} = 0$ for all $0 \leq t' < t'' < \infty$.

The following lemma is an analogue of Lemma 1.

Lemma 3. *The condition **I** is necessary for **H_W** to hold for some set W dense in $[0, \infty)$, and sufficient for **H_W** to hold for some set W which is $[0, \infty)$ except for at most a countable set.*

Condition **I** guarantees that the set of weak convergence $S = (V \cup V_0) \cap W'_0$, which appears in Theorem 4, coincides with $[0, \infty)$ except for at most a countable set.

It should be noted that condition **I** does not guarantee that set S contains a pre-assigned point $w \in [0, \infty)$. In order to include a point w in the set of convergence, one should assume that condition **H_w** holds and also to require that $w \in V \cup V'_0$. Since the point $0 \in V$, in this case, one should require condition **H₀** to hold, in order to include this point in the set of weak convergence.

The following variant of Theorem 4 gives conditions (weaker than in Theorem 3) that guarantee that the basic condition of weak convergence **A** holds for compositions of càdlàg processes.

Theorem 5. *Let conditions **C**, **D**, **I**, and **H₀** hold. Then the set $S = (V \cup V_0) \cap W'_0$ is $[0, \infty)$ except for at most a countable set, contains the point 0, and,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \in S \Rightarrow \zeta_0(t) = \xi_0(\nu_0(t)), t \in S \text{ as } \varepsilon \rightarrow 0.$$

As was mentioned above, Theorems 2 and 3, based on the continuity condition **G**, do not cover the cases where the limiting internal process stops the limiting external process at its discontinuity points. This case is covered by Theorems 4 and 5. In Theorem 5, condition **G** is replaced by the weaker condition **I** that ensures the right positioning of the pre-limiting stopping moments on the right-hand side of the moments where the pre-limiting external processes experience large jumps. This condition does involve the pre-limiting joint distributions of stopping processes and external processes not only via their joint distributions but also via the joint distributions of stopping processes and moments

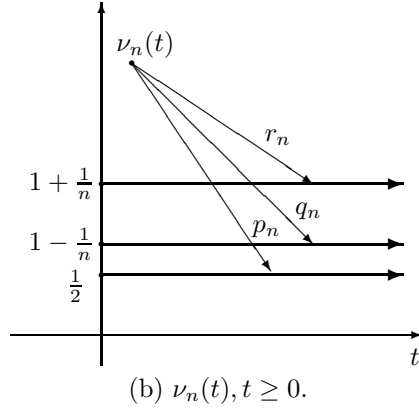
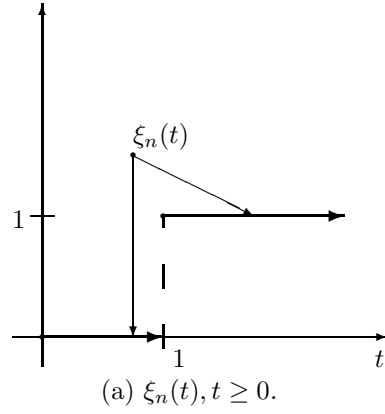


FIGURE 2. Second-type continuity condition.

of large jumps for external processes. The latter distributions are not so complicated and the corresponding conditions can effectively be verified in some important cases.

In conclusion of discussion concerned the conditions for weak convergence of compositions of càdlàg processes, let us consider the following example shown in Figures 2, 3, and 4.

Let $\xi_n(t) = \chi_{[1, \infty)}(t)$, $t \geq 0$, for $n \geq 1$. Let also, for $n \geq 1$, the process $\nu_n(t), t \geq 0$ have three possible realisations that occur with probabilities p_n , q_n and r_n , where $p_n + q_n + r_n = 1$. These realisations are $\frac{1}{2}$ for $t \geq 0$, $1 - n^{-1}$ for $t \geq 0$, and $1 + n^{-1}$ for $t \geq 0$.

We assume that probabilities p_n , q_n and r_n converge as $n \rightarrow \infty$ to the limiting values p_0 , q_0 and r_0 , respectively.

In this case, condition **C** obviously holds. The limiting process $\xi_0(t) = \chi_{[1, \infty)}(t)$, $t \geq 0$. The limiting stopping process $\nu_0(t), t \geq 0$ has only two possible realisations that occur with the probabilities p_0 and $q_0 + r_0$, respectively. These realisations are $\frac{1}{2}$ for $t \geq 0$, and 1 for $t \geq 0$. The condition of **J**-compactness **D** also holds. For $n \geq 1$, the composition $\xi_n(\nu_n(t)), t \geq 0$ has two possible realisations that occur with the probabilities $p_n + q_n$ and r_n , respectively. These realisations are 0 for $t \geq 0$, and 1 for $t \geq 0$. The limiting composition $\xi_0(\nu_0(t)), t \geq 0$ has the same two possible realisations but they occur with the probabilities p_0 and $q_0 + r_0$, respectively.

Conditions **G** and **F**₀ hold if and only if (a) $p_0 = 1$. In this case, the limiting composition $\xi_0(\nu_0(t)), t \geq 0$ has with probability 1 only one realisation, 0 for $t \geq 0$. The compositions $\xi_n(\nu_n(t))$ weakly converge to $\xi_0(\nu_0(t))$ on the set $S = [0, \infty)$.

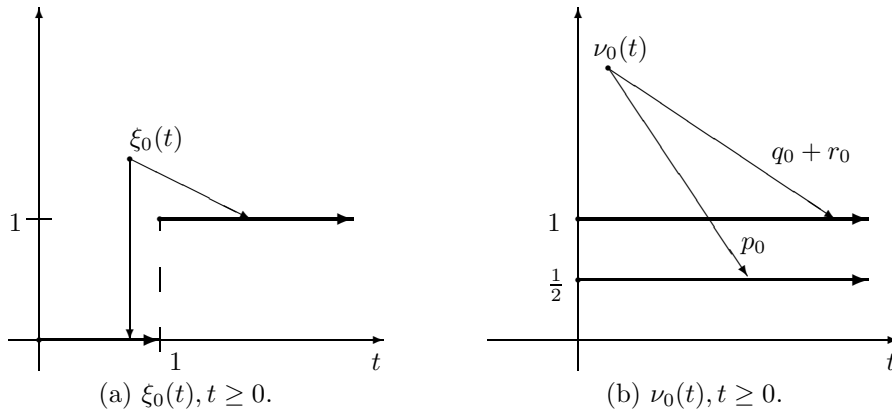


FIGURE 3. Second-type continuity condition.

Conditions **I** and **H**₀ hold if and only if (b) $q_0 = 0$. If also $p_0 < 1$, then conditions **I** and **H**₀ hold but **G** and **F**₀ do not. If $q_0 = 0$, the limiting composition $\xi_0(\nu_0(t)), t \geq 0$ has two possible realisations 0 for $t \geq 0$, and 1 for $t \geq 0$. They occur with the probabilities p_0 and r_0 , respectively. Again, the $\xi_n(\nu_n(t))$ weakly converge to $\xi_0(\nu_0(t))$ on the set $S = [0, \infty)$.

Neither conditions **G** and **F**₀ nor conditions **I** and **H**₀ hold, if (c) $q_0 > 0$. In this case $p_n + q_n \not\rightarrow p_0$ and $r_n \not\rightarrow q_0 + r_0$ as $n \rightarrow \infty$. The compositions $\xi_n(\nu_n(t))$ do not weakly converge to $\xi_0(\nu_0(t))$ for every $t \in [0, \infty)$.

These statements are consistent with the remarks above.

There exists a huge literature concerned weak limit theorems for randomly stopped stochastic processes mainly related to the models, where stochastic processes stopped at random moments that are independent or asymptotically independent of the external processes. The originating works here are Wald (1945), Robbins (1948), Kolmogorov and Prokhorov (1949), Anscombe (1952), Dobrushin (1955), and Rényi (1957). We refer to books by Kruglov and Korolev (1990), Gnedenko and Korolev (1996), and Bening and Korolev (2002) which give an extended presentation of results and bibliographies of works related to these models. An extended bibliography is also given in Silvestrov (2004).

As far as the model, where the limiting process is the composition of the limiting external process and the limiting internal stopping process that can be dependent in an arbitrary way, is concerned, the first general result was given in Billingsley (1968). There, the author deal with the case when the external limiting process is continuous. This result was improved to the form of Theorem 2, where the external limiting process is a càdlàg process, in Silvestrov (1971a, 1972a). Theorems 4 and 5 present new results from Silvestrov (2004).

4. **J**-CONVERGENCE FOR COMPOSITIONS OF CÀDLÀG PROCESSES

As was mentioned in above, conditions **C**, **D**, and **E**, which are natural candidates that are expected to provide **J**-convergence of compositions of càdlàg processes are not sufficient for this. As follows from the examples given in Section 3, the continuity condition **G** or the weaker continuity condition **I** should be additionally assumed to provide the required basic condition **A** of weak convergence for compositions on some set dense in $[0, \infty)$.

The following example let us clarify situation with another basic condition **B** of **J**-compactness for compositions of càdlàg processes.

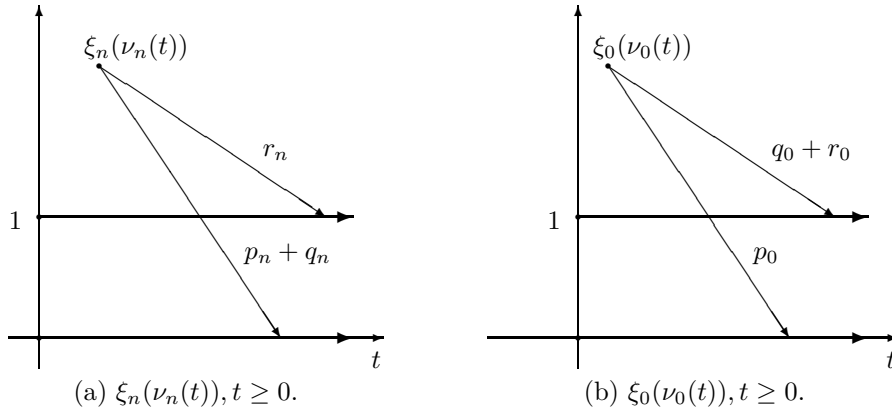


FIGURE 4. Second-type continuity condition.

Let us consider the following example illustrated in Figures 5 and 6. We define $\xi_n(t) = \chi_{[1, \frac{7}{4})}(t)$, $t \geq 0$, for $n \geq 1$, and $\nu_n(t) = t + n^{-1}$ if $t < 1$ and $t + 1$ if $t \geq 1$, for $n \geq 1$.

In this case, condition **C** obviously holds. The corresponding limiting process $\xi_0(t) = \chi_{[1, \frac{7}{4})}(t)$, $t \geq 0$, and the limiting stopping process $\nu_0(t) = t$ if $t < 1$ and $t + 1$ if $t \geq 1$. Conditions **D** and **E** also hold.

In this case, the composition $\xi_n(\nu_n(t)) = \chi_{[1-n^{-1}, 1)}(t)$, $t \geq 0$, while $\xi_0(\nu_0(t)) = 0$, $t \geq 0$.

The process $\xi_n(\nu_n(t))$, $t \geq 0$ has two jumps with the values 1 and -1 in the close points $1 - n^{-1}$ and 1, respectively. So, $\Delta_J(\xi_n(\nu_n(\cdot)), c, T) = 1$ if $n^{-1} < c$ and, therefore, $\lim_{c \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} \Delta_J(\xi_n(\nu_n(\cdot)), c, T) = 1$. This shows that the condition of **J**-compactness **B** does not hold for the processes $\xi_n(\nu_n(t))$, $t \geq 0$ since the left limiting value of the limiting stopping process $\nu_0(t)$, $t \geq 0$, at point 1, which is a point of discontinuity for the limiting stopping process, is $\nu_0(1 - 0) = 1$. This value is a point of discontinuity for the external limiting processes $\xi_0(t)$, $t \geq 0$. The example can be easily modified such that the right limiting value of the limiting stopping process at a discontinuity point would cause the same effect.

The example considered above leads to the following hypothesis. In order to provide **J**-compactness of compositions it is enough to add, to the conditions **C**, **D** and **E**, the following condition:

$$\mathbf{J}: \mathbf{P}\{\nu_0(t \pm 0) \notin R[\xi_0(\cdot)] \text{ for } t \in R[\nu_0(\cdot)]\} = 1.$$

This hypothesis is true as shows the following theorem from Silvestrov (1974).

Theorem 6. *Let conditions **C**, **D**, **E**, and **J** hold. Then,*

$$\lim_{c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \mathbf{P}\{\Delta_J(\xi_\varepsilon(\nu_\varepsilon(\cdot)), c, T) > \delta\} = 0, \quad \delta, T > 0.$$

Now we can give general conditions for **J**-convergence of compositions of càdlàg processes. This can be done by combining conditions for weak convergence of compositions given in Theorem 3 or in Theorem 5, with conditions for **J**-compactness given in Theorem 6.

The following theorem is from Silvestrov (1974).

Theorem 7. *Let conditions **C**, **D**, **E**, **G**, **F**₀, and **J** hold. Then,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \rightarrow 0.$$

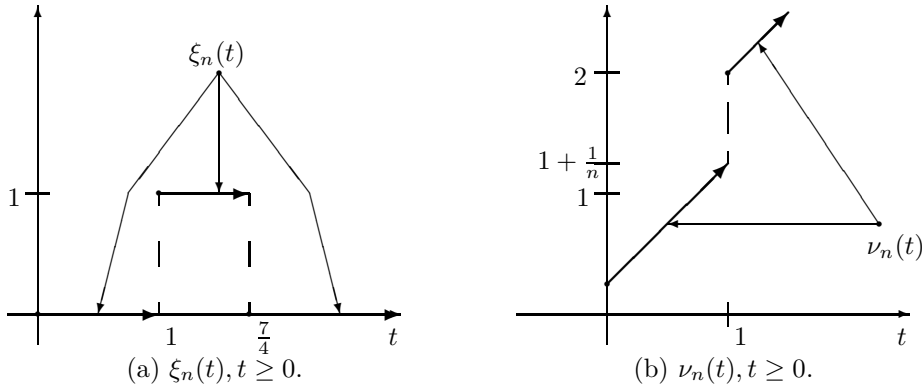


FIGURE 5. Third-type continuity condition.

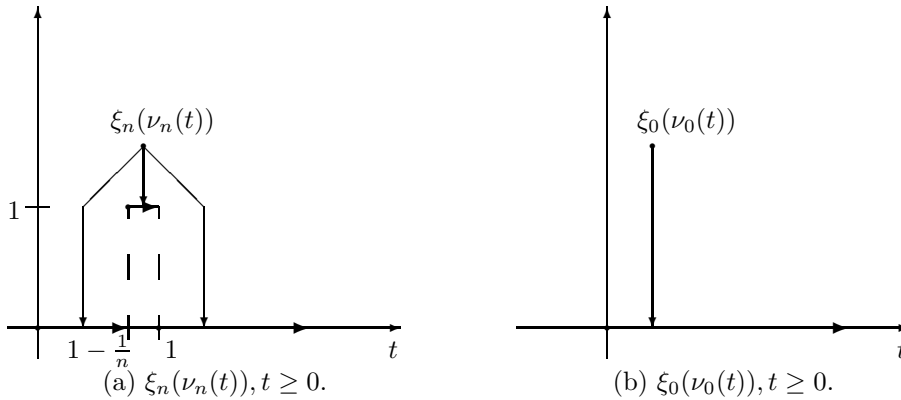


FIGURE 6. Third-type continuity condition.

In the example given above, the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ **J**-converge. However, the compositions $\xi_n(\nu_n(t)), t \geq 0$ can **J**-converge even if the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ do not **J**-converge.

Let us modify the example considered above and shown in Figures 5 and 6. Figure 7 illustrates this modified example. We use the same external processes $\xi_n(t) = \chi_{[1, \frac{7}{4}]}(t), t \geq 0$, for $n \geq 1$, but define new internal stopping processes $\nu_n(t) = \frac{1}{2}(1 - n^{-1})^{-1}t$ if $t < 1 - n^{-1}$ and $t + 1 + n^{-1}$ if $t \geq 1 - n^{-1}$, for $n \geq 1$.

In this case, the corresponding limiting process $\xi_0(t) = \chi_{[1, \frac{7}{4}]}(t), t \geq 0$, and the limiting stopping process $\nu_0(t) = \frac{1}{2}t$ if $t < 1$ and $t + 1$ if $t \geq 1$. Hence, $\xi_n(\nu_n(t)) = 0, t \geq 0$, for $n \geq 1$ as well as for $n = 0$. Therefore, the compositions $\xi_n(\nu_n(t)), t \geq 0$ **J**-converge. This is consistent with Theorem 7, since all conditions of this theorem hold.

However, the vector processes $(\nu_n(t), \xi_n(t)), t \geq 0$ do not **J**-converge, since the process $(\nu_n(t), \xi_n(t))$ has two large jumps with the absolute values $\frac{3}{2}$ and 1 in the close points $1 - n^{-1}$ and 1, respectively.

Theorem 7 serves properly for the models, where limiting external processes are randomly stopped at continuity points. The models, where external processes are randomly stopped at discontinuity points, do require to improve this result. In this case, the following theorem from Silvestrov (2004) provides effective conditions for **J**-convergence of compositions of càdlàg processes.

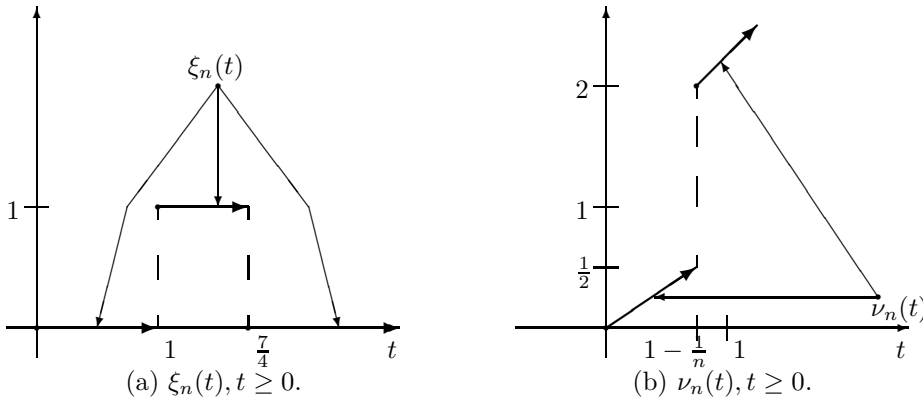


FIGURE 7. Third-type continuity condition.

Theorem 8. *Let conditions **C**, **D**, **E**, **I**, **H**₀, and **J** hold. Then,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \rightarrow 0.$$

Let us go back to the example considered in Subsection 3 and shown in Figures 2, 3, and 4. In this example, conditions **C**, **D**, **E**, and the third-type continuity conditions **J** hold.

In the case (a) $p_0 = 1$, conditions **G** and **F**₀ hold. Therefore, according to Theorem 7, the compositions $\xi_n(\nu_n(t)), t \geq 0$ **J**-converge to $\xi_0(\nu_0(t)), t \geq 0$ as $n \rightarrow \infty$.

In the case (b) $q_0 = 0, p_0 < 1$, conditions **G** and **F**₀ do not hold. However, in this case, the weaker conditions **I** and **H**₀ hold. Therefore, according to Theorem 8, the compositions $\xi_n(\nu_n(t)), t \geq 0$ **J**-converge to $\xi_0(\nu_0(t)), t \geq 0$ as $n \rightarrow \infty$.

This example is, of course, an artificial one. In the next two sections, we illustrate the results given in Theorems 7 and 8, in particular, clarify the difference in the conditions of these theorems, by applying these theorems for two classical models of randomly stopped processes with random scaling of time and renewal type stopping.

The preceding results mainly relate to Theorem 7. The simplest case, where both limiting external and stopping processes are continuous was considered in Billingsley (1968). This result was extended in various directions in Silvestrov (1971b, 1972c, 1972d), Serfozo (1973), and Whitt (1980). The case, where the limiting external process is continuous, was considered in Silvestrov (1974) and Whitt (1980). The case, where the limiting stopping process is continuous, is important for many applications. This case was considered in Silvestrov (1972b, 1974). The general case, where both the limiting external and stopping processes can be discontinuous, was considered in Silvestrov (1974). These results are presented in Theorem 7. Theorem 8 presents a new result from Silvestrov (2004).

5. J-CONVERGENCE OF COMPOSITIONS WITH RANDOM SCALING OF TIME

In this section, we consider the model, where stopping process has the following simple form $\nu_\varepsilon(t) = t\nu_\varepsilon, t \geq 0$, where ν_ε is a non-negative random variable. In this case, the composition has the form $\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon), t \geq 0$.

The most well known variant is represented by the model of random sums. Let, for every $\varepsilon > 0$, $\xi_{\varepsilon,n}, n = 1, 2, \dots$ be a sequence of real-valued random variables and μ_ε be a non-negative random variable. Further, we need a non-random function $n_\varepsilon > 0$ of parameter ε such that $n_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Consider a sum-process with non-random stopping index, $\xi_\varepsilon(t) = \sum_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}, t \geq 0$, and an analogue of this process the stopping

index of which is random, $\zeta_\varepsilon(t) = \sum_{k \leq t\mu_\varepsilon} \xi_{\varepsilon,k}$, $t \geq 0$. Denote by $\nu_\varepsilon = \mu_\varepsilon/n_\varepsilon$ the normalised random stopping index. Then the process $\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon)$, $t \geq 0$ can be represented in the form of composition of two processes $\xi_\varepsilon(t)$, $t \geq 0$ and $\nu_\varepsilon(t) = t\nu_\varepsilon$, $t \geq 0$.

Another classical variant is represented by the model of extremes with random sample size. It is defined in the way similar with the case of random sums. The only difference is that the external process is defined as $\xi_\varepsilon(t) = \max_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}$, $t \geq 0$ while the stopping process again is $\nu_\varepsilon(t) = t\nu_\varepsilon$, $t \geq 0$.

Consider the following weak convergence condition:

K: $(\nu_\varepsilon, \xi_\varepsilon(t)), t \in U \Rightarrow (\nu_0, \xi_0(t)), t \in U$ as $\varepsilon \rightarrow 0$, where (a) ν_0 is a non-negative random variable, (b) $\xi_0(t)$, $t \geq 0$ is a càdlàg process, (c) U is a subset of $[0, \infty)$ that is dense in this interval and contains the point 0.

Condition **K** obviously implies condition **C** to hold. The processes $\nu_\varepsilon(t) = t\nu_\varepsilon$, $t \geq 0$ are monotonic and the corresponding limiting process $\nu_0(t) = t\nu_0$, $t \geq 0$ is continuous. These imply that the processes $\nu_\varepsilon(t)$ are compact in **J** topology, i.e., condition **E** holds. Also conditions **G** and **F**₀ hold because of the process $\nu_0(t)$ is continuous. Therefore, conditions of Theorems 3 and 7 are reduced to only two conditions **K** and **D**.

In this case, we can give an explicit description of the set of weak convergence $S = (V_0 \cup V'_0) \cap W_0$. Obviously, the sets $V_0 = V'_0 = [0, \infty)$ and, thus, the set of weak convergence $S = W_0$.

By the definition, set W_0 is the set of $t \geq 0$ such that $P\{\tau_{kn}/\nu_0 = t\} = 0$ for all k , $n = 1, 2, \dots$, where τ_{kn} , $k = 1, 2, \dots$ are successive moments of jumps of the process $\xi_0(t)$, $t \geq 0$, with absolute values of the jumps lying in the interval $[\frac{1}{n}, \frac{1}{n-1})$. Note that the random variables τ_{kn} take values in the interval $(0, \infty]$ and the random variable ν_0 takes values in the interval $[0, \infty)$. So, the random variable τ_{kn}/ν_0 takes values in the interval $(0, \infty]$, that is, it is positive and, possibly, improper.

The set W_0 coincides with $[0, \infty)$ except for at most a countable set. Also, $0 \in W_0$. Indeed, the set $\overline{W}_0 = [0, \infty) \setminus W_0$ coincides with the set of all atoms of the distribution functions of the random variables τ_{kn}/ν_0 , $k, n = 1, 2, \dots$. This set is at most countable and $0 \notin \overline{W}_0$. Therefore, the set W_0 equals $[0, \infty)$ except for the set \overline{W}_0 . Also, $0 \in W_0$. Note also that W_0 is the set of points of stochastic continuity for the process $\xi_0(t\nu_0)$, $t \geq 0$.

The following theorem is a direct corollary of the results in Silvestrov (1971a, 1972a, 1972b).

Theorem 9. *Let conditions **K** and **D** hold. Then,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(t\nu_\varepsilon), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \xi_0(t\nu_0), t \geq 0 \text{ as } \varepsilon \rightarrow 0.$$

We consider a general model where the corresponding external processes $\xi_\varepsilon(t)$, $t \geq 0$ and stopping indices ν_ε can be dependent in an arbitrary way. As follows from Theorem 9, weak convergence as well as **J**-convergence of the corresponding randomly stopped processes $\xi_\varepsilon(t\nu_\varepsilon)$, $t \geq 0$ can be obtained under only two conditions. The first one is the condition **K** of joint weak convergence of the random stopping indices and external processes with non-random stopping indices, and the second the condition **D** of **J**-compactness of the external processes. No extra assumptions on their independence or even on their asymptotic independence are required.

Note also that in the classical case of sum-processes with random indices and i.i.d. summands, the condition of **J**-compactness **D** can be omitted since it is implied, in this case, by condition **K**.

In some sense, the result of Theorem 9 is surprising. It gives a unified approach to various concrete models including those mentioned above sum- and max-processes with

random indices. Being applied to these models, Theorem 9 covers the most of the preceding results, without any additional assumptions about independence or asymptotical independence of random stopping indices and external processes.

6. J-CONVERGENCE OF COMPOSITIONS WITH RENEWAL TYPE STOPPING

Let $\alpha_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$ be, for every $\varepsilon \geq 0$, a two-dimensional càdlàg process with real-valued components. In this section we consider the model, where stopping process has the form $\nu_\varepsilon(t) = \inf(s \geq 0 : \kappa_\varepsilon(s) > t), t \geq 0$ and, therefore, the composition has the form $\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0$. In Silvestrov (1974, 2000, 2004) the process $\nu_\varepsilon(t)$ is referred as an exceeding time process and $\zeta_\varepsilon(t)$ as a generalised exceeding time process.

The most well known variant is represented by the model of generalised renewal process. Let, for every $\varepsilon > 0$, $\alpha_{\varepsilon,n} = (\kappa_{\varepsilon,n}, \xi_{\varepsilon,n}), n = 1, 2, \dots$ be a sequence of random vectors with real-valued components. Further, we need a non-random function $n_\varepsilon > 0$ of parameter ε such that $n_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Consider a sum-process $\alpha_\varepsilon(t) = (\kappa_\varepsilon(t), \xi_\varepsilon(t)) = \sum_{k \leq tn_\varepsilon} \alpha_{\varepsilon,k}, t \geq 0$. Then, the process $\nu_\varepsilon(t) = \inf(s \geq 0 : \kappa_\varepsilon(s) > t)$ is a renewal process and $\zeta_\varepsilon(t)$ is a generalised renewal process.

Another variant of a generalised exceeding process is an extremal processes with renewal stopping. It is defined in the way similar with the case of generalised renewal process. The only difference is that the external process is defined as $\xi_\varepsilon(t) = \max_{k \leq tn_\varepsilon} \xi_{\varepsilon,k}, t \geq 0$. In this case, the process $\zeta_\varepsilon(t)$ can be referred as a max-process with renewal stopping.

An interesting is also model, in which the process $\xi_\varepsilon(t)$ is defined as a sum-process while the process $\kappa_\varepsilon(t) = \max_{k \leq tn_\varepsilon} \kappa_{\varepsilon,k}, t \geq 0$ is a max-process. In this case, the process $\zeta_\varepsilon(t)$ is usually referred as a shock process.

Let us also introduce the process $\kappa_\varepsilon^+(t) = \sup_{0 \leq s \leq t} \kappa_\varepsilon(s), t \geq 0$.

To avoid some additional side effects at time point 0 and the need of considering the case, where the random variables $\nu_\varepsilon(t)$ can be improper, let us assume the following condition:

L: (a) $\kappa_\varepsilon(0) \equiv 0$, and (b) $\kappa_\varepsilon^+(t) \xrightarrow{P} \infty$ as $t \rightarrow \infty$, for every $\varepsilon \geq 0$.

Let us also assume that the processes $\alpha_\varepsilon(t), t \geq 0$ **J**-converge, i.e., the following conditions hold:

M: $\alpha_\varepsilon(t) \in Y \Rightarrow \alpha_0(t), t \in Y$ as $\varepsilon \rightarrow 0$, where Y is a subset of $[0, \infty)$, dense in this interval and containing the point 0;

N: $\lim_{c \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} P\{\Delta_J(\alpha_\varepsilon(\cdot), c, T) > \delta\} = 0, \delta, T > 0$.

In this paper, we also restrict consideration by the most important case, where the following condition holds:

O: $\kappa_0^+(t), t \geq 0$ is a strictly monotonic process with probability 1.

Using the definition of the stopping process $\nu_\varepsilon(t) = \inf(s \geq 0 : \kappa_\varepsilon(s) > t) = \inf(s \geq 0 : \kappa_\varepsilon^+(s) > t), t \geq 0$ it is not difficult to show that conditions **L**, **M**, and **O** imply that condition **C** holds for the processes $(\nu_\varepsilon(t), \xi_\varepsilon(t)), t \geq 0$. Moreover, in this case, the limiting random variable $\nu_0(0) = 0$ with probability 1. Since, the process $\xi_0(t), t \geq 0$ is a càdlàg process, condition **F**₀ and therefore condition **H**₀ holds. Condition **N** obviously implies condition **D**. Condition **O** implies that the limiting process $\nu_0(t), t \geq 0$ is continuous with probability 1. The processes $\nu_\varepsilon(t), t \geq 0$ are monotonic, and, therefore, their weak convergence to a continuous process implies that these processes are **J**-compact, i.e., condition **E** holds. Also condition **J** automatically holds due to continuity of the process $\nu_0(t), t \geq 0$.

The following useful lemma from Silvestrov (2004) completes the analysis of conditions which should provide **J**-convergence of generalised exceeding processes.

Lemma 4. *Conditions **L**, **M**, **N**, and **O** imply condition **I** to hold.*

In fact, condition **O** can be weakened in this lemma and replaced by the assumption that $\nu_0(0) = 0$ with probability 1.

The following theorem proved in Silvestrov (2004) follows from Theorem 8 and the remarks above.

Theorem 10. *Let conditions **L**, **M**, **N**, and **O** hold. Then,*

$$\zeta_\varepsilon(t) = \xi_\varepsilon(\nu_\varepsilon(t)), t \geq 0 \xrightarrow{\mathbf{J}} \zeta_0(t) = \xi_0(\nu_0(t)), t \geq 0 \text{ as } \varepsilon \rightarrow 0.$$

Theorem 10 gives very natural and general conditions of weak and **J**-convergence for generalised exceeding processes.

We refer to the works Silvestrov (2000, 2004), where one can find a detailed analysis of conditions for weak and **J**-convergence of generalised exceeding processes, including models, where condition **O** does not hold.

The generalised exceeding processes supply examples of the models, where the first-type continuity condition **G** does not work while the second-type continuity condition **I** can be used.

Let us assume that there exist $\delta > 0$ and a random moment τ such that $\mathbf{P}(A_\delta) = p > 0$, where $A_\delta = \{\kappa_0^+(\tau) - \kappa_0^+(\tau - 0) \geq \delta, |\xi_0(\tau) - \xi_0(\tau - 0)| \geq \delta, \tau < \infty\}$. Then, obviously, there exist $t' < t''$ such that $\mathbf{P}\{A_\delta, \kappa_0^+(\tau - 0) \leq t', \kappa_0^+(\tau) > t''\} = p' > 0$. But, in this case, $\mathbf{P}\{\nu_0(t') = \nu_0(t'') = \tau \in R[\xi_0(\cdot)]\} = p'' > 0$. Thus, the first-type continuity condition **G** does not hold.

In applications to generalised renewal processes constructed from the sequences of i.i.d. random vectors $(\kappa_{\varepsilon,n}, \xi_{\varepsilon,n})$, $n = 1, 2, \dots$ with non-negative first components, the limiting process $\alpha_0(t) = (\kappa_0(t), \xi_0(t))$, $t \geq 0$ is a càdlàg homogeneous process with independent increments, which the first component is non-negative. In this case, the processes $\kappa_0(t) = \kappa_0'(t) + \kappa_0''(t)$, $t \geq 0$ and $\xi_0(t) = \xi_0'(t) + \xi_0''(t)$, $t \geq 0$ can be decomposed in the sums of continuous and jump components. If the jump components $\kappa_0''(t)$, $t \geq 0$ and $\xi_0''(t)$, $t \geq 0$ are independent then condition **G** holds. However, in the case, where the jump components $\kappa_0''(t)$, $t \geq 0$ and $\xi_0''(t)$, $t \geq 0$ are dependent, there exists $\delta > 0$ such that moment $\tau = \inf\{s > 0 : \kappa_0(s) - \kappa_0(s - 0) \geq \delta, |\xi_0(s) - \xi_0(s - 0)| \geq \delta\} < \infty$ with probability 1. Thus, condition **G** does not hold, while condition **I**, according Lemma 4, can hold.

The works by Silvestrov and Teugels (1998, 2004) and Silvestrov (1974, 2000, 2004) contain a detailed presentation of applications to generalised renewal processes, extremal processes with renewal stopping and shock processes, including models constructed from sequences of i.i.d. random variables, as well as bibliographical remarks related to publications in this area.

7. CONCLUSION

The theory of limit theorems for randomly stopped stochastic processes still contains many problems that do require further studies. These are limit theorems for vector compositions of stochastic processes and randomly stopped stochastic fields. Also the models with Markov and martingale type stopping do require additional studies. These limit theorems may be applied to models of optimal stopping for stochastic processes and have important financial and statistical applications. Inverse limit theorems for randomly stopped stochastic processes do also require additional investigation. These theorems would allow deducing convergence of non-randomly stopped stochastic processes from their counterpart for randomly stopped processes. Such theorems play a key role in necessity statements of the limit theorems for processes with semi-Markov modulation. The latter processes themselves supply examples of many interesting models of randomly stopped stochastic processes with queuing, reliability, and other applications.

In conclusion, we would like to point out, together with books by Silvestrov (1974, 2004), used as the basis for the present survey, other books that contain results on limit

theorems for randomly stopped processes. These are Csörgő and Révész (1981), Anisimov (1988, 2007), Gut (1988), Kruglov and Korolev (1990), Csörgő and Horváth (1993, 1997), Rahimov (1995), Gnedenko and Korolev (1996), Kalashnikov (1997), Bening and Korolev (2002), Whitt (2002), and Korolyuk and Linnios (2005).

The book by Silvestrov (2004) also contains a comprehensive bibliography of works related to limit theorems for randomly stopped stochastic processes, with more than 800 references.

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