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ON THE φ-ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

In this paper we study the a.s. asymptotic behaviour of the solution of the stochastic differential equation $dX(t) = g(X(t))dt + \sigma(X(t))dW(t)$, $X(0) = b > 0$, where g and σ are positive continuous functions and W is a Wiener process. Making use of the theory of pseudo-regularly varying (PRV) functions, we find conditions on g, σ and φ, under which $\varphi(X(\cdot))$ can be approximated a.s. by $\varphi(\mu(\cdot))$, where μ is the solution of the ordinary differential equation $d\mu(t) = g(\mu(t))dt$, $\mu(0) = b$. As an application of these results we discuss the problem of φ-asymptotic equivalence for solutions of stochastic differential equations.

1. Introduction

Gihman and Skorohod [16], Keller et al. [19], and later Buldygin et al. [7–11] considered the asymptotic behaviour, as $t \to \infty$, of solutions of certain stochastic differential equations (SDE's) and gave conditions, under which the asymptotics of these solutions are determined by nonrandom functions. In this paper, we reconsider this problem and study conditions, under which solutions of *two* SDE's are asymptotically equivalent.

Consider, for $k = 1, 2$, the stochastic differential equations

$$dX_k(t) = g_k(X_k(t))\,dt + \sigma_k(X_k(t))\,dW_k(t),\ t \ge 0, \quad X_k(0) = b_k > 0. \tag{1.1}$$

Here $\{W_k,\ k = 1, 2\}$ are standard Wiener processes defined on a common probability space; $\{b_k,\ k = 1, 2\}$ are nonrandom positive constants; $\{g_k, \sigma_k,\ k = 1, 2\}$ are continuous functions defined on the set $\mathbf{R} = (-\infty, \infty)$ and such that, for each $k = 1, 2$, the functions σ_k and $(g_k(u), u > 0)$ are positive, and (1.1) has almost surely (a.s.) a unique and continuous Itô-solution $X_k = (X_k(t), t \ge 0)$ with

$$\lim_{t\to\infty} X_k(t) = \infty \quad \text{a.s.} \tag{1.2}$$

For $k = 1, 2$ denote by $\mu_k = (\mu_k(t), t \ge 0)$ the solution of the Cauchy problem for the ordinary differential equations (ODE's) corresponding to (1.1) with $\sigma_k \equiv 0$, i.e.

$$d\mu_k(t) = g_k(\mu_k(t))\,dt,\ t \ge 0, \quad \mu_k(0) = b_k > 0 \quad (k = 1, 2). \tag{1.3}$$

We assume that, for each $k = 1, 2$, the function g_k is such that the solution μ_k exists, is unique and satisfies

$$\lim_{t\to\infty} \mu_k(t) = \infty. \tag{1.4}$$

The following four main problems will be considered in this paper for given functions φ_1 and φ_2.

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The first problem (Problem I) is to investigate, under which conditions it follows that solutions of the SDE's (1.1) and their corresponding ODE's (1.3) are φ-asymptotically equivalent, that is

$$\lim_{t\to\infty} \frac{\varphi_k(X_k(t))}{\varphi_k(\mu_k(t))} = 1 \quad \text{a.s.}, \quad k = 1, 2. \tag{1.5}$$

The second problem (Problem II) is to study, under which conditions it holds that solutions of the ODE's (1.3) are φ-asymptotically equivalent, that is

$$\lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1. \tag{1.6}$$

The next problem (Problem III) is a modification of Problem I. The question is, under which conditions it follows that the solution of the first SDE in (1.1) is φ-asymptotically equivalent to the solution of the second ODE in (1.3), that is

$$\lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = 1 \quad \text{a.s.} \tag{1.7}$$

And finally, Problem IV is to verify, under which conditions it holds that solutions of the SDE's (1.1) are φ-asymptotically equivalent, that is

$$\lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = 1 \quad \text{a.s.} \tag{1.8}$$

All these problems are closely connected, of course, and it is clear that the solutions of Problems III and IV follow from those of Problems I and II.

Gihman and Skorohod [16], §17, and Keller et al. [19] considered some versions of Problem I for a single equation, while Buldygin et al. [7–12] considered some versions of all problems above. Here, we further study these problems in more detail.

In order to solve Problems I and II, we follow the general approach developed in Buldygin et al. [5–7]. This approach allows for solving the following general problem: Find conditions on a given function, under which its inverse or quasi-inverse function preserves the equivalence of functions.

The paper is organized as follows. In Section 2, we formulate and discuss the results concerning Problem I. Subsequently, Problems II, III and IV are considered in Sections 3, 4 and 5, respectively. The main problems of this paper are closely connected with the relations between limits of ratios of functions and their (quasi-) inverse functions from various classes of regularly varying (RV) functions and their extensions. These relations are discussed in Section 6. In Section 7, some of our main results are proved.

2. The φ-Asymptotic Equivalence of Solutions of SDE's and ODE's.

In this section we consider the asymptotic behaviour, as $t \to \infty$, of the Itô-solution $X = (X(t), t \geq 0)$ of the SDE

$$dX(t) = g(X(t))dt + \sigma(X(t))dW(t), \ t \geq 0, \quad X(0) = b > 0. \tag{2.1}$$

Here W is a standard Wiener process. We assume that $\sigma = (\sigma(x), -\infty < x < \infty)$ is a positive function and $g = (g(x), -\infty < x < \infty)$ is positive on $(0, \infty)$ (or ultimately positive), and we shall only be interested in situations, in which $\lim_{t\to\infty} X(t) = \infty$ a.s. and such that infinity will not be reached in finite time.

Denote by $\mu = (\mu(t), t \geq 0)$ the solution of the ODE corresponding to (2.1) for $\sigma \equiv 0$, i.e.

$$d\mu(t) = g(\mu(t))dt, \ t \geq 0, \quad \mu(0) = b. \tag{2.2}$$

We assume that the function g is such that the solution μ exists, is unique, tends to ∞ as $t \to \infty$, and that infinity will not be reached in finite time.

Put

$$G(x)=\int_b^x \frac{ds}{g(s)}\,,\quad x\in[b,\infty). \tag{2.3}$$

Note that $G=(G(x),x\ge b)$ is the inverse function of μ, i.e., $G=\mu^{-1}$, if g is positive and continuous for $x\ge b$, and $\lim_{t\to\infty}\mu(t)=\infty$ if and only if $\lim_{t\to\infty}G(t)=\infty$.

The main problem in this section is to study conditions, under which

$$\lim_{t\to\infty}\frac{\varphi(X(t))}{\varphi(\mu(t))}=1 \quad \text{a.s.} \tag{2.4}$$

for a given function $(\varphi(x),-\infty<x<\infty)$.

As a first step for solving this problem we use the Skorohod method. In Gihman and Skorohod [16], §17, Theorem 4 (see also Keller et al. [19], Theorem 5) the process

$$Y(t)=G(X(t)),\ t\ge 0,$$

is studied and it is proved that

$$\lim_{t\to\infty}\frac{G(X(t))}{t}=1 \quad \text{a.s.} \tag{2.5}$$

under certain conditions (see Remark 2.3 below).

The General Statement. Let g, σ and φ be functions satisfying the following conditions:

(A1) g is continuous and positive on $(0,\infty)$ and σ is continuous and positive on $(-\infty,\infty)$ and such that (2.1) has a.s. a unique and continuous solution as well as (2.2) has a unique and continuously differentiable solution;

(A2) $\varphi=(\varphi(x),x>0)$ is a positive and continuously differentiable function, strictly increasing to infinity as $x\to\infty$.

Put

$$G^{(\varphi)}(\cdot)=G(\varphi^{-1}(\cdot)),\quad g^{(\varphi)}(\cdot)=g(\varphi^{-1}(\cdot))\varphi'(\varphi^{-1}(\cdot)),$$

where G is as in (2.3), the function φ^{-1} is inverse to φ, and φ' is the first derivative of φ.

Observe that $(G^{(\varphi)}(t),t\ge\varphi(b))$ is the inverse function of $\varphi(\mu(\cdot))$.

For example, if $\varphi(\cdot)=\log(\cdot)$, then $G^{(\log)}(\cdot)=G(e^{(\cdot)})$ and $g^{(\log)}(\cdot)=e^{-(\cdot)}g(e^{(\cdot)})$.

If $\varphi(x)\equiv x$, then $G^{(\varphi)}=G$ and $g^{(\varphi)}=g$.

Now our goal is to find conditions on g, σ and φ, under which relation (2.4) holds. To do so, we first consider the following general statement, which describes extra conditions for relation (2.5) to imply or being equivalent to (2.4). Note that the result below holds for nonrandom functions.

Theorem 2.1. *Assume conditions* (A1), (A2) *and*

$$\int_b^\infty \frac{du}{g(u)}=\infty. \tag{2.6}$$

Let g and φ be such that

$$\liminf_{t\to\infty}\int_t^{ct}\frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)}=\liminf_{t\to\infty}\int_{\varphi^{-1}(t)}^{\varphi^{-1}(ct)}\frac{du}{g(u)G(u)}>0 \quad \textit{for all } c>1. \tag{2.7}$$

Then,

1) *if* (2.5) *holds, then* (2.4) *follows*;

2) *if*

$$\lim_{c\downarrow 1}\limsup_{t\to\infty}\int_t^{ct}\frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)}=\lim_{c\downarrow 1}\limsup_{t\to\infty}\int_{\varphi^{-1}(t)}^{\varphi^{-1}(ct)}\frac{du}{g(u)G(u)}=0, \tag{2.8}$$

then (2.5) *and* (2.4) *are equivalent.*

Recall that, by condition (2.6), $\mu(t)\to\infty$ as $t\to\infty$. Moreover, (2.6) excludes the possibility of explosions, that is, the solution does not reach infinity in finite time.

Theorem 2.1 with $\varphi(x)\equiv x$ describes extra conditions for relation (2.5) to imply or being equivalent to

$$\lim_{t\to\infty}\frac{X(t)}{\mu(t)}=1 \quad \text{a.s.} \tag{2.9}$$

(see Buldygin et al. [8]).

Corollary 2.1. *Assume* (A1), (A2) *and* (2.6). *Let* g *and* G *be such that*

$$\liminf_{t\to\infty}\int_t^{ct}\frac{du}{g(u)G(u)}>0 \qquad \textit{for all } c>1.$$

Then,

1) *if* (2.5) *holds, then* (2.9) *follows*;
2) *if*

$$\lim_{c\downarrow 1}\limsup_{t\to\infty}\int_t^{ct}\frac{du}{g(u)G(u)}=0,$$

then (2.5) *and* (2.9) *are equivalent.*

Next, we consider some sufficient conditions, for both (2.7) (Proposition 2.1) and (2.8) (Proposition 2.2), which may be more suitable for practical use.

Proposition 2.1. *Let* g *be a positive and continuous function on* $(0,\infty)$ *such that* (2.6) *holds, and let* φ *satisfy* (A2). *Assume that at least one of the following conditions holds*:

(i) $\limsup_{t\to\infty} g^{(\varphi)}(t)G^{(\varphi)}(t)/t=\limsup_{t\to\infty} g(t)G(t)\varphi'(t)/\varphi(t)<\infty$;
(ii) $g(\cdot)\varphi'(\cdot)$ *is eventually nonincreasing*;
(iii) *there exists* $\alpha<1$ *such that* $0<\inf_{t\ge 1} g^{(\varphi)}(t)t^{-\alpha}$, $\sup_{t\ge 1} g^{(\varphi)}(t)t^{-\alpha}<\infty$;
(iv) $(g^{(\varphi)})^*(c)<c$ *for all* $c>1$, *with* $(g^{(\varphi)})^*(c)=\limsup_{t\to\infty} g^{(\varphi)}(ct)/g^{(\varphi)}(t)$;
(v) $g^{(\varphi)}$ *is an RV function with index* $\alpha<1$ *(see Section 6 below).*

Then, condition (2.7) *is satisfied.*

Remark 2.1. Under (2.6), condition (i) of Proposition 2.1 is equivalent to (2.7), if the function g is eventually nondecreasing.

Remark 2.2. Condition (i) of Proposition 2.1 does not hold for $g^{(\varphi)}(t)\equiv t$, since

$$\limsup_{t\to\infty} g^{(\varphi)}(t)G^{(\varphi)}(t)/t=\lim_{t\to\infty}\int_1^t\frac{ds}{s}=\infty.$$

Moreover, this condition does not hold for any regularly varying function $g^{(\varphi)}$ of index 1, that is, for a function $g^{(\varphi)}$ such that $g^{(\varphi)}(t)\equiv t\ell(t)$, where ℓ is slowly varying. This is due to a result of Parameswaran [24], which proves that

$$\lim_{t\to\infty}\ell(t)\int_1^t\frac{ds}{s\ell(s)}=\infty.$$

Proposition 2.2. *Let g be a positive and continuous function on* $(0,\infty)$ *such that* (2.6) *holds, and let φ satisfy* (A2). *Assume that at least one of the following conditions holds*:

(i) $\liminf_{t\to\infty} g^{(\varphi)}(t)G^{(\varphi)}(t)/t = \liminf_{t\to\infty} g(t)G(t)\varphi'(t)/\varphi(t) > 0$;
(ii) $g(\cdot)\varphi'(\cdot)$ *is eventually nondecreasing*;
(iii) $\int_{0+}^{1} dc/\left(g^{(\varphi)}\right)^{*}(c) > 0$, *with* $\left(g^{(\varphi)}\right)^{*}(c) = \limsup_{t\to\infty} g^{(\varphi)}(ct)/g^{(\varphi)}(t)$;
(iv) *the set* $\{c \in (0,1] : \left(g^{(\varphi)}\right)^{*}(c) < \infty\}$ *has positive Lebesgue measure*;
(v) *at least one of the conditions* (iii), (iv), *or* (v) *of Proposition* 2.1 *holds.*

Then, condition (2.8) *is satisfied.*

Remark 2.3. Under (2.6), condition (i) of Proposition 2.2 is equivalent to (2.8), if the function g is eventually nonincreasing.

Example 2.1. Let $g(x) = \varphi(x) = x, x > 0$. Clearly condition (2.6) holds, but condition (2.7) does not, since, for all $c > 1$,

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g^{(\varphi)}(u)G^{(\varphi)}(u)} = \liminf_{t\to\infty} \int_t^{ct} \frac{du}{u\log u} \le \liminf_{t\to\infty} \frac{c-1}{\log t} = 0.$$

Next, if $g(x) = x, x > 0$, and $\varphi(x) = \log x, x > 0$, then

$$\lim_{t\to\infty} \frac{g^{(\varphi)}(t)G^{(\varphi)}(t)}{t} = 1, \quad t > 0.$$

Thus, by Propositions 2.1 and 2.2, conditions (2.6), (2.7) and (2.8) hold.

The Gihman–Skorohod Condition. Theorem 2.2 below provides some conditions, under which relation (2.4) holds true.

First, consider the following condition of Gihman and Skorohod [16], §17:

(GS) g is continuous and positive on $(0,\infty)$, σ is continuous and positive on $(-\infty,\infty)$, and g and σ are such that (2.1) has a.s. a unique and continuous solution with arbitrary initial condition and with $\lim_{t\to\infty} X(t) = \infty$, as well as (2.2) has a unique and continuously differentiable solution with arbitrary positive initial condition. Let σ/g be bounded and let $g'(x)$ exist for all $x > 0$ with $g'(x) \to 0$ as $x \to \infty$.

Remark 2.4. Recall that, under (GS), relation (2.5) holds true a.s., that is

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.s.}$$

(see Gihman and Skorohod [16], §17, Theorem 4 and Remark 1).

Remark 2.5. Problem (2.1) has a.s. a unique and continuous solution X with arbitrary initial condition and with $\lim_{t\to\infty} X(t) = \infty$ a.s., as well as problem (2.2) has a unique and continuous solution with arbitrary positive initial condition, if, for example, the functions g and σ satisfy the following assumptions:

a) for some K and for all $x \in (-\infty,\infty)$,

$$|g(x)| + |\sigma(x)| \le K(1+|x|);$$

b) for each $C > 0$ there exists an L_C such that, for $|x| \le C$ and $|y| \le C$,

$$|g(x) - g(y)| + |\sigma(x) - \sigma(y)| \le L_C|x-y|;$$

c) for all $x \in (-\infty,\infty)$,

$$\int_{-\infty}^{x} \exp\left\{-\int_0^z \frac{2g(u)}{\sigma^2(u)}du\right\}dz = \infty \quad \text{and} \quad \int_{x}^{\infty} \exp\left\{-\int_0^z \frac{2g(u)}{\sigma^2(u)}du\right\}dz < \infty$$

(see Gihman and Skorohod [16], §15, and §16, Theorem 1).

Theorem 2.2. *Assume conditions* (GS), (A2) *and* (2.6), *and let condition* (2.7) *or at least one of the conditions* (i)–(v) *of Proposition* 2.1 *hold. Then relation* (2.4) *follows.*

The Keller–Kersting–Rösler Conditions. Theorem 2.3 below provides further conditions, under which relation (2.4) holds true.

Here we discuss the conditions of Keller et al. [19]. For $t > 0$, put

$$h(t) = \frac{g'(t)\sigma^2(t)}{2g^2(t)}\,, \quad \psi(t) = \int_1^t \frac{\sigma^2(u)}{g^3(u)} du\,.$$

First consider the following general condition:

(K0) g is continuous and positive on $(0, \infty)$, σ is continuous and positive on $(-\infty, \infty)$, and g and σ are such that (2.1) has a.s. a unique and continuous solution with arbitrary initial condition and $\lim_{t\to\infty} X(t) = \infty$ with positive probability, as well as (2.2) has a unique and continuously differentiable solution with arbitrary positive initial condition.

The following five conditions have been used in Keller et al. [19]:

(K1) $g : (0, \infty) \to (0, \infty)$ is strictly positive and twice continuously differentiable such that $\int_1^\infty (g(u))^{-1} du = \infty$.

(K2) $h(t) \to 0$ as $t \to \infty$.

(K3) $\sigma : (0, \infty) \to (0, \infty)$ is strictly positive and continuously differentiable such that $\int_0^\infty (tg(\mu(t)))^{-2}\sigma^2(\mu(t))dt < \infty$.

(K4) The functions $g(\cdot)$, $g'(\cdot)$, $\sigma^2(\mu(\cdot))/g^2(\mu(\cdot))$ and $h(\mu(\cdot))$ are eventually concave or convex. If $\psi(\infty) = \infty$, we require the same behaviour for the function $h(\psi^{-1}(\cdot))$.

(K5) There is a constant $C > 0$ such that $\log \mu(2t) \le C \log \mu(t)$ for large t. Furthermore, the function $e^{-(\cdot)} g(e^{(\cdot)})$ together with its derivative is eventually concave or convex.

Remark 2.6. Under the above conditions, the following two statements hold (see Theorem 1 and Theorem 5 in Keller et al. [19]).

I) Under (K0)–(K4), relation (2.5) holds true.

II) Under (K0)–(K5), relation (2.4) holds true with $\varphi(t) = \log t, t > 0$.

Theorem 2.3. *Assume conditions* (K0)–(K4), *and* (A2), *and let condition* (2.7) *or at least one of the conditions* (i)–(v) *of Proposition* 2.1 *holds. Then relation* (2.4) *follows.*

3. The φ-Asymptotic Equivalence of the Solutions of ODE's.

In this section we consider the ODE's (1.3) and discuss conditions under which it holds that solutions μ_1 and μ_2 of these ODE's are φ-asymptotically equivalent, that is (1.6) holds true.

Consider functions g_k and φ_k, $k = 1, 2$, satisfying the following conditions: for each $k = 1, 2$,

(B1) g_k is continuous and positive on $(0, \infty)$ and such that (1.3) has a unique and continuously differentiable solution;

(B2) $\varphi_k = (\varphi_k(x), x > 0)$ is a positive and continuously differentiable function, strictly increasing to infinity as $x \to \infty$.

Put

$$G_k^{(\varphi_k)}(\cdot) = G_k(\varphi_k^{-1}(\cdot)), \quad g_k^{(\varphi_k)} = g_k(\varphi_k^{-1}(\cdot))\varphi'_k(\varphi_k^{-1}(\cdot)), \quad k = 1, 2,$$

where, for each $k = 1, 2$,

$$G_k(x) = \int_{b_k}^x \frac{ds}{g_k(s)}\,, \quad x \in [b_k, \infty),$$

the function φ_k^{-1} is inverse to φ_k, and φ_k' is the first derivative of φ_k.

Note that, for $k = 1, 2$, the function $G_k = (G_k(x), x \geq b_k)$ is inverse to μ_k, i.e., $G_k = \mu_k^{-1}$, and $(G_k^{(\varphi_k)}(x), x \geq \varphi_k(b_k))$ is the inverse function of $\varphi_k(\mu_k(\cdot))$, that is

$$G_k^{(\varphi_k)}(x) = \int_{\varphi_k(b_k)}^{x} \left(\frac{ds}{g_k^{(\varphi_k)}(s)} \right), \quad x \in [\varphi_k(b_k), \infty).$$

In the sequel we make use of the condition

$$\int_{b_k}^{\infty} \frac{du}{g_k(u)} = \infty, \quad k = 1, 2. \tag{3.1}$$

It follows from (B1) that (3.1) is equivalent to

$$\int_{b_k}^{\infty} \frac{du}{g_k^{(\varphi_k)}(u)} = \infty, \quad k = 1, 2.$$

The latter condition means that $\lim_{x\to\infty} G_k^{(\varphi_k)}(x) = \infty$, $k = 1, 2$. Thus, under (B1), condition (3.1) holds if and only if (1.4) holds.

Our goal in this section is to find conditions on g_k and φ_k, $k = 1, 2$, under which relation (1.6) holds. Theorem 3.1 below gives conditions, under which the following three relations hold:

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1, \tag{3.2}$$

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longleftarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1, \tag{3.3}$$

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1. \tag{3.4}$$

Consider the next two conditions: for $k = 1, 2$,

$$\liminf_{t\to\infty} \int_t^{ct} \frac{du}{g_k^{(\varphi_k)}(u) G_k^{(\varphi_k)}(u)} = \liminf_{t\to\infty} \int_{\varphi_k^{-1}(t)}^{\varphi_k^{-1}(ct)} \frac{du}{g_k(u) G_k(u)} > 0 \quad \text{for all } c > 1; \tag{3.5}$$

$$\lim_{c\downarrow 1} \limsup_{t\to\infty} \int_t^{ct} \frac{du}{g_k^{(\varphi_k)}(u) G_k^{(\varphi_k)}(u)} = \lim_{c\downarrow 1} \limsup_{t\to\infty} \int_{\varphi_k^{-1}(t)}^{\varphi_k^{-1}(ct)} \frac{du}{g_k(u) G_k(u)} = 0. \tag{3.6}$$

Remark 3.1. Note that, for $k = 1, 2$, conditions (3.5) and (3.6), respectively, coincide with conditions (2.7) and (2.8), where $g = g_k^{(\varphi_k)}$ and $G = G_k^{(\varphi_k)}$. Hence, if at least one of the conditions (i) – (v) of Proposition 2.1 [Proposition 2.2] holds with $g = g_k^{(\varphi_k)}$ and $G = G_k^{(\varphi_k)}$, then condition (3.5) [(3.6)] follows.

Example 3.1. For $k = 1, 2$, let the functions g_k and φ_k be positive and continuous on $(0, \infty)$ and such that condition (B2) holds, and let $g_k^{(\varphi_k)}$ be an RV function with index $\alpha < 1$ (see Section 6 below). Then condition (v) of both Propositions 2.1 and 2.2 holds together with condition (3.1) and, by Remark 3.1, conditions (3.5) and (3.6) are satisfied.

Theorem 3.1. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (B1), (B2) *and* (3.1) *hold. Then,*

1) *if, at least for one $k = 1, 2$, condition* (3.5) *holds, then* (3.2) *follows*;
2) *if, at least for one $k = 1, 2$, condition* (3.6) *holds, then* (3.3) *follows*;
3) *if, at least for one $k = 1, 2$, condition* (3.5) *holds and also, at least for one $k = 1, 2$, condition* (3.6) *holds, then* (3.4) *follows.*

By Theorem 3.1 and Example 3.1, we have the following result.

Corollary 3.1. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (B1) *and* (B2) *hold. If at least one of the functions $g_1^{(\varphi_1)}$ and $g_2^{(\varphi_2)}$ is an RV function with index less than* 1*, then* (3.4) *holds true.*

On the Mutual Relation Between the φ-Asymptotic Equivalence of the Functions g_1, g_2 and the Solutions of ODE's.

Here we discuss a new Problem II*. The question is to find conditions, under which it holds that the following three relations are satisfied:

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1, \tag{3.7}$$

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longleftarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1, \tag{3.8}$$

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} = 1. \tag{3.9}$$

It is clear that, in general, these relations do not hold. For instance, consider the following counterexample to relation (3.7).

Example 3.2. Let $\varphi_1(x) = \varphi_2(x) = x$,

$$g_1(x) = x, \quad g_2(x) = x + \sqrt{x}, \quad x > 0,$$

and $\mu_1(0) = \mu_2(0) = 1$. Then

$$\mu_1(t) = e^t, \quad \mu_2(t) = \left(2e^{t/2} - 1\right)^2, \quad t \ge 0.$$

Thus

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1, \quad \text{but} \quad \lim_{t\to\infty} \frac{\mu_2(t)}{\mu_1(t)} = 4.$$

Observe that g_1 and g_2 are both RV functions with index 1.

On an Application of Karamata's Theorem. Theorem 3.1 shows that Problem II* is directly connected with the next one.

Consider two functions $(f_1(t), t > 0)$ and $(f_2(t), t > 0)$, which are nonnegative and Lebesgue-integrable on finite intervals, and, for given positive numbers a_1 and a_2, put

$$F_k(t) = \int_{a_k}^t f_k(u)\, du, \quad t \ge a_k, \quad k = 1, 2.$$

Assume that $\lim_{t\to\infty} F_k(t) = \infty$, $k = 1, 2$. The question is, under which conditions the following three relations hold:

$$\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{F_1(t)}{F_2(t)} = 1, \tag{3.10}$$

$$\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = 1 \quad \Longleftarrow \quad \lim_{t\to\infty} \frac{F_1(t)}{F_2(t)} = 1, \tag{3.11}$$

$$\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{F_1(t)}{F_2(t)} = 1, \tag{3.12}$$

It is clear that (3.10) always holds. But the inverse relation (3.11) does not hold in general. A simple counterexample is the following one: $f_1(t) = 2t$, $f_2(t) = 2t(1 + \cos t^2)$,

$t \geq 0$, and $F_1(t) = t^2$, $F_2(t) = t^2 + \sin t^2$, $t \geq 0$. Note that the function f_1 is an RV function with index 1.

So, for the relation (3.11) to hold one needs additional conditions. On applying Karamata's theorem (see Bingham et al. [4], p. 26) we get the next result.

Lemma 3.1. *If f_1 and f_2 are RV functions with indices α_1 and α_2 greater than -1, then* (3.11) *and* (3.12) *hold true.*

Now we return to Problem II*. By relation (3.10) and Theorem 3.1, the following result holds.

Theorem 3.2. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (B1), (B2) *and* (3.1) *hold. If, at least for one $k = 1, 2$, condition* (3.5) *holds, then* (3.7) *follows.*

From Theorem 3.2, with $\varphi_1 = \varphi_2 = \varphi$, we conclude the following result.

Corollary 3.2. *Let g_1, g_2 and φ be such that conditions* (B1), (3.1) *and* (A2) *hold. If, at least for one $k = 1, 2$, condition* (3.5) *holds, with $\varphi_k = \varphi$, then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi(\mu_1(t))}{\varphi(\mu_2(t))} = 1.$$

Lemma 3.1 in combination with Corollary 3.1 gives the next theorem.

Theorem 3.3. *Let g_k and φ_k, $k = 1, 2$, be such that conditions* (B1) *and* (B2) *hold. If $g_1^{(\varphi_1)}$ and $g_2^{(\varphi_2)}$ are RV functions with indices less than* 1, *then* (3.9) *follows.*

From Theorem 3.3, with $\varphi_1 = \varphi_2 = \varphi$, we conclude the following results.

Corollary 3.3. *Let g_1, g_2 and φ be such that conditions* (B1) *and* (A2) *hold. If $g_1^{(\varphi)}$ and $g_2^{(\varphi)}$ are RV functions with indices less than* 1, *then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi(\mu_1(t))}{\varphi(\mu_2(t))} = 1.$$

Corollary 3.4. *Let g_1 and g_2 be RV functions with indices less than* 1, *and let* (B1) *hold. Then,*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\mu_1(t)}{\mu_2(t)} = 1.$$

Remark 3.2. Example 3.2 shows that the RV functions g_1 and g_2 in Corollary 3.4 (and in other statements above) cannot have indices equal to 1.

4. More about Asymptotic Equivalence of the Solutions of SDE's and Their Corresponding ODE's.

On applying the results above, we can now discuss, under which conditions it holds that the solution X_1 of the first SDE in (1.1) and the solution μ_2 of the second ODE in (1.3) are φ-asymptotically equivalent, i.e. that (1.7) holds true.

This problem is more general than Problem I (see Section 1), but its solution follows from the results of Sections 2 and 3, since

$$\lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_1(\mu_1(t))} \cdot \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))}\,. \tag{4.1}$$

The following results demonstrate that, under certain conditions, the statements of Theorems 2.1, 2.2 and 2.3 are stable with respect to a change of the initial condition and a change of the function $g^{(\varphi)}$ to an asymptotically equivalent version.

Theorem 4.1. *Assume* (B1), (B2) *and* (3.1), *and let* $g = g_1$ *and* $\sigma = \sigma_1$ *be such that* (A1) *and* (2.5) *hold. Then,*

1) *if, for at least one* $k = 1, 2$, *condition* (3.5) *holds, then*

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = 1 \quad a.s., \tag{4.2}$$

and moreover,

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = 1 \quad a.s.; \tag{4.3}$$

2) *if, for each* $k = 1, 2$, *conditions* (3.5) *and* (3.6) *hold, then*

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = 1 \quad a.s. \tag{4.4}$$

Remark 4.1. Theorem 4.1 remains valid if (2.5) is replaced by (GS) or (K0)–(K4).

We consider some corollaries of Theorem 4.1 under the Gihman–Skorohod condition (GS).

Theorem 4.2. *Assume* (B1), (B2) *and* (3.1), *and let* $g = g_1$ *and* $\sigma = \sigma_1$ *be such that* (GS) *holds.*

1) *If, for at least one* $k = 1, 2$, *condition* (3.5) *holds, then* (4.2) *and* (4.3) *follow;*
2) *If, for each* $k = 1, 2$, *conditions* (3.5) *and* (3.6) *hold, then* (4.4) *follows.*

Theorem 4.3. *Assume* (B1) *and* (B2), *and let* $g = g_1$ *and* $\sigma = \sigma_1$ *be such that* (GS) *holds.*

1) *If at least one of* $g_1^{(\varphi_1)}$ *and* $g_2^{(\varphi_2)}$ *is an RV function with index less than* 1 *(see Section 6 below), then* (4.2) *and* (4.3) *follow;*
2) *If both* $g_1^{(\varphi_1)}$ *and* $g_2^{(\varphi_2)}$ *are RV functions with indices less than* 1, *then* (4.4) *follows and, moreover,*

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(\mu_2(t))} = 1 \quad a.s. \tag{4.5}$$

Observe that Theorems 2.2, 4.2 and 4.3 generalize and complement Theorem 4 in Gihman and Skorohod [16], §17.

By Theorem 4.3, with $\varphi_1 = \varphi_2 = \varphi$, we have the following results.

Corollary 4.1. *Assume* (GS) *with* $g = g_1$ *and* $\sigma = \sigma_1$, *and let* g_2 *and* φ *be such that conditions* (B1) *and* (A2) *hold. If* $g_1^{(\varphi)}$ *and* $g_2^{(\varphi)}$ *are RV functions with indices less than* 1, *then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi(X_1(t))}{\varphi(\mu_2(t))} = 1 \quad a.s.$$

Corollary 4.2. *Assume* (B1) *and* (GS), *with* $g = g_1$ *and* $\sigma = \sigma_1$. *If* g_1 *and* g_2 *are RV functions with indices less than* 1, *then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{X_1(t)}{\mu_2(t)} = 1 \quad a.s.$$

Example 4.1. (See Gihman and Skorohod [16], §17, Corollary 1). Assume (GS) with $g(x) = g_1(x) \sim Cx^\beta$ as $x \to \infty$, where $0 \le \beta < 1$ and $C > 0$. Then, by Corollary 4.2, with $g_2(x) = Cx^\beta$ for $x > 0$, we have

$$\lim_{t\to\infty} \frac{X_1(t)}{(C(1-\beta)t)^{1/(1-\beta)}} = 1 \quad \text{a.s.},$$

since $\mu_2(t) \sim (C(1-\beta)t)^{1/(1-\beta)}$ as $t \to \infty$.

Observe that, in view of Remark 2.2, we cannot use Theorem 4.2 with $\varphi_1(x) \equiv \varphi_2(x) \equiv x$ and $g_1(x) \sim Cx$ as $x \to \infty$.

Example 4.2. Assume (GS) with $g(x) = g_1(x) \sim Cx/(\log x)^\gamma$ as $x \to \infty$, where $\gamma > 0$ and $C > 0$. Put $\varphi_1(x) = \varphi_2(x) = (\log(x+1))^{1+\gamma}$ for $x > 0$. Then $g_1^{(\varphi_1)}(t) \sim C(1+\gamma)$ as $t \to \infty$. Thus, by Corollary 4.1, with $g_2(x) = C(x+1)/(\log(x+1))^\gamma$ for $x > 0$, we have

$$\lim_{t\to\infty} \frac{(\log X_1(t))^{1+\gamma}}{C(1+\gamma)t} = 1 \quad \text{a.s.},$$

since $\varphi_2(\mu_2(t)) \sim (C(1+\gamma)t)$ as $t \to \infty$.

Example 4.3. Assume (GS) with $g(x) = g_1(x) \sim Cx \exp\left(-(\log x)^r\right)$ as $x \to \infty$, where $0 < r < 1$ and $C > 0$. Note that $\exp\left((\log x)^r\right)$, $x > 1$, is a slowly varying function, and $\exp\left((\log x)^r\right)/(\log x)^\gamma \to \infty$ as $x \to \infty$ for all $\gamma > 0$. Put $\varphi_1(x) = \varphi_2(x) = \exp\left((\log x)^r\right)$ for $x > 0$. Then $g_1^{(\varphi_1)}(t) \sim r(\log t)^{(r-1)/r}$ as $t \to \infty$. Thus, by Corollary 4.1, with $g_2(x) = Cx \exp\left(-(\log x)^r\right)$ for $x > 0$, we have

$$\lim_{t\to\infty} \frac{\exp\left((\log X_1(t))^r\right)}{\exp\left((\log \mu_2(t))^r\right)} = 1 \quad \text{a.s.}$$

Remark 4.2. Theorems 4.2 and 4.3, and Corollaries 4.1 and 4.2 remain valid if (2.5) is replaced by (K0)–(K4).

Example 4.4. (See Gihman and Skorohod [16], §17, Corollary 2). Assume (K0)–(K4) with $g(x) = g_1(x) \sim Cx$ as $x \to \infty$, where $C > 0$. Put $\varphi_1(x) = \varphi_2(x) = \log x$ and $g_2(x) = Cx$ for $x > 0$. Then $g_1^{(\varphi_1)}(t) \sim C$ as $t \to \infty$. Thus, by Theorem 4.3 and Remark 4.2, we have

$$\lim_{t\to\infty} \frac{\log X_1(t)}{Ct} = 1 \quad \text{a.s.},$$

since $\varphi_2(\mu_2(t)) \sim Ct$ as $t \to \infty$.

5. The φ-Asymptotic Equivalence of the Solutions of SDE's.

In this section we consider the SDE's (1.1) and discuss, under which conditions it holds that the solutions X_1 and X_2 of these SDE's are φ-asymptotically equivalent, i.e. that (1.8) holds true. This problem is the last one in the list of our main problems (see Section 1), and its solution follows from the results of Sections 2 and 3, since

$$\lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_1(\mu_1(t))} \cdot \lim_{t\to\infty} \frac{\varphi_1(\mu_1(t))}{\varphi_2(\mu_2(t))} \cdot \lim_{t\to\infty} \frac{\varphi_2(\mu_2(t))}{\varphi_2(X_2(t))}\,. \tag{5.1}$$

In this section, we study some new statements under the Gihman-Skorokhod condition (GS) only (see Section 2). Consider the following version of (GS):

(GS*) Condition (GS) holds for the functions $g = g_k$ and $\sigma = \sigma_k$, $k = 1, 2$.

The following statements provide conditions, under which solutions of the SDE's (1.1) are φ-asymptotically equivalent.

Theorem 5.1. *Assume* (GS*), (B2) *and* (3.1).

1) *If, for at least one* $k = 1, 2$, *condition* (3.5) *holds, then*

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = 1 \quad a.s., \tag{5.2}$$

and moreover,

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = 1 \quad a.s.; \tag{5.3}$$

2) *if, for each* $k = 1, 2$, *conditions* (3.5) *and* (3.6) *hold, then*

$$\lim_{t\to\infty} \frac{G_1^{(\varphi_1)}(t)}{G_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = 1 \quad a.s. \tag{5.4}$$

Theorem 5.2. *Assume* (GS*) *and* (B2).

1) *If at least one of* $g_1^{(\varphi_1)}$ *and* $g_2^{(\varphi_2)}$ *is an RV function with index less than* 1 *(see Section 6 below), then* (5.2) *and* (5.3) *follow;*
2) *If both* $g_1^{(\varphi_1)}$ *and* $g_2^{(\varphi_2)}$ *are RV functions with indices less than* 1, *then* (5.4) *follows and, moreover,*

$$\lim_{t\to\infty} \frac{g_1^{(\varphi_1)}(t)}{g_2^{(\varphi_2)}(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi_1(X_1(t))}{\varphi_2(X_2(t))} = 1 \quad a.s. \tag{5.5}$$

Theorem 5.2, with $\varphi_1 = \varphi_2 = \varphi$, implies the following result.

Corollary 5.1. *Assume* (GS*) *and* (A2). *If* $g_1^{(\varphi)}$ *and* $g_2^{(\varphi)}$ *are RV functions with indices less than* 1, *then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{\varphi(X_1(t))}{\varphi(X_2(t))} = 1 \quad a.s.$$

For $\varphi(x) \equiv x$ one has the following statement.

Corollary 5.2. *Assume* (GS*). *If* g_1 *and* g_2 *are RV functions with indices less than* 1, *then*

$$\lim_{t\to\infty} \frac{g_1(t)}{g_2(t)} = 1 \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{X_1(t)}{X_2(t)} = 1 \quad a.s.$$

6. Properties of Some Classes of Functions

In this section we recall the definitions and some properties of various classes of regularly varying functions and their extensions. Also asymptotic quasi-inverse functions and relations between limits of the ratio of functions and their quasi-inverse functions are discussed.

Let $\mathbf{R}_+$ be the set of nonnegative reals. Also let $\mathbb{F} = \mathbb{F}(\mathbf{R}_+)$ be the space of real-valued functions $f = (f(t), t \geq 0)$, and $\mathbb{F}_+ = \bigcup_{A>0}\{f \in \mathbb{F} : f(t) > 0, t \in [A, \infty)\}$. Thus $f \in \mathbb{F}_+$ if and only if f is eventually positive.

Let $\mathbb{F}^{(\infty)}$ be the space of functions $f \in \mathbb{F}_+$ such that

(i) $\sup_{0\leq t\leq T} f(t) < \infty \quad \forall\, T > 0$;
(ii) $\limsup_{t\to\infty} f(t) = \infty$.

Further let $\mathbb{F}^{\infty}$ be the space of functions $f \in \mathbb{F}^{(\infty)}$ such that $\lim_{t\to\infty} f(t) = \infty$. We also make use of the subspaces $\mathbb{C}^{(\infty)}$ and $\mathbb{C}^{\infty}$ of continuous functions in $\mathbb{F}^{(\infty)}$ and $\mathbb{F}^{\infty}$, respectively.

Finally, the space $\mathbb{C}^{\infty}_{\text{inc}}$ contains all functions $f \in \mathbb{C}^{\infty}$, which are strictly increasing for large t.

For a given $f \in \mathbb{F}_+$, we consider the *upper* and *lower limit functions*

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

which take values in $[0, \infty]$.

RV and ORV Functions. Recall that a measurable function $f \in \mathbb{F}_+$ is called *regularly varying* (RV) if $f_*(c) = f^*(c) = \varkappa(c) \in \mathbf{R}_+$ for all $c > 0$ (see Karamata [17]). In particular, if $\varkappa(c) = 1$ for all $c > 0$, then the function f is called *slowly varying* (SV). For any RV function f, $\varkappa(c) = c^{\alpha}$, $c > 0$, for some number $\alpha \in \mathbf{R}$, which is called the *index* of the function f. Moreover, $f(t) = t^{\alpha}\ell(t)$, $t > 0$, where ℓ is a slowly varying function.

A measurable function $f \in \mathbb{F}_+$ is called *O-regularly varying* (ORV) if $f^*(c) < \infty$ for all $c > 0$ (see Avakumović [1] and Karamata [18]). It is obvious that any RV function is an ORV function. The theory of RV functions and later extensions and generalizations turned out to be fruitful in various fields of mathematics (cf. Seneta [25] and Bingham et al. [4] for excellent surveys on this topic and for the history of its theory and applications).

PRV Functions. For any RV function f, we have $f^*(c) \to 1$ as $c \to 1$. In order to generalize this property to a wider class of functions, we introduce the following notion (see Buldygin et al. [5]).

Definition 6.1. A measurable function $f \in \mathbb{F}_+$ is called *pseudo-regularly varying* (PRV) if

$$\limsup_{c\to 1} f^*(c) = 1. \tag{6.1}$$

Any PRV function is ORV, but not vice versa. Moreover, any RV function is PRV, but not vice versa. Corresponding examples have been given in Buldygin et al. [5].

PRV functions and their various applications have been studied by Korenblyum [21], Matuszewska [22], Matuszewska and Orlicz [23], Stadtmüller and Trautner [26], Berman [2, 3], Yakymiv [28], Cline [13], Djurčić [14], Djurčić and Torgašev [15], Klesov et al. [20], and Buldygin et al. [5–10]. Note that PRV functions are called *regularly oscillating* in Berman [2], *weakly oscillating* in Yakymiv [28], *intermediate regularly varying* in Cline [13] and CRV in Djurčić [14]. We stick to the notion PRV introduced in Buldygin et al. [5].

One of the well-known properties of PRV functions is that they and only they preserve the equivalence of functions (see, for example, Buldygin et al. [5]).

Two functions u and v are called (*asymptotically*) *equivalent* if $u(t) \sim v(t)$ as $t \to \infty$, that is, $\lim_{t\to\infty} u(t)/v(t) = 1$. The equivalence of functions is denoted by $u \sim v$. Recall that a function f *preserves the equivalence of functions* if $f(u(t))/f(v(t)) \to 1$ as $t \to \infty$ for all nonnegative functions u and v such that $u \sim v$ and $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) = \infty$.

Lemma 6.1. *A measurable function $f \in \mathbb{F}_+$ preserves the equivalence of functions if and only if it is PRV.*

SQI Functions. Next we define a further class of functions playing an important role in the context of this paper (see also Buldygin et al. [5, 6]).

Definition 6.2. A measurable function $f \in \mathbb{F}_+$ is called *sufficiently quickly increasing* (SQI) if

$$f_*(c) > 1 \qquad \text{for all } c > 1. \tag{6.2}$$

These functions have also been used by Yakymiv [27], Djurčić and Torgašev [15], and Buldygin et al. [5–10].

Note that any slowly varying function f *cannot* be an SQI function. On the other hand, any RV function of positive index as well as any *quickly increasing* monotone function, for example $f(t) = e^t,\ t \geq 0$, is SQI.

Quasi-Inverse Functions. First, we recall the definition of a *quasi-inverse function*, which will be useful for our considerations below (cf. Buldygin et al. [5, 6]).

Definition 6.3. Let $f \in \mathbb{F}^{(\infty)}$. A function $f^{(-1)} \in \mathbb{F}^\infty$ is called a *quasi-inverse function* of f if $f(f^{(-1)}(s)) = s$ for all large s.

For any $f \in \mathbb{C}^{(\infty)}$, a quasi-inverse function exists, but may not be unique (see Buldygin et al. [5, 6]). If $f \in \mathbb{C}^\infty_{\text{inc}}$, then its *inverse function* f^{-1} exists, that is, $f(f^{-1}(s)) = s$ and $f^{-1}(f(t)) = t$ for all sufficiently large s and t.

Quasi-Inverse Functions Preserving the Equivalence of Functions. Next we discuss conditions under which quasi-inverse functions preserve the equivalence of functions (see Buldygin et al. [5, 6]).

Theorem 6.1. *Assume $f \in \mathbb{C}^\infty_{\text{inc}}$. Then, its inverse function f^{-1} preserves the equivalence of functions if and only if condition* (6.2) *holds.*

Finally we consider relations between limits of the ratio of functions and their quasi-inverse functions (see Buldygin et al. [5, 6]).

Theorem 6.2. *Assume $f \in \mathbb{C}^\infty_{\text{inc}}$ and let f satisfy condition* (6.2). *If, for some function $x \in \mathbb{F}^\infty$,*

$$\lim_{t\to\infty} \frac{x(t)}{f(t)} = a \qquad \text{with some } a \in (0, \infty),$$

then, for any quasi-inverse function $x^{(-1)}$ of x, we have

$$\lim_{s\to\infty} \frac{x^{(-1)}(s)}{f^{-1}(s/a)} = 1,$$

where f^{-1} is the inverse function of f.

7. Auxiliary Results

The proofs of some statements in this paper are closely connected with the questions of when differentiable functions satisfy PRV or SQI conditions (see Section 6). These questions were studied in Buldygin et al. [8, 9, 11]. In this section, some results from these papers are collected.

Conditions for Differentiable Functions to be PRV or SQI. Consider the following five conditions on a function f and its derivative f':

(D) $f \in \mathbb{F}^\infty$ and f is positive and continuously differentiable for all $t \geq t_0 > 0$;
(DM) Condition (D) holds and $f'(t) \geq 0$ for all $t \geq t_0 > 0$;
(DM+) Condition (D) holds and $f'(t) > 0$ for all $t \geq t_0 > 0$;
(DM1) Condition (DM+) holds and f' is nonincreasing for all $t \geq t_0 > 0$;
(DM2) Condition (DM+) holds and f' is nondecreasing for all $t \geq t_0 > 0$.

For a function f satisfying condition (D), the following integral representation holds:

$$f(t) = f(t_0) \exp\left\{ \int_{t_0}^{t} \frac{f'(u)}{f(u)} du \right\} \tag{7.1}$$

for any $t > t_0$.

The next result provides some simple inequalities between the limit functions of f and f'.

Proposition 7.1. *If condition (DM+) holds, then*

$$c(f')_*(c) \le f_*(c) \le f^*(c) \le c(f')^*(c)$$

for all $c \ge 0$.

In particular, Proposition 7.1 demonstrates that under condition (DM+), if $f'(\cdot)$ is an ORV (PRV, SQI, RV) function, then f possesses the same property. Below, these results will be strengthened for PRV and SQI functions.

Lemma 7.1. *Assume condition (D). Then f is a PRV function if and only if*

$$\lim_{c \to 1} \limsup_{t \to \infty} \int_{t}^{ct} \frac{f'(u)}{f(u)} du = 0.$$

Lemma 7.2. *Assume condition (DM). Then f is a PRV function if and only if*

$$\lim_{c \downarrow 1} \limsup_{t \to \infty} \int_{t}^{ct} \frac{f'(u)}{f(u)} du = 0.$$

Let us consider some corollaries of the above lemmas.

Corollary 7.1. *Assume condition (D).*

1) *If*

$$\limsup_{t \to \infty} \frac{t|f'(t)|}{f(t)} < \infty,$$

then f is a PRV function.

2) *If f is a PRV function, then*

$$\liminf_{t \to \infty} \frac{tf'(t)}{f(t)} < \infty.$$

3) *If condition (DM) holds and*

$$\limsup_{t \to \infty} \frac{tf'(t)}{f(t)} < \infty, \tag{7.2}$$

then f is a PRV function.

4) *If condition (DM1) holds, then f is a PRV function.*

Remark 7.1. If condition (D) holds and $\limsup_{t \to \infty} t|f'(t)| < \infty$, then $f^*(c) = 1$ for all $c > 0$. This means that f is an SV function, and hence it is a PRV function. Thus we can confine ourselves to the case, when $\limsup_{t \to \infty} t|f'(t)| = \infty$.

Corollary 7.2. *Assume condition (DM2). Then f is a PRV function if and only if* (7.2) *holds true.*

The integral in the next statement means the Lebesgue integral.

Corollary 7.3. *Assume condition $(DM+)$. If*

$$\int_{0+}^{1} (f')_*(c)dc > 0, \tag{7.3}$$

then f is a PRV function.

On applying Corollary 7.3, we get the following result.

Corollary 7.4. *Assume condition $(DM+)$. If the set $\{c \in (0,1] : (f')_*(c) > 0\}$ has positive Lebesgue measure, then f is a PRV function. In particular, this condition holds if f' is an ORV function.*

Now we discuss conditions for differentiable functions to be SQI.

Lemma 7.3. *Assume condition (D). Then f is an SQI function if and only if*

$$\liminf_{t\to\infty} \int_t^{ct} \frac{f'(u)}{f(u)} du > 0 \quad \textit{for all } c > 1.$$

Next, we consider some corollaries of Lemma 7.3.

Corollary 7.5. *Assume condition (DM).*

1) *If*

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} > 0, \tag{7.4}$$

then f is an SQI function.

2) *If f is an SQI function, then*

$$\limsup_{t\to\infty} \frac{tf'(t)}{f(t)} > 0.$$

3) *If f is an SQI function, then*

$$\limsup_{t\to\infty} tf'(t) = \infty.$$

4) *If condition $(DM2)$ holds, then f is an SQI function.*

Corollary 7.6. *Assume condition $(DM1)$. Then f is an SQI function if and only if* (7.4) *holds true.*

The next result gives a condition in terms of the function $(f')_*(\cdot)$.

Corollary 7.7. *Assume condition $(DM+)$. If*

$$c(f')_*(c) > 1 \quad \textit{for all } c > 1,$$

then f is an SQI function.

8. Proofs of the Main Results

Proof of Theorem 2.1. By conditions (2.6), (2.7) and Lemma 7.3, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, we have that $G^{(\varphi)}$ is an SQI function, that is, it satisfies (6.2). Moreover, $G^{(\varphi)} \in \mathbb{C}_{\mathrm{inc}}^{\infty}$. Hence, by Theorem 6.1, the function $\varphi(\mu(\cdot)) = (G^{(\varphi)})^{-1}(\cdot)$ preserves the equivalence of functions (see Section 6). Therefore, in view of (2.5),

$$\lim_{t\to\infty} \frac{\varphi(X(t))}{\varphi(\mu(t))} = \lim_{t\to\infty} \frac{\varphi(\mu(G(X(t))))}{\varphi(\mu(t))} = \lim_{t\to\infty} \frac{G(X(t))}{t} = 1 \quad \text{a.s.},$$

since $\mu = G^{-1}$. Thus, relation (2.4) holds and statement 1) is proved.

By conditions (2.6), (2.8) and Lemma 7.2, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, we have that $G^{(\varphi)}$ is a PRV function (see Definition 6.1). Hence, by Lemma 6.1, the function $G^{(\varphi)} = G(\varphi^{-1})$ preserves the equivalence of functions. Therefore, in view of (2.4),

$$\lim_{t\to\infty} \frac{G(X(t))}{t} = \lim_{t\to\infty} \frac{G^{(\varphi)}(\varphi(X(t)))}{G^{(\varphi)}(\varphi(\mu(t)))} = \lim_{t\to\infty} \frac{\varphi(X(t))}{\varphi(\mu(t))} = 1 \quad \text{a.s.,}$$

that is, relation (2.5) holds. Thus, statement 2) follows from the last implication in combination with 1). □

Proof of Proposition 2.1. Condition (2.7) follows from

a) (2.6) and (i), in view of Corollary 7.5, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, since

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} = \liminf_{t\to\infty} \frac{t}{G^{(\varphi)}(t)g^{(\varphi)}(t)} = \liminf_{t\to\infty} \frac{\varphi(t)}{G^{(\varphi)}(\varphi(t))g^{(\varphi)}(\varphi(t))}$$
$$= \liminf_{t\to\infty} \frac{\varphi(t)}{G(t)g(t)\varphi'(t)} = \left(\limsup_{t\to\infty} \frac{G(t)g(t)\varphi'(t)}{\varphi(t)}\right)^{-1} > 0;$$

b) (2.6) and (ii), since, by (ii), $g^{(\varphi)}$ is eventually nonincreasing and thus (i) holds;
c) (iii), since (i) (and also (2.6)) follows from (iii);
d) (2.6) and (iv), in view of Corollary 7.7, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, since $\left(1/g^{(\varphi)}\right)_* = 1/(g^{(\varphi)})^*$;
e) (2.6) and (v), since (iv) follows from (v).

□

Proof of Proposition 2.2. Condition (2.8) follows from

a) (2.6) and (i), in view of Corollary 4.1, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, since

$$\limsup_{t\to\infty} \frac{tf'(t)}{f(t)} = \limsup_{t\to\infty} \frac{t}{g^{(\varphi)}(t)G^{(\varphi)}(t)} = \limsup_{t\to\infty} \frac{\varphi(t)}{G^{(\varphi)}(\varphi(t))g^{(\varphi)}(\varphi(t))}$$
$$= \limsup_{t\to\infty} \frac{\varphi(t)}{G(t)g(t)\varphi'(t)} < \infty = \left(\liminf_{t\to\infty} \frac{G(t)g(t)\varphi'(t)}{\varphi(t)}\right)^{-1} < \infty;$$

b) (2.6) and (ii), since, by (ii), $g^{(\varphi)}$ is eventually nondecreasing and thus (i) holds;
c) (2.6) and (iii), in view of Corollary 7.3, with $f = G^{(\varphi)}$ and $f' = 1/g^{(\varphi)}$, since $(1/g^{(\varphi)})_*(c) = 1/(g^{(\varphi)})^*(c)$ for all $c > 0$;
d) (2.6) and (iv), since (iii) follows from (iv);
e) (2.6) and (v), since (iv) follows from (v).

□

Proof of Theorem 2.2. Theorem 2.2 follows from Theorem 2.1 in combination with Remark 2.4 and Proposition 2.1. □

Proof of Theorem 2.3. Theorem 2.3 follows from Theorem 2.1 in combination with Remark 2.6 and Proposition 2.1. □

Proof of Theorem 3.1. Assume that (3.5) holds for at least one $k = 1, 2$. Then, by conditions (3.1), (3.5) and Lemma 7.3, with $f = G_k^{(\varphi_k)}$ and $f' = 1/g_k^{(\varphi_k)}$, we have that $G_k^{(\varphi_k)}$ is an SQI function, i.e. it satisfies (6.2). Moreover, $G_j^{(\varphi_j)} \in \mathbb{C}_{\text{inc}}^{\infty}$ and $(G_j^{(\varphi_j)})^{-1} = \varphi_j \circ \mu_j$, $j = 1, 2$. Hence, by Theorem 6.2, relation (3.2) follows and statement 1) is proved.

In order to prove statement 2) we assume that (3.6) holds for at least one $k = 1, 2$. Then, by conditions (3.1), (3.6) and Lemma 7.2, with $f = G_k^{(\varphi_k)}$ and $f' = 1/g_k^{(\varphi_k)}$, we have that $G_k^{(\varphi_k)}$ is a PRV function (see Definition 6.1). Hence, by Lemma 6.1, the function $G_k^{(\varphi_k)}$ preserves the equivalence of functions. Therefore, by Theorem 6.1, the function $\varphi_k \circ \mu_k$ is an SQI function, since $G_j^{(\varphi_j)} \in \mathbb{C}_{\text{inc}}^\infty$ and $(G_j^{(\varphi_j)})^{-1} = \varphi_j \circ \mu_j$, $j = 1, 2$. Hence, by Theorem 6.2, relation (3.3) follows and statement 2) is proved.

Statement 3) follows from statements 1) and 2). □

Proof of Lemma 3.1. For each $k = 1, 2$, the function F_k is an RV function with index $\alpha_k + 1 > 0$, since f_k is an RV function with index $\alpha_k > -1$. Assume that

$$\lim_{t\to\infty} \frac{F_1(t)}{F_2(t)} = 1.$$

Then $\alpha_1 + 1 = \alpha_2 + 1 = \beta > 0$ and, by Karamata's theorem (see Bingham et al. [4], p. 26), we have that

$$\lim_{t\to\infty} \frac{tf_k(t)}{\beta F_k(t)} = 1, \quad k = 1, 2.$$

Hence

$$\lim_{t\to\infty} \frac{f_1(t)}{f_2(t)} = \lim_{t\to\infty} \frac{tf_1(t)}{\beta F_1(t)} \cdot \lim_{t\to\infty} \frac{F_1(t)}{F_2(t)} \cdot \lim_{t\to\infty} \frac{\beta F_2(t)}{tf_2(t)} = 1.$$

Thus relation (3.11) is proved.

Relation (3.12) follows from relations (3.10) and (3.11). □

Proof of Theorem 4.1. In view of relation (4.1), Theorem 4.1 follows from Theorems 2.1, 3.1 and 3.2. □

Proof of Theorem 4.2. Theorem 4.2 follows from Theorem 4.1 and Remark 2.4. □

Proof of Theorem 4.3. Theorem 4.3 follows from Theorem 4.2 in combination with Example 3.1 and Lemma 3.1, with $f_k = 1/g_k^{(\varphi_k)}$, $k = 1, 2$. □

Proof of Theorem 5.1. In view of relation (5.1), Theorem 5.1 follows from Theorems 2.2, 3.1 and 3.2. □

Proof of Theorem 5.2. Theorem 5.2 follows from Theorem 5.1 in combination with Example 3.1 and Lemma 3.1, with $f_k = 1/g_k^{(\varphi_k)}$, $k = 1, 2$. □

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