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THE ROSENBLATT COEFFICIENT OF DEPENDENCE FOR m–DEPENDENT RANDOM SEQUENCES WITH APPLICATIONS TO THE ASCLT

We prove a new bound for the Rosenblatt coefficient of the normalized partial sums of a sequence of m-dependent random variables; this bound is used to prove a general result, from which the Almost Sure Central Limit Theorem can be deduced.

Introduction

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of normalized centered i. i. d random variables. Put

$$S_n = X_1 + \cdots + X_n, \quad U_n = \frac{S_n}{\sqrt{n}}.$$

In paper [4], it was proved that

$$\sup_{A,x} \left| P(U_p \in A, U_q \leq x) - P(U_p \in A)P(U_q \leq x) \right| \leq H \sqrt[4]{\frac{p}{q}}, \tag{1.1}$$

where H is a suitable constant depending on the sequence $(X_n)_{n\in\mathbb{N}}$ only and where the sup is taken over $A \in B(\mathbb{R})$ and $x \in \mathbb{R}$.

It is well known that covariance inequalities of the Rosenblatt type such as (1.1) are a crucial tool in the proof of Almost Sure Limit Theorems, see papers [2], [5], and [9] for some literature on this topic.

Here, we deal with a more general case than the one, considered in [4], of a sequence of i.i.d random variables. More precisely, the aim of the present paper is twofold: first, in Theorem (2.3), we prove an inequality similar to (1.1) for the case of a sequence of m–dependent random variables $(X_n)_{n\in\mathbb{N}}$. Note that we do not assume the identical distribution of $(X_n)_{n\in\mathbb{N}}$; note, moreover, that the constant H in the second member of our inequality (see the statement of Theorem (2.3)) is absolute.

Using the inequality of Theorem (2.3), we prove a general result [Theorem (2.5) of this paper] which is, in some sense, a generalization of the ASCLT to some kind of Borel sets A such that ∂A is not necessarily of Lebesgue measure 0. We deduce the ASCLT as a corollary of Theorem (2.5) (Corollary (2.6)).

The paper is organized as follows: Section 2 contains the statements of the main results [i.e. Theorem (2.3), Theorem (2.5), and Corollary (2.6)]. In Section 3, we prove Theorem (2.3). In Section 4, we prove Theorem (2.5) and Corollary (2.6).

Throughout the whole paper, the symbol H denotes a constant which may not have the same value in all cases.

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1. The main results

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of m-dependent real centered random variables with

$$\sup_n E[X_n^{2+\delta}] < +\infty \tag{2.1}$$

for a suitable $\delta \in (0,1]$.

In the sequel, we put $\alpha = \delta(6\delta+8)^{-1}$. Moreover, we set $S_n = X_1 + X_2 + \cdots + X_n$, $v_n = Var S_n$,

$$U_n = \frac{S_n}{\sqrt{v_n}}$$

and assume that

$$\liminf_{n\to\infty} \frac{v_n}{n} > 0. \tag{2.2}$$

The first result proved in this paper is

(2.3) Theorem. *There exists an absolute constant H such that, for every pair of integers p, q with $p \le q$, the following bound holds:*

$$\sup_{A,x} \left| P(U_p \in A,\, U_q \le x) - P(U_p \in A)P(U_q \le x) \right| \le H\Big(\sqrt[4]{\frac{v_p}{v_q}} + \frac{1}{q^\alpha} \Big),$$

where the sup is taken over $A \in B(\mathbb{R})$ and $x \in \mathbb{R}$.

Theorem (2.3) will be used to prove the second main result of this paper [Theorem (2.5) below].

For a fixed Borel set $A \subseteq \mathbb{R}$, consider the two sequences (T_n) and (W_n) defined, respectively, as

$$T_n = \frac{\sum_{i=1}^n 1_A(U_{2^i})}{n}; \quad W_n = \frac{\sum_{i=1}^n \frac{1}{i} 1_A(U_i)}{\log n}, \quad n \ge 1.$$

Put

$$\phi(n) = \frac{v_n}{n}. \tag{2.4}$$

(2.5) Theorem. *In addition to the hypotheses of Theorem (2.3), assume that the sequence $(\phi(n))$ defined in (2.4) is not decreasing, and let $A \subseteq \mathbb{R}$ be a finite union of intervals. Then, P–a.s. the two sequences $(T_n)_{n\ge1}$ and $(W_n)_{n\ge1}$ have the same limit points as $n \to \infty$.*

Denote, by λ, the Lebesgue measure on $\mathbb{R}$ and, by μ, the Gaussian measure on $\mathbb{R}$, i.e.

$$\mu(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, \lambda(dx), \qquad A \in B(\mathbb{R}).$$

Theorem (2.5) has the following consequence:

(2.6) Corollary (ASCLT). *There exists a P–null set Γ such that, for every $\omega \in \Gamma^c$, we have*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \frac{1}{i} 1_A(U_i)}{\log n} = \mu(A)$$

for every Borel set $A \subseteq \mathbb{R}$ such that $\lambda(\partial A) = 0$.

2. The proof of Theorem (2.3)

We start with some preparatory results.

For every integer $n \geq 1$, we put

$$\Pi_n = \sup_{x \in \mathbb{R}} \Big| P\big(U_n \leq x\big) - \Phi(x) \Big|,$$

where Φ is the distribution function of the standard normal law. In [6], the following Berry-Esseen-type result is proved:

(3.1) Theorem. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of m-dependent random variables verifying (2.1) and (2.2). Then, for every integer n,*

$$\Pi_n \leq \frac{H}{n^\alpha},$$

where H is an absolute constant.

(3.2) Definition. The *concentration function* of a r.v. S is defined as

$$Q(\epsilon) = \sup_{x \in \mathbb{R}} P(x < S \leq x + \epsilon), \quad \epsilon \in \mathbb{R}^+.$$

In the sequel, we denote, by Q_n, the concentration function of U_n.

The following result gives an estimate of Q_n. It is similar to the one given in [8] for a sequence of i.i.d. random variables, but here the constant H is absolute (i.e. it doesn't depend on the sequence $(X_n)_{n \in \mathbb{N}}$).

(3.3) Lemma. *There is an absolute constant H such that, for every $\epsilon \in \mathbb{R}^+$,*

$$Q_n(\epsilon) \leq H \left(\epsilon + \frac{1}{n^\alpha} \right).$$

Proof. Denoting the distribution function of U_n by F_n, Theorem (3.1) yields

$$\max \left\{ \big| F_n(x+\epsilon) - \Phi(x+\epsilon) \big|, \big| F_n(x) - \Phi(x) \big| \right\} \leq \Pi_n \leq \frac{H}{n^\alpha}.$$

Hence,

$$\begin{aligned} & P(x < U_n \leq x + \epsilon) = F_n(x+\epsilon) - F_n(x) \\ & \leq \big| F_n(x+\epsilon) - \Phi(x+\epsilon) \big| + \big| F_n(x) - \Phi(x) \big| + \Phi(x+\epsilon) - \Phi(x) \\ & \leq \frac{H}{n^\alpha} + \frac{1}{\sqrt{2\pi}}\, \epsilon \leq H \left(\epsilon + \frac{1}{n^\alpha} \right). \end{aligned}$$

The following lemma is stated in [1] without proof:

(3.4) Lemma. *If S and T are random variables, then, for every pair of real numbers a, b with $b \geq 0$, we have*

$$\begin{aligned} & P\big(S + T \leq a - b\big) - P\big(|T| > b\big) \leq P\big(S \leq a\big) \\ & \leq P\big(S + T \leq a + b\big) + P\big(|T| > b\big). \end{aligned}$$

Proof. The first inequality follows from the inclusion

$$\{S + T \leq a - b\} \subseteq \{S \leq a\} \cup \{|T| > b\}.$$

The second inequality follows from the first one applied to the pair of random variables $S + T, -T$ and to the pair of numbers $a + b, b$.

We now begin the proof of Theorem (2.3).

Let p, q be two integers with $p \leq q$; let $(Y_n)_{n\in\mathbb{N}}$ be an independent copy of $(X_n)_{n\in\mathbb{N}}$, and put

$$V_q = \frac{Y_1 + \cdots Y_p + X_{p+1} + \cdots X_q}{\sqrt{v_q}}.$$

Put, moreover,

$$Z = V_q - U_q = \frac{(Y_1 - X_1) + \cdots + (Y_p - X_p)}{\sqrt{v_q}} = \frac{R_p}{\sqrt{v_q}}.$$

If we set

$$H = \{U_p \in A\}, \quad K = \{U_q \leq x\},$$

our aim is to give a bound for $|P(H \cap K) - P(H)P(K)|$.

Let $\epsilon > 0$ be any positive real number, and put

$$K_1 = \{V_q \leq x - \epsilon\}, \quad K_2 = \{V_q \leq x + \epsilon\}, \quad F = \{|Z| > \epsilon\}.$$

By Lemma (3.4) (applied to $S = U_q$, $T = Z$, $a = x$, $b = \epsilon$), we can write

$$P(K_1) - P(F) \leq P(K) \leq P(K_2) + P(F).$$

Hence,

$$\begin{aligned} &|P(H\cap K) - P(H)P(K)| \leq \max\big\{|P(H\cap K) - P(K_1)P(H) + P(F)P(H)|,\\ &|P(H\cap K) - P(K_2)P(H) - P(F)P(H)|\big\} \\ &\leq \max\big\{|P(H\cap K) - P(K_1)P(H)|, |P(H\cap K) - P(K_2)P(H)|\big\} + P(F). \end{aligned} \tag{3.5}$$

In what follows, we estimate the three quantities in the last member, i.e. $|P(H\cap K) - P(K_1)P(H)|$, $|P(H \cap K) - P(K_2)P(H)|$ and $P(F)$.

We start with $P(F)$. We have

$$P(F) = P(|R_p| > \epsilon\sqrt{v_q}) \leq \frac{E\big[|R_p|\big]}{\epsilon\sqrt{v_q}} \leq \frac{Var^{1/2}(R_p)}{\epsilon\sqrt{v_q}}. \tag{3.6}$$

Now, since $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ are independent and have the same law,

$$Var(R_p) = 2Var(S_p) = 2v_p \tag{3.7}$$

.

From (3.6) and the (3.7), we conclude that

$$P(F) \leq \frac{H}{\epsilon}\sqrt{\frac{v_p}{v_q}}. \tag{3.8}$$

We now pass to the terms $|P(H \cap K) - P(K_1)P(H)|$ and $|P(H \cap K) - P(K_2)P(H)|$. We give the details only for $|P(H \cap K) - P(K_2)P(H)|$, since the proof is identical for the other quantity.

We need some more lemmas.

(3.9) Lemma. *Let g be a Lipschitzian function defined on $\mathbb{R}$, with Lipschitz constant β. Then*

$$\big|E[g(U_q)] - E[g(V_q)]\big| \leq H\,\beta\sqrt{\frac{v_p}{v_q}}.$$

Proof. Arguing as for relation (3.6) and using (3.7), we get

$$\begin{aligned} \big|E[g(U_q)] - E[g(V_q)]\big| &\leq E\big[|g(U_q) - g(V_q)|\big] \leq \beta\, E[|U_q - V_q|] \\ &= \beta\,\frac{E[|R_p|]}{\sqrt{v_q}} \leq \frac{\beta\, Var^{1/2}(R_p)}{\sqrt{v_q}} \leq H\,\beta\sqrt{\frac{v_p}{v_q}}. \end{aligned}$$

In the sequel, we denote, by $\tilde{Q}_q$, the concentration function of V_q.

(3.10) Lemma. *Let $z \in \mathbb{R}$ and $g = 1_{(-\infty,z]}$. Then, for every $\eta > 0$, we have*

$$\big|E[g(U_q)] - E[g(V_q)]\big| \le \frac{H}{\eta}\sqrt{\frac{v_p}{v_q}} + Q_q(\eta) + \tilde{Q}_q(\eta).$$

Proof. Put

$$h(t) = \left(1 + \frac{z-t}{\eta}\right) 1_{(z,z+\eta]}(t), \qquad \tilde{g}(t) = g(t) + h(t).$$

Then $\tilde{g}$ is Lipschitzian with the Lipschitz constant $1/\eta$. So, by Lemma (3.9),

$$\big|E[\tilde{g}(U_q)] - E[\tilde{g}(V_q)]\big| \le \frac{H}{\eta}\sqrt{\frac{v_p}{v_q}}. \tag{3.11}$$

On the other hand, h has support contained in $(z, z+\eta]$ and is bounded by 1. Hence, we have trivially

$$\big|E[h(U_q) - h(V_q)]\big| \le Q_q(\eta) + \tilde{Q}_q(\eta). \tag{3.12}$$

Now, recalling that $g = \tilde{g} - h$, we can write

$$\begin{aligned} &\big|E[g(U_q)] - E[g(V_q)]\big| = \big|E[(\tilde{g}-h)(U_q)] - E[(\tilde{g}-h)(V_q)]\big| \\ &\le \big|E[\tilde{g}(U_q)] - E[\tilde{g}(V_q)]\big| + \big|E[h(U_q) - h(V_q)]\big| \end{aligned},$$

and the conclusion follows from relations (3.11) and (3.12).

The next lemma concerns the concentration function $\tilde{Q}_n$ of V_n. Its proof is identical to the proof of Lemma (3.3), since it is immediate to see that also the sequence $(Y_1, Y_2, \dots, Y_p, X_{p+1}, \dots)$ is m-dependent.

(3.13) Lemma. *There is an absolute constant H such that, for every $\epsilon \in \mathbb{R}^+$,*

$$\tilde{Q}_n(\epsilon) \le H\left(\epsilon + \frac{1}{n^\alpha}\right).$$

We go back to the proof of the main result (2.3). Since H and K_2 are independent, we can write

$$\begin{aligned} |P(H \cap K) - P(K_2)P(H)| &= P(H)\big|P(K|H) - P(K_2|H)\big| \\ &= P(H)\big|E_H[f(U_q)] - E_H[g(V_q)]\big|, \end{aligned}$$

where $f = 1_{(-\infty,x]}$ and $g = 1_{(-\infty,x+\epsilon]}$. We denote, by E_H, the expectation with respect to the probability law $P(\cdot|H)$. By summing and subtracting $E_H[g(U_q)]$, we see that the above quantity is not greater than

$$\begin{aligned} &P(H)\big|E_H[g(U_q)] - E_H[g(V_q)]\big| + P(H)E_H[|f-g|(U_q)] \\ &= |E[g(U_q)] - E[g(V_q)]\big| + E[|f-g|(U_q)] \\ &\le \frac{H}{\epsilon}\sqrt{\frac{v_p}{v_q}} + 2\,Q_q(\epsilon) + \tilde{Q}_q(\epsilon), \end{aligned} \tag{3.14}$$

using Lemma (3.10) and observing that the function $f - g$ is bounded by 1 and has the interval $(x, x+\epsilon]$ as its support.

Estimate (3.14) holds not only for $|P(H \cap K) - P(K_2)P(H)|$, but also for $|P(H \cap K) - P(K_1)P(H)|$.

We now insert relations (3.8) and (3.14) into (3.5) and obtain

$$
\begin{aligned}
|P(H\cap K)-P(H)P(K)| &\le \frac{H}{\epsilon}\sqrt{\frac{v_p}{v_q}}+2\,Q_q(\epsilon)+\tilde{Q}_q(\epsilon)\\
&\le H\left(\frac{1}{\epsilon}\sqrt{\frac{v_p}{v_q}}+\epsilon+\frac{1}{q^\alpha}\right)
\end{aligned}
$$

by Lemmas (3.3) and (3.13). The above inequality holds for every $\epsilon > 0$; by passing to the infimum in ϵ, we get

$$|P(H\cap K)-P(H)P(K)|\le H\left(\sqrt[4]{\frac{v_p}{v_q}}+\frac{1}{q^\alpha}\right).$$

4. The proof of Theorem (2.5) and the ASCLT

Let's start with the proof of Theorem (2.5). It is sufficient to consider the case where A is of the form $A=(-\infty,x]$. The proof is split in two steps: (i) and (ii).

Put

$$a_n=\log_2\big(1+\frac{1}{n}\big). \tag{4.1}$$

(i) Here, we prove that (S_n) and (H_n) have the same limit points, where

$$H_n=\frac{\sum_{i=1}^{2^n}a_i 1_A(U_i)}{n};$$

This is equivalent to proving that the sequence

$$T_n-H_n+\frac{a_{2^n}1_A(U_{2^n})}{n}=\frac{\sum_{i=1}^{n}1_A(U_{2^i})-\sum_{i=1}^{2^n-1}a_i 1_A(U_i)}{n}$$

tends to 0 as $n\to\infty$, P–a.s. Now, the numerator of the fraction in the second member above can be written as

$$
\begin{aligned}
\sum_{i=1}^{n}1_A(U_{2^i})-\sum_{i=1}^{n}\sum_{j=2^{i-1}}^{2^i-1}a_j 1_A(U_j) &= \sum_{i=1}^{n}\Big(1_A(U_{2^i})-\sum_{j=2^{i-1}}^{2^i-1}a_j 1_A(U_j)\Big)\\
&=\sum_{i=1}^{n}\sum_{j=2^{i-1}}^{2^i-1}a_j\big(1_A(U_{2^i})-1_A(U_j)\big)
\end{aligned}
$$

(note that $\sum_{j=2^{i-1}}^{2^i-1}a_j=\log_2(2^i)-\log_2(2^{i-1})=1$). Put now

$$R_i=\sum_{j=2^{i-1}}^{2^i-1}a_j\big(1_A(U_{2^i})-1_A(U_j)\big). \tag{4.2}$$

Then we must prove that, P-a.s.

$$\lim_{n\to\infty}\frac{\sum_{i=1}^{n}R_i}{n}=0.$$

We write

$$\frac{\sum_{i=1}^{n}R_i}{n}=\frac{\sum_{i=1}^{n}\big(R_i-E[R_i]\big)}{n}+\frac{\sum_{i=1}^{n}E[R_i]}{n}=\frac{\sum_{i=1}^{n}\tilde{R}_i}{n}+\frac{\sum_{i=1}^{n}E[R_i]}{n}$$

and consider separately the two summands above.

For the first one, we apply the Gaal–Koksma law (see [8], p. 134) to the sequence $(\tilde{R}_n)_n$:

(4.3) Theorem (Gaal–Koksma Strong Law of Large Numbers). *Let $(X_n)_n$ be a sequence of centered random variables with finite variance. Suppose that there exists a constant $\beta > 0$ such that, for all integers $m \geq 0$, $n \geq 0$,*

$$E\Big[\Big(\sum_{i=m+1}^{m+n} X_i\Big)^2\Big] \leq H\big((m+n)^\beta - m^\beta\big), \tag{4.4}$$

for a suitable constant H independent of m and n. Then, for each $\rho > 0$,

$$\sum_{i=1}^{n} X_i = O\Big(n^{\beta/2}(\log n)^{2+\rho}\Big), \quad P - a.s.$$

We need a bound for $Cov(\tilde{R}_i, \tilde{R}_j)$. It is easily seen that, for $i \leq j$,

$$Cov(\tilde{R}_i, \tilde{R}_j) = \sum_{h=2^{i-1}}^{2^i-1} \sum_{k=2^{j-1}}^{2^j-1} a_h a_k \Big(C(2^i, 2^j) - C(h, 2^j) - C(2^i, k) + C(h, k)\Big),$$

where

$$C(p,q) = Cov(1_A(U_p), 1_A(U_q)) = P(U_p \in A, U_q \in A) - P(U_p \in A)P(U_q \in A).$$

By Theorem (2.3), there exists a constant H such that, for every p, q with $2^{i-1} \leq p \leq 2^i$ and $2^{j-1} \leq q \leq 2^j$,

$$C(p,q) \leq H\left(\sqrt[4]{\frac{v_p}{v_q}} + \frac{1}{q^\alpha}\right) = H\left(\sqrt[4]{\frac{p\,\phi(p)}{q\,\phi(q)}} + \frac{1}{q^\alpha}\right) \leq H\left(\frac{p}{q}\right)^\alpha \leq H\, 2^{-\alpha|i-j|},$$

so that we obtain

$$Cov(\tilde{R}_i, \tilde{R}_j) \leq H\, 2^{-\alpha|i-j|} \sum_{h=2^{i-1}}^{2^i-1} a_h \sum_{k=2^{j-1}}^{2^j-1} a_k = H\, 2^{-\alpha|i-j|}.$$

In particular, $E[\tilde{R}_i^2] \leq H$. In order to use the Gaal–Koksma law, we evaluate

$$E\Big[\Big(\sum_{i=m+1}^{m+n} \tilde{R}_i\Big)^2\Big] = E\Big[\sum_{i=m+1}^{m+n} \tilde{R}_i^2 + 2 \sum_{m+1 \leq i < j \leq m+n} \tilde{R}_i \tilde{R}_j\Big]$$

$$\leq Hn + 2\,H \sum_{m+1 \leq i < j \leq m+n} 2^{-\alpha|i-j|} = H\,n + 2\,H \sum_{r=1}^{n-1} (n-r)(2^\alpha)^{-r}$$

$$\leq H\,n + 2\,H\,n \sum_{r=0}^{n-1} (2^\alpha)^{-r} \leq H\,n = H\,\big[(m+n) - m\big].$$

Hence, the condition in the Gaal–Koksma law holds with $\beta = 1$, and we obtain

$$\sum_{i=1}^{n} \tilde{R}_i = O\big(\sqrt{n}(\log n)^{2+\rho}\big), \qquad P - a.s.,$$

which implies

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n \tilde{R}_i}{n} = 0, \qquad P - a.s.$$

We now prove that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n E[R_i]}{n} = 0.$$

By Cesaro's theorem, it will be sufficient to prove that

$$\lim_{n\to\infty} E[R_n] = \lim_{n\to\infty} \sum_{j=2^{n-1}}^{2^n-1} a_j\big(P(U_{2^n}\in A) - P(U_j\in A)\big) = 0$$

[recall formula (4.2)]. This is immediate by the relation

$$\sum_{j=2^{i-1}}^{2^i-1} a_j = \log_2(2^i) - \log_2(2^{i-1}) = 1$$

and by Theorem (3.1), which implies

$$\lim_{n\to\infty} P(U_n\in A) = \mu(A).$$

(ii) We now prove that (H_n) and (W_n) have the same limit points. First, observe that

$$W_n = \frac{\sum_{i=1}^n \frac{1}{i\log 2} 1_A(U_i)}{\log_2 n}.$$

Since the sequences (a_n) [see definition (4.1)] and (b_n), where $b_n = \frac{1}{n\log 2}$, are equivalent as $n\to\infty$, this amounts to show that (H_n) has the same limit points as

$$V_n = \frac{\sum_{i=1}^n a_i 1_A(U_i)}{\log_2 n}.$$

This is easy since, for $2^r \le n < 2^{r+1}$, we can write

$$\frac{\sum_{i=1}^{2^r} a_i 1_A(U_i)}{r+1} \le V_n \le \frac{\sum_{i=1}^{2^{r+1}} a_i 1_A(U_i)}{r}.$$

We pass to the proof of the ASCLT [Corollary (2.6)]. Consider first a Borel set A of the form $A = (-\infty, x]$. The Gaal–Koksma law applied to the sequence

$$1_A(U_{2^i}) - P(U_{2^i}\in A)$$

gives, P–a.s,

$$\lim_{n\to\infty}\left(T_n - \frac{\sum_{i=1}^n P(U_{2^i}\in A)}{n}\right) = \lim_{n\to\infty} \frac{\sum_{i=1}^n \big(1_A(U_{2^i}) - P(U_{2^i}\in A)\big)}{n} = 0.$$

by an argument similar to that used above for the sequence $(\tilde{R}_n)$ [see below for the definition of $(\tilde{R}_n)$]. On the other hand, again by Cesaro's theorem and Theorem (3.1), we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P(U_{2^i}\in A)}{n} = \lim_{n\to\infty} P(U_{2^n}\in A) = \mu(A).$$

Hence, we get

$$\lim_{n\to\infty} T_n = \mu(A), \qquad P - a.s. \tag{4.5}$$

Now, the classical techniques (similar to those used in the Glivenko–Cantelli theorem; see, e.g., [3], p. 59) yield that the P-null set Γ such that (4.5) holds for $\omega\in\Gamma^c$ is independent of A, and it is henceforth immediate that, on Γ^c, (4.5) holds also for Borel sets A that are finite unions of disjoint intervals.

For a general set A with $\lambda(\partial A) = \mu(\partial A) = 0$, fix $\epsilon > 0$ and let A_ϵ and B_ϵ be finite unions of disjoint intervals such that

$$A_\epsilon \subseteq A \subseteq B_\epsilon \qquad \text{and} \qquad \mu(B_\epsilon \setminus A_\epsilon) < \epsilon.$$

Then

$$\frac{\sum_{i=1}^n 1_{A_\epsilon}(U_{2^i})}{n} \le T_n \le \frac{\sum_{i=1}^n 1_{B_\epsilon}(U_{2^i})}{n};$$

hence, by passing to the limit as $n \to \infty$, we get, for $\omega \in \Gamma^c$,

$$\mu(A_\epsilon) \le \liminf_{n\to\infty} T_n \le \limsup_{n\to\infty} T_n \le \mu(B_\epsilon); \tag{4.6}$$

since

$$\mu(A_\epsilon) \le \mu(A) \le \mu(B_\epsilon) \le \mu(A_\epsilon) + \epsilon \tag{4.7}$$

by passing to the limit as $\epsilon \to 0$ in (4.7) and then in (4.6), we deduce that $\lim_{n\to\infty} T_n$ exists for $\omega \in \Gamma^c$ and, moreover,

$$\lim_{n\to\infty} T_n = \mu(A), \qquad \omega \in \Gamma^c.$$

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