Theory of Stochastic Processes
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LONG-TERM RETURNS IN STOCHASTIC INTEREST RATE MODELS

We consider the behavior of integral functional of the solution of stochastic differential equation with coefficients contained small parameter. The dependence on the order of small parameter in every term of equation with Wiener process and Poisson measure term is studied. We observe the convergence of the long-term return, using an extension of the Cox-Ingersoll-Ross stochastic model of the short interest rate. Obtained results are applied for studying of two-factor stochastic interest rate model.

1. Introduction

Controlling the risk induced by interest rate fluctuation is of crucial importance for banks and insurance companies. In this light, we think it is interesting to study and to model the long-term return in a mathematical way.

Interest to the behavior of long-term return leads to the investigation of limit behavior of integral functionals of solution of stochastic differential equation. In [1] properties of integral functionals of Brownian motion with drift are considered. In [2] the boundary classification of diffusion is used in order to derive a criterion for the convergence of perpetual integral functionals of transient real-valued diffusion. In [3] the behavior of integral functional of the solution to stochastic differential equation with Wiener process and Poisson measure term, and with coefficients containing small parameter is studied. The first part of this paper contains some generalization of results from [3].

In the Cox-Ingersoll-Ross stochastic model [4] the dynamics of the short interest rate $(r_t)_{t\geq 0}$ is expressed by stochastic differential equation

$$dr_t = k(\gamma - r_t)dt + \sigma\sqrt{r_t}dw_t,$$

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with $(w_t)_{t\geq 0}$ one-dimensional Wiener process and k, γ and σ positive constants. In this model r_t never becomes negative and converges to the long-term constant value γ [5]. Further, it is reasonable to conjecture that the market will constantly change this level γ and the volatility σ. Therefore it is natural to consider γ as the stochastic process and to generalize the volatility.

In mentioned models the short interest rate satisfies the stochastic differential equation of diffusion type. In this paper we consider the stochastic differential equation for the processes with discontinuities. We observe the convergence of the long-term return $\frac{1}{t}\int_0^t r_s\,ds$, where $(r_t)_{t\geq 0}$ satisfies the stochastic differential equation with Wiener process and Poisson measure term, which is the generalization of the Cox-Ingersoll-Ross stochastic model of the short interest rate.

2. The limit behavior of integral functional of the solution of stochastic differential equation

We study the behavior, as $\varepsilon \to 0$, of the integral functional $\eta_\varepsilon(t) = (\varepsilon^k/t)\int\limits_0^{t/\varepsilon^k} d(s,\xi(s))ds$, where $\xi(t)$ is the solution of stochastic differential equation

$$d\xi(t) = \varepsilon^{k_1} f(t,\xi(t))dt + \varepsilon^{k_2} g(t,\xi(t))dw(t) + \varepsilon^{k_3}\int_{\mathrm{R}^d} q(t,\xi(t),y)\tilde{\nu}(dt,dy), \quad (1)$$

$$\xi(0) = \xi_0;$$

$\varepsilon > 0$ is the small parameter; $k > 0,\ k_i > 0, i = 1,2,3$; $d(t,x)$ is non-random function; $f(t,x) = \{f_i(t,x), i = \overline{1,d}\}$, $q(t,x,y) = \{q_i(t,x,y), i = \overline{1,d}\}$ are non-random vector-valued functions; $g(t,x) = \{g_{ij}(t,x), i,j = \overline{1,d}\}$ is non-random matrix-valued function; $t \in [0,T], x, y \in \mathrm{R}^d$; $w(t)$ is d-dimensional Wiener process; $\tilde{\nu}(dt,dy) = \nu(dt,dy) - \Pi(dy)dt$, $\nu(dt,dy)$ is the Poisson measure independent on $w(t)$, $E\nu(dt,dy) = \Pi(dy)dt$; $\Pi(\cdot)$ is a sigma-finite measure on the σ-algebra of Borel sets in R^d; ξ_0 is the random vector independent on $w(t)$ and $\tilde{\nu}(t,\cdot)$.

We need the following result.

Lemma 1. *Let* $\int_{\mathrm{R}^d} q(t,x,y)\Pi(dy)$ *is bounded and uniformly continuous in* x *with respect to* $t \in [0,\infty)$ *in every compact set* $|x| \leq C$. *Let* $\Pi(\cdot)$ *be a sigma-finite measure on the* σ*-algebra of Borel sets in* R^d. *For each* $x \in \mathrm{R}^d$ *there exists limit*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{\mathrm{R}^d} q(t,x,y)\,\Pi(dy)dt = \int_{\mathrm{R}^d} \bar{q}(x,y)\,\Pi(dy),$$

where $\int_{\mathrm{R}^d} \bar{q}(x,y)\,\Pi(dy)$ *is bounded and continuous. Then for any stochastically continuous process* $\xi(t)$ *we have*

$$\mathrm{P}-\lim_{\varepsilon\to 0}\int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon,\xi(s),y)\,\Pi(dy)ds = \int_0^t\int_{\mathrm{R}^d}\bar{q}(\xi(s),y)\,\Pi(dy)ds$$

for all arbitrary $t\in[0,T]$.

Proof. Since the process $\xi(t)$ is stochastically continuous then [6, pp.218-219] for any $\delta_1>0$ there exists such constant $C>0$ that

$$\sup_{t\in[0,T]}\mathrm{P}\{|\xi(t)|>C\}\le\delta_1$$

and for arbitrary $\delta_1>0$ and $\delta_2>0$ there exists such $\delta_3>0$ that

$$\mathrm{P}\{|\xi(t_1)-\xi(t_2)|>\delta_2\}\le\delta_1$$

for all $|t_1-t_2|<\delta_3, t_1,t_2\in[0,T]$. We choose δ_2 such that $|\int\limits_{\mathrm{R}^d} q(t,x_1,y)\,\Pi(dy) - \int\limits_{\mathrm{R}^d} q(t,x_2,y)\,\Pi(dy)|<\delta_1$ and $|\int_{\mathrm{R}^d}\bar{q}(x_1,y)\,\Pi(dy)-\int_{\mathrm{R}^d}\bar{q}(x_2,y)\,\Pi(dy)|<\delta_1$ for all $t\in[0,T]$, as $|x_1-x_2|\le\delta_2,\ |x_1|\le C, |x_2|\le C$.

Let us consider partition $0=t_0<t_1<\ldots<t_n=t,\ t\in[0,T]$ such that $\max\limits_{0\le k\le n-1}|t_{k+1}-t_k|<\delta_3$. For any $\delta>0$ we have

$$\mathrm{P}\Big\{\Big|\int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon,\xi(s),y)\,\Pi(dy)ds-\int_0^t\int_{\mathrm{R}^d}\bar{q}(\xi(s),y)\,\Pi(dy)ds\Big|>\delta\Big\}\le$$

$$\le\mathrm{P}\Big\{\Big|\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(s),y)-q(s/\varepsilon,\xi(t_{k-1}),y)]\,\Pi(dy)ds\Big|>\delta/3\Big\}+$$

$$+\mathrm{P}\Big\{\Big|\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(t_{k-1}),y)-\bar{q}(\xi(t_{k-1}),y)]\,\Pi(dy)ds\Big|>\delta/3\Big\}+$$

$$+\mathrm{P}\Big\{\Big|\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\int_{\mathrm{R}^d}[\bar{q}(\xi(s),y)-\bar{q}(\xi(t_{k-1}),y)]\,\Pi(dy)ds\Big|>\delta/3\Big\}=\mathrm{P}_1+\mathrm{P}_2+\mathrm{P}_3.$$

For estimation of P_1 and P_3 we use Chebyshev inequality, properties of chosen partition and above mentioned inequalities. Let us estimate P_1 :

$$\mathrm{P}_1\le\frac{3}{\delta}E\Big(\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\Big|\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(s),y)-q(s/\varepsilon,\xi(t_{k-1}),y)]\,\Pi(dy)\Big|ds\Big)\le$$

$$\le\frac{3}{\delta}E\Big(\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\Big(\Big|\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(s),y)-q(s/\varepsilon,\xi(t_{k-1}),y)]\,\Pi(dy)\Big|\times$$

$$\times\chi\{|\xi(s)-\xi(t_{k-1})|\le\delta_2\}\cdot\chi\{|\xi(s)|\le C,\ |\xi(t_{k-1})|\le C\}+$$

$$+C_1(\chi\{|\xi(s)-\xi(t_{k-1})|>\delta_2\}+\chi\{|\xi(s)|>C\}+\chi\{|\xi(t_{k-1})|>C\})\Big)ds\Big)\le$$

$$\le\frac{3}{\delta}\Big[E\Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\delta_1\,ds\Big)+C_1\Big(\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathrm{P}\{|\xi(s)-\xi(t_{k-1})|>\delta_2\}\,ds+$$

$$+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathrm{P}\{|\xi(s)|>C\}\,ds+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\mathrm{P}\{|\xi(t_{k-1})|>C\}\,ds\Big)\Big]\le\frac{Ct\delta_1}{\delta}.$$

Similarly we obtain $\mathrm{P}_3\le Ct\delta_1/\delta$, where we use notation C for any constant independent on ε. For each $k=\overline{1,n}$ from conditions of lemma we have

$$\lim_{\varepsilon\to0}\int_{t_{k-1}}^{t_k}\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(t_{k-1}),y)-\bar q(\xi(t_{k-1}),y)]\,\Pi(dy)ds=0\qquad\text{a.s.}$$

Therefore $\lim\limits_{\varepsilon\to0}\mathrm{P}_2=0$, and for arbitrary $\delta_1>0,\delta>0$

$$\lim_{\varepsilon\to0}\mathrm{P}\Big\{\Big|\int_0^t\int_{\mathrm{R}^d}[q(s/\varepsilon,\xi(s),y)-\bar q(\xi(s),y)]\,\Pi(dy)ds\Big|>\delta\Big\}\le\frac{Ct\delta_1}{\delta},$$

whence we obtain the statement of the lemma. □

Lemma 2. *Let for each $x\in\mathrm{R}^d$ there exists $\lim\limits_{T\to\infty}\frac{1}{T}\int_0^T b(t,x)\,dt=\bar b(x)$. The function $\bar b(x)$ is bounded and continuous, function $b(t,x)$ is bounded and continuous in x uniformly with respect to (t,x) in any region $t\in[0,\infty),|x|\le C$, and stochastic process $\xi(t)$ is stochastically continuous, then*

$$\mathrm{P}-\lim_{\varepsilon\to0}\int_0^t b(s/\varepsilon,\xi(s))\,ds=\int_0^t\bar b(\xi(s))\,ds$$

for all arbitrary $t\in[0,T]$.
The proof of this statement is similar to the proof of lemma 1.

We suppose that coefficients of equation (1) satisfy the following conditions:

1) $|f(t,x)|^2+\|g(t,x)\|^2+\int_{\mathrm{R}^d}|q(t,x,y)|^2\Pi(dy)\le C$, where $|f|^2=\sum\limits_{i=1}^{d}f_i^2$, $\|g\|^2=\sum\limits_{i,j=1}^{d}g_{ij}^2$;

2) For any $N>0$ there exists $L_N>0$ such that

$$|f(t,x_1)-f(t,x_2)|^2+\|g(t,x_1)-g(t,x_2)\|^2+$$

$$+\int_{\mathrm{R}^d} |q(t,x_1,y)-q(t,x_2,y)|^2\,\Pi(dy) \le L_N|x_1-x_2|^2,$$

for all $x_i \in \mathrm{R}^d, i=1,2$ such that $|x_i| \le N, i=1,2$.

3) Functions $f(t,x)$ and $g(t,x)$ are continuous in x uniformly with respect to $t \in [0,\infty)$ and x in every set $|x| \le C$. For each $x \in \mathrm{R}^d$ there exist the following limits

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t,x)\,dt = \bar f(x), \lim_{T\to\infty}\frac{1}{T}\int_0^T g(t,x)g^*(t,x)\,dt = \bar G(x),$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_{\mathrm{R}^d} q(t,x,y)q^*(t,x,y)\Pi(dy)dt = \int_{\mathrm{R}^d}\bar Q(x,y)\Pi(dy).$$

Here g^* is the matrix (vector) transpose to g, therefore for vector-valued function $q(t,x,y)$ the product $q(t,x,y)q^*(t,x,y)$ is the $d\times d$-matrix-valued function.

4) The functions $\bar f(x), \bar G(x)$, $\int_{\mathrm{R}^d}\bar Q(x,y)\,\Pi(dy)$ are bounded, continuous in x. Matrix $\bar B(x) = \bar G(x) + \int_{\mathrm{R}^d}\bar Q(x,y)\,\Pi(dy)$ is uniformly parabolic.

5) $\int_{\mathrm{R}^d} q(t,x,y)q^*(t,x,y)\,\Pi(dy)$ is bounded, continuous in x uniformly with respect to $t\in[0,\infty)$ in every compact set $|x|\le C$.
$\int_{\mathrm{R}^d}|q(t,x,y)|^i\,\Pi(dy) \le C,\ i=\overline{1,6}$.

Theorem 1. *Let conditions 1)-5) be fulfilled,* $k=\min(k_1,2k_2,2k_3)$ *and the function* $d(t,x)$ *is bounded, continuous in* x *uniformly with respect to* (t,x) *in any region* $t\in[0,\infty), |x|\le C$. *For each* $x\in\mathrm{R}^d$ *there exists* $\lim_{T\to\infty}\frac{1}{T}\int_0^T d(t,x)\,dt = \bar d(x)$. *The function* $\bar d(x)$ *is bounded and continuous. Let us consider* $\eta_\varepsilon(t) = (\varepsilon^k/t)\int_0^{t/\varepsilon^k} d(s,\xi(s))ds$, *where* $\xi(t)$ *is the solution of equation (1).*
1. If $k_1 = 2k_2 = 2k_3$, *then stochastic process* $\eta_\varepsilon(t)$ *converges in law, as* $\varepsilon\to 0$, *to stochastic process* $\bar\eta(t) = \frac{1}{t}\int_0^t \bar d(\bar\xi(s))\,ds$, *where process* $\bar\xi(t)$ *is the solution of stochastic differential equation*

$$d\bar\xi(t) = \bar f(\bar\xi(t))dt + \bar\sigma(\bar\xi(t))d\bar w(t), \quad \bar\xi(0)=\xi_0, \tag{2}$$

$\bar\sigma(x) = \bar B^{1/2}(x)$; $\bar w(t)$ *is some* d*-dimensional Wiener process.*
2. If $k<k_1$, *then in equation (2) the drift coefficient* $\bar f(x)$ *is absent; if* $k<2k_2$, *then in equation (2) the diffusion matrix* $\bar B(x)$ *does not depend on* $\bar G(x)$*; and if* $k<2k_3$, *then* $\bar B(x)$ *does not contain the term* $\int_{\mathrm{R}^d}\bar Q(x,y)\,\Pi(dy)$.

Proof. We can rewrite $\eta_\varepsilon(t)$ in the form $\eta_\varepsilon(t) = (1/t)\int_0^t d(s/\varepsilon^k,\xi(s/\varepsilon^k))\,ds$. Let us denote $\xi_\varepsilon(t)=\xi(t/\varepsilon^k)$, $w_\varepsilon(t)=\varepsilon^{k/2}w(t/\varepsilon^k)$, $\tilde\nu_\varepsilon(t,\cdot)=\nu(t/\varepsilon^k,\cdot)-$

$(t/\varepsilon^k)\Pi(\cdot)$. It worth to note that for any $\varepsilon > 0$ $w_\varepsilon(t)$ is the Wiener process and $\tilde{\nu}_\varepsilon(t,\cdot)$ is the centered Poisson measure. With these notations from equation (1) we obtain

$$\xi_\varepsilon(t) = \xi_0 + \varepsilon^{k_1-k}\int_0^t f(s/\varepsilon^k, \xi_\varepsilon(s))\,ds + \varepsilon^{k_2-k/2}\int_0^t g(s/\varepsilon^k, \xi_\varepsilon(s))\,dw_\varepsilon(s) + \\ + \varepsilon^{k_3}\int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon^k, \xi_\varepsilon(s), y)\,\tilde{\nu}_\varepsilon(ds, dy). \quad (3)$$

It follows from conditions 1), 2) that the solution of equation (3) exists and unique for each $\varepsilon > 0$.

Let us check that following conditions are fulfilled:

a) $\lim_{h\downarrow 0}\overline{\lim_{\varepsilon\to 0}}\sup_{|t-s|<h} \mathrm{P}\{|\xi_\varepsilon(t) - \xi_\varepsilon(s)| > \delta\} = 0$ for any $\delta > 0, t, s \in [0,T]$;

b) $\lim_{N\to\infty}\overline{\lim_{\varepsilon\to 0}}\sup_{t\in[0,T]} \mathrm{P}\{|\xi_\varepsilon(t)| > N\} = 0.$

Using properties of stochastic integrals, we can obtain the estimates

$$E|\xi_\varepsilon(t)|^2 \le C[E|\xi_0|^2 + (\varepsilon^{2(k_1-k)}T + \varepsilon^{2k_2-k} + \varepsilon^{2k_3-k})t],$$

$$E|\xi_\varepsilon(t) - \xi_\varepsilon(s)|^2 \le C[\varepsilon^{2(k_1-k)}|t-s| + \varepsilon^{2k_2-k} + \varepsilon^{2k_3-k}]|t-s|.$$

From Chebyshev inequality and obtained estimates we have fulfillment of conditions a) and b). Similarly we can check conditions a) and b) for stochastic process

$$\zeta_\varepsilon(t) = \varepsilon^{k_2-k/2}\int_0^t g(s/\varepsilon^k, \xi_\varepsilon(s))\,dw_\varepsilon(s) + \varepsilon^{k_3}\int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon^k, \xi_\varepsilon(s), y)\,\tilde{\nu}_\varepsilon(ds, dy).$$

Therefore [7, pp.13-18], for any sequence $\varepsilon_n \to 0, n = 1, 2, \ldots$ there exists a subsequence $\varepsilon_m = \varepsilon_{n_m} \to 0, m = 1, 2, \ldots$, probability space, stochastic processes $\tilde{\xi}_{\varepsilon_m}(t), \tilde{\zeta}_{\varepsilon_m}(t), \bar{\xi}(t), \bar{\zeta}(t)$ defined on this space, such that $\tilde{\xi}_{\varepsilon_m}(t) \to \bar{\xi}(t), \tilde{\zeta}_{\varepsilon_m}(t) \to \bar{\zeta}(t)$ in probability, as $\varepsilon_m \to 0$, and finite-dimensional distributions of $\tilde{\xi}_{\varepsilon_m}(t), \tilde{\zeta}_{\varepsilon_m}(t)$ coincide with finite-dimensional distributions of $\xi_{\varepsilon_m}(t), \zeta_{\varepsilon_m}(t)$. Since we are interested in limit behavior of distributions, we can consider processes $\xi_{\varepsilon_m}(t)$ and $\zeta_{\varepsilon_m}(t)$ instead of $\tilde{\xi}_{\varepsilon_m}(t), \tilde{\zeta}_{\varepsilon_m}(t)$. From (3) we obtain equation

$$\xi_{\varepsilon_m}(t) = \xi_0 + \varepsilon_m^{k_1-k}\int_0^t f(s/\varepsilon_m^k, \xi_{\varepsilon_m}(s))\,ds + \zeta_{\varepsilon_m}(t). \quad (4)$$

From this point we will omit the sub-index m in ε_m for simplicity of notation. It worth to note that processes $\xi_\varepsilon(t)$ and $\zeta_\varepsilon(t)$ are stochastically continuous without discontinuity of second kind. Let us obtain some estimates for processes $\xi_\varepsilon(t)$ and $\zeta_\varepsilon(t)$:

$$E|\xi_\varepsilon(t)-\xi_\varepsilon(s)|^4 = E\Big|\varepsilon^{k_1-k}\int_s^t f(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,d\tau+\zeta_\varepsilon(t)-\zeta_\varepsilon(s)\Big|^4 \le$$

$$\le C[\varepsilon^{4(k_1-k)}|t-s|^4+E|\zeta_\varepsilon(t)-\zeta_\varepsilon(s)|^4]. \tag{5}$$

$$E|\zeta_\varepsilon(t)-\zeta_\varepsilon(s)|^4 \le 8\Big(\varepsilon^{4k_2-2k}E\Big|\int_s^t g(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,dw_\varepsilon(\tau)\Big|^4+$$

$$+\varepsilon^{4k_3}E\Big|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^4\Big).$$

If we use Jensen's inequality and properties of one-dimensional Wiener process, we obtain

$$E\Big|\int_s^t g(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,dw_\varepsilon(\tau)\Big|^4 \le d\sum_{i=1}^d E\Big|\sum_{j=1}^d\int_s^t g_{ij}(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,dw^j_\varepsilon(\tau)\Big|^4 \le$$

$$\le d^4\sum_{i,j=1}^d E\Big|\int_s^t g_{ij}(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,dw^j_\varepsilon(\tau)\Big|^4 \le C(t-s)^2.$$

Let us estimate

$$E\Big|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^{2m} \quad \text{for } m=2,3. \tag{6}$$

Since

$$E\Big|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^{2m} \le$$

$$\le d^{m-1}\sum_{i=1}^d E\Big|\int_s^t\int_{\mathrm{R}^d} q_i(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^{2m},$$

it is sufficient to estimate $E|\int_s^t\int_{\mathrm{R}^d} q_i(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)|^{2m}$, $i=\overline{1,d}$. In view of this later on, estimating (6), we will consider one-dimensional case. Therefore for simplicity of notations we will omit the sub-index i. Let $\hat\xi_\varepsilon(t) = \int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)$. If we apply generalized Ito formula to $|\hat\xi_\varepsilon(t)|^{2m}$ and take mathematical expectation, we get

$$E|\hat\xi_\varepsilon(t)|^{2m} = E\int_s^t\int_{\mathrm{R}^d}\Big\{|\hat\xi_\varepsilon(\tau)+q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)|^{2m}-|\hat\xi_\varepsilon(\tau)|^{2m}-$$

$$-2m|\hat\xi_\varepsilon(\tau)|^{2m-1}\operatorname{sign}\hat\xi_\varepsilon(\tau)\cdot q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\Big\}\frac{\Pi(dy)}{\varepsilon^k}\,d\tau.$$

We obtain

$$E\Big|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^4\le$$

$$\le C[(t-s)^2\varepsilon^{-2k}+(t-s)^{3/2}\varepsilon^{-3k/2}+(t-s)\varepsilon^{-k}],$$

$$E\Big|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)\Big|^6\le$$

$$\le C[(t-s)^3\varepsilon^{-3k}+(t-s)^{5/2}\varepsilon^{-5k/2}+(t-s)^2\varepsilon^{-2k}+(t-s)\varepsilon^{-k}].$$

Taking into account obtained estimates of $E|\int_s^t g(\tau/\varepsilon^k,\xi_\varepsilon(\tau))\,dw_\varepsilon(\tau)|^4$ and $E|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)|^4$, we have:

$$E|\zeta_\varepsilon(t)-\zeta_\varepsilon(s)|^4\le C[(\varepsilon^{4k_2-2k}+\varepsilon^{4k_3-2k})|t-s|^2+$$

$$+\varepsilon^{4k_3-3k/2}|t-s|^{3/2}+\varepsilon^{4k_3-k}|t-s|]. \tag{7}$$

Similarly $E|\xi_\varepsilon(t)-\xi_\varepsilon(s)|^6\le C(\varepsilon^{6(k_1-k)}|t-s|^6+E|\zeta_\varepsilon(t)-\zeta_\varepsilon(s)|^6)$. Using the estimate of $E|\int_s^t\int_{\mathrm{R}^d} q(\tau/\varepsilon^k,\xi_\varepsilon(\tau),y)\,\tilde\nu_\varepsilon(d\tau,dy)|^6$ and taking into consideration that $k=\min(k_1,2k_2,2k_3)$, we obtain:

$$E|\xi_\varepsilon(t)-\xi_\varepsilon(s)|^6\le C,\quad E|\zeta_\varepsilon(t)-\zeta_\varepsilon(s)|^6\le C. \tag{8}$$

Since $\xi_\varepsilon(t)\to\bar\xi(t),\zeta_\varepsilon(t)\to\bar\zeta(t)$ in probability, as $\varepsilon\to 0$, then, using (8), from (5) and (7) we obtain estimates

$$E|\bar\xi(t)-\bar\xi(s)|^4\le C(|t-s|^4+|t-s|^2),\quad E|\bar\zeta(t)-\bar\zeta(s)|^4\le C|t-s|^2.$$

Therefore processes $\bar\xi(t)$ and $\bar\zeta(t)$ satisfy Kolmogorov's continuity condition [8, pp.235-237]. It should be noted that process $\zeta_\varepsilon(t)$ is the vector-valued square integrable martingale with matrix characteristic

$$\langle\zeta_\varepsilon,\zeta_\varepsilon\rangle(t)=\varepsilon^{2k_2-k}\int_0^t g(s/\varepsilon^k,\xi_\varepsilon(s))g^*(s/\varepsilon^k,\xi_\varepsilon(s))\,ds+$$

$$+\varepsilon^{2k_3-k}\int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon^k,\xi_\varepsilon(s),y)q^*(s/\varepsilon^k,\xi_\varepsilon(s),y)\,\Pi(dy)ds. \tag{9}$$

For any $\delta>0$

$$\mathrm{P}\Big\{\Big|\int_0^t d(s/\varepsilon^k,\xi_\varepsilon(s))\,ds-\int_0^t\bar d(\bar\xi(s))\,ds\Big|>\delta\Big\}\le$$

$$\le\frac{2}{\delta}E\Big|\int_0^t[d(s/\varepsilon^k,\xi_\varepsilon(s))-d(s/\varepsilon^k,\bar\xi(s))]\,ds\Big|+$$

$$+\mathrm{P}\Big\{\Big|\int_0^t d(s/\varepsilon^k,\bar\xi(s))\,ds-\int_0^t\bar d(\bar\xi(s))\,ds\Big|>\delta/2\Big\}=\frac{2}{\delta}I_1+I_2.$$

Since the function $d(t,x)$ is continuous in x uniformly with respect to (t,x) in any region $t \in [0,\infty), |x| \le N$, then for any $\delta_1 > 0$ there exists $\delta_2 > 0$ such, that $\sup_{t\ge 0} |d(t,x) - d(t,y)| \le \delta_1$ as $|x-y| \le \delta_2, |x| \le N, |y| \le N$. Therefore from boundedness of $d(t,x)$ we have

$$I_1 \le E\int_0^t |d(s/\varepsilon^k, \xi_\varepsilon(s)) - d(s/\varepsilon^k, \bar\xi(s))|\chi\{|\xi_\varepsilon(s) - \bar\xi(s)| \le \delta_2\}\times$$

$$\times\chi\{|\xi_\varepsilon(s)| \le N, |\bar\xi(s)| \le N\}\, ds + C\left(\int_0^t \mathrm{P}\{|\xi_\varepsilon(s) - \bar\xi(s)| > \delta_2\}\, ds + \right.$$

$$\left. + \int_0^t \mathrm{P}\{|\xi_\varepsilon(s)| > N\}\, ds + \int_0^t \mathrm{P}\{|\bar\xi(s)| > N\}\, ds\right) \le$$

$$\le \delta_1 t + \frac{C}{N^2} + C\int_0^t \mathrm{P}\{|\xi_\varepsilon(s) - \bar\xi(s)| > \delta_2\}\, ds.$$

Since $\mathrm{P}-\lim\limits_{\varepsilon\to 0} \xi_\varepsilon(s) = \bar\xi(s)$, $\delta_1 > 0$ and $N > 0$ are arbitrary, then $\lim\limits_{\varepsilon\to 0} I_1 = 0$.

The process $\bar\xi(s)$ is continuous and function $d(t,x)$ satisfies the conditions of lemma 2. Therefore $\lim\limits_{\varepsilon\to 0} I_2 = 0$ and

$$\lim_{\varepsilon\to 0}\int_0^t d(s/\varepsilon^k, \xi_\varepsilon(s))\, ds = \int_0^t \bar d(\bar\xi(s))\, ds \tag{10}$$

in law (because the distributions of $\xi_{\varepsilon_m}(t), \zeta_{\varepsilon_m}(t)$ coincide with distributions of stochastic processes $\tilde\xi_{\varepsilon_m}(t), \tilde\zeta_{\varepsilon_m}(t)$ and in fact we have proved that $\mathrm{P}-\lim_{\varepsilon_m\to 0}\int_0^t d(s/\varepsilon_m^k, \xi_{\varepsilon_m}(s))\, ds = \int_0^t \bar d(\bar\xi(s))\, ds)$.

Let us consider the case $k_1 = 2k_2 = 2k_3$. From (4) we obtain

$$\xi_\varepsilon(t) = \xi_0 + \int_0^t f(s/\varepsilon^k, \xi_\varepsilon(s))\, ds + \zeta_\varepsilon(t),$$

where martingale $\zeta_\varepsilon(t)$ has a matrix characteristic

$$\langle\zeta_\varepsilon, \zeta_\varepsilon\rangle(t) = \int_0^t g(s/\varepsilon^k, \xi_\varepsilon(s))g^*(s/\varepsilon^k, \xi_\varepsilon(s))\, ds +$$

$$+ \int_0^t\int_{\mathrm{R}^d} q(s/\varepsilon^k, \xi_\varepsilon(s), y)q^*(s/\varepsilon^k, \xi_\varepsilon(s), y)\,\Pi(dy)ds.$$

Using lemma 1 and lemma 2 it is easy to show that

$$\mathrm{P}-\lim_{\varepsilon\to 0}\int_0^t f(s/\varepsilon^k, \xi_\varepsilon(s))\, ds = \int_0^t \bar f(\bar\xi(s))\, ds,$$

$$\mathrm{P}-\lim_{\varepsilon\to 0}\langle\zeta_\varepsilon, \zeta_\varepsilon\rangle(t) = \int_0^t \bar B(\bar\xi(s))\, ds.$$

Hence $\bar{\zeta}(t)$ is a vector-valued continuous square integrable martingale with matrix characteristic $\langle\bar{\zeta},\bar{\zeta}\rangle(t) = \int_0^t \bar{B}(\bar{\xi}(s))\,ds$. It follows from conditions 4)-5) and [9, pp.446-449] that there exists a d-dimensional Wiener process $\bar{w}(t)$ such that $\bar{\zeta}(t) = \int_0^t \bar{\sigma}(\bar{\xi}(s))\,d\bar{w}(s)$, where $\bar{\sigma}(x)\bar{\sigma}^*(x) = \bar{B}(x)$. Therefore the process $\bar{\xi}(t)$ is the solution of stochastic differential equation

$$\bar{\xi}(t) = \xi_0 + \int_0^t \bar{f}(\bar{\xi}(s))\,ds + \int_0^t \bar{\sigma}(\bar{\xi}(s))\,d\bar{w}(s). \tag{11}$$

Furthermore, equation (11) has unique weak solution. Hence for any sequence $\varepsilon_m \to 0$ the stochastic process $\xi_{\varepsilon_m}(t)$ converges in probability to the solution $\bar{\xi}(t)$ of equation (11). From this and (10) we have proof of statement 1) of the theorem.

When $k < k_1$ the boundedness of $f(t,x)$ implies that $E\left|\int_0^t f(s/\varepsilon^k, \xi_\varepsilon(s))\,ds\right| \le C$, therefore the second term in the right side of (3) converges to 0 in probability, as $\varepsilon \to 0$, and we obtain the first statement in 2). From boundedness of $g(t,x)$ and $\int_{\mathrm{R}^d} q(t,x,y)q^*(t,x,y)\,\Pi(dy)$ we obtain that either first or second term in the right side of (9) converges to 0 in probability (respectively to the cases $k < 2k_2$ or $k < 2k_3$), as $\varepsilon \to 0$. Then we can complete the proof of statement 2) of the theorem as the proof of statement 1). □

3. Long-term returns in stochastic interest rate models

Suppose that a stochastic process X_t satisfies the stochastic differential equation

$$dX_t = (2\beta X_t + \delta_t)dt + g(X_t)dw_t + \int_{\mathrm{R}} q(X_t,y)\tilde{\nu}(dt,dy) \quad \forall\, t \in \mathrm{R}_+ \tag{12}$$

$\beta < 0$; $g(x)$, $q(x,y)$ are non-random functions; $x, y \in \mathrm{R}$; w_t is one-dimensional Wiener process; $\tilde{\nu}(dt,dy) = \nu(dt,dy) - \Pi(dy)dt$, $\nu(dt,dy)$ is the Poisson measure independent on w_t, $E\nu(dt,dy) = \Pi(dy)dt$; $\Pi(\cdot)$ is a sigma-finite measure on the σ-algebra of Borel sets in R; X_0 is the random variable independent on w_t and $\tilde{\nu}(t,\cdot)$ and such that there is a constant $c > 0$ with $EX_0^2 \le c$.

We suppose that the following conditions are fulfilled:

1) $g(0) = 0$, $q(0,y) = 0 \quad \forall\, y \in \mathrm{R}$;
2) there is a constant $b > 0$ with $|g(x_1) - g(x_2)|^2 \le b|x_1 - x_2|$ and $\int_{\mathrm{R}} |q(x_1,y) - q(x_2,y)|^2\,\Pi(dy) \le b|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathrm{R}$;
3) δ_t is non-random bounded function.

It follows from these conditions that equation (12) has weak solution [10, p.357]. If we use generalized Ito formula, we get the following representation of the solution:

$$X_t = e^{2\beta t}\Big(X_0 + \int_0^t \delta_s e^{-2\beta s}\,ds + \int_0^t g(X_s)e^{-2\beta s}\,dw_s + \\ + \int_0^t \int_{\mathrm{R}} q(X_s, y)e^{-2\beta s}\,\tilde\nu(ds, dy)\Big) \quad (13)$$

Theorem 2. *Suppose that X_t satisfies the stochastic differential equation (12). Let conditions 1)-3) be fulfilled and $\frac{1}{t}\int_0^t \delta_s\,ds \to \bar\delta$, as $t \to \infty$. Then $\frac{1}{t}\int_0^t X_s ds \to \frac{\bar\delta}{-2\beta}$ in mean square, as $t \to \infty$.*

Proof. Using representation (13), let us estimate EX_t^2, $t \in \mathrm{R}_+$.

$$EX_u^2 \le 4Ee^{4\beta u}\Big(X_0^2 + \Big(\int_0^u \delta_s e^{-2\beta s}\,ds\Big)^2 + \Big(\int_0^u g(X_s)e^{-2\beta s}\,dw_s\Big)^2 + \\ + \Big(\int_0^u \int_{\mathrm{R}} q(X_s, y)e^{-2\beta s}\,\tilde\nu(ds, dy)\Big)^2\Big) = I.$$

$$\int_0^u Eg^2(X_s)e^{-4\beta s}\,ds \le be^{-4\beta u}\int_0^u E|X_s|\,ds \le be^{-4\beta u}\int_0^u (EX_s^2)^{1/2}\,ds.$$

The same estimate we have for $\int_0^u \int_{\mathrm{R}} Eq^2(X_s, y)e^{-4\beta s}\,\Pi(dy)ds$. It follows from [10, p.370] that in conditions of the theorem EX_s^2 is bounded on $[0, u]$, therefore $\int_0^u Eg^2(X_s)e^{-4\beta s}\,ds < \infty$ and $\int_0^u \int_{\mathrm{R}} Eq^2(X_s, y)e^{-4\beta s}\,\Pi(dy)ds < \infty$. Then

$$I = 4Ee^{4\beta u}X_0^2 + 4e^{4\beta u}\Big(\int_0^u \delta_s e^{-2\beta s}\,ds\Big)^2 + \\ +4e^{4\beta u}\int_0^u Eg^2(X_s)e^{-4\beta s}ds + 4e^{4\beta u}\int_0^u \int_{\mathrm{R}} Eq^2(X_s, y)e^{-4\beta s}\,\Pi(dy)ds.$$

$$e^{4\beta u}\Big(\int_0^u \delta_s e^{-2\beta s}\,ds\Big)^2 \le Ce^{4\beta u}\Big(\frac{e^{-2\beta u}-1}{-2\beta}\Big)^2 \le Ce^{4\beta u}\cdot e^{-4\beta u} = C.$$

$$\int_0^u Eg^2(X_s)e^{-4\beta s}ds + \int_0^u \int_{\mathrm{R}} Eq^2(X_s, y)e^{-4\beta s}\,\Pi(dy)ds \le \\ \le 2b\int_0^u E|X_s|e^{-4\beta s}\,ds \le 2b\int_0^u (EX_s^2)^{1/2}e^{-4\beta s}\,ds.$$

$$\sup_{u\le t} EX_u^2 \le 4EX_0^2 + C + 8b\sup_{u\le t} e^{4\beta u}\int_0^u (EX_s^2)^{1/2}e^{-4\beta s}\,ds \le$$

$$\le C_1 + C_2 \sup_{u\le t} e^{4\beta u} \int_0^u (\sup_{\tau\le t} EX_\tau^2)^{1/2} e^{-4\beta s}\, ds \le$$

$$\le C_1 + (\sup_{\tau\le t} EX_\tau^2)^{1/2} \cdot C_2 \sup_{u\le t} e^{4\beta u} \frac{e^{-4\beta u}-1}{-4\beta} \le C_1 + C_3 (\sup_{u\le t} EX_u^2)^{1/2}.$$

Therefore $\sup_{u\le t} EX_u^2 - C_3(\sup_{u\le t} EX_u^2)^{1/2} \le C_1$, whence $\sup_{u\le t} EX_u^2 \le C$, where constant C is independent on t, and thus $EX_t^2 \le C$.

From the equation (12) we obtain

$$\frac{X_t - X_0}{t} = \frac{1}{t}\int_0^t 2\beta X_s\, ds + \frac{1}{t}\int_0^t \delta_s\, ds +$$

$$+ \frac{1}{t}\int_0^t g(X_s)\, dw_s + \frac{1}{t}\int_0^t \int_{\mathrm{R}} q(X_s, y)\, \tilde{\nu}(ds, dy). \tag{14}$$

Using that $EX_t^2 \le C$, we have

$$E\Big(\frac{X_t - X_0}{t}\Big)^2 \le 2\Big(\frac{EX_t^2}{t^2} + \frac{EX_0^2}{t^2}\Big) \to 0,\ t\to\infty.$$

$$\frac{1}{t^2} E\Big(\int_0^t g(X_s)\, dw_s\Big)^2 = \frac{1}{t^2}\int_0^t Eg^2(X_s)\, ds \le \frac{1}{t^2}\int_0^t bE|X_s|\, ds \le$$

$$\le \frac{1}{t^2}\int_0^t b(EX_s^2)^{1/2}\, ds \le \frac{C}{t} \to 0,\ t\to\infty.$$

$$\frac{1}{t^2} E\Big(\int_0^t \int_{\mathrm{R}} q(X_s, y)\tilde{\nu}(ds, dy)\Big)^2 = \frac{1}{t^2}\int_0^t \int_{\mathrm{R}} Eq^2(X_s, y)\, \Pi(dy) ds \le$$

$$\le \frac{1}{t^2}\int_0^t bE|X_s|\, ds \le \frac{1}{t^2}\int_0^t b(EX_s^2)^{1/2}\, ds \le \frac{C}{t} \to 0,\ t\to\infty.$$

$$\frac{1}{t}\int_0^t \delta_s\, ds \to \bar{\delta},\ t\to\infty.$$

Therefore $\frac{1}{t}\int_0^t X_s\, ds \to \frac{\bar{\delta}}{-2\beta}$ in mean square □

Remark. In conditions of theorem 2 there is a constant $C > 0$ with $EX_t^2 \le C$.

Let us generalize theorem 2 to the case of the stochastic process δ_t.

Theorem 3. *Suppose that the stochastic differential equation (12) has a solution X_t such that $EX_t^2 < \infty$ and let conditions 1)-2) be fulfilled. Suppose that δ_t is a stochastic process and that there is a constant $k > 0$ such that $\int_0^t E\delta_s^2\, ds \le k(1+t) \quad \forall t \in \mathrm{R}_+$ and $\frac{1}{t}\int_0^t \delta_s\, ds \to \bar{\delta}$ in mean square, as $t\to\infty$. Then $\frac{1}{t}\int_0^t X_s ds \to \frac{\bar{\delta}}{-2\beta}$ in mean square, as $t\to\infty$.*

Proof. Similarly to the proof of theorem 2 we obtain

$$\sup_{u\le t} EX_u^2 \le C_1 + \sup_{u\le t} Ee^{4\beta u}\Big(\int_0^u \delta_s e^{-2\beta s}\,ds\Big)^2 + C_3\Big(\sup_{u\le t} EX_u^2\Big)^{1/2}.$$

Since

$$\sup_{u\le t} Ee^{4\beta u}\Big(\int_0^u \delta_s e^{-2\beta s}\,ds\Big)^2 \le \sup_{u\le t} Ee^{4\beta u}\int_0^u \delta_s^2\,ds\int_0^u e^{-4\beta s}\,ds \le$$

$$\le \sup_{u\le t} Ee^{4\beta u}\frac{e^{-4\beta u}}{-4\beta}\int_0^u \delta_s^2\,ds \le \frac{1}{-4\beta}\int_0^t E\delta_s^2\,ds \le C_4(t+1),$$

then

$$\sup_{u\le t} EX_u^2 - C_3(\sup_{u\le t} EX_u^2)^{1/2} \le C_1 + C_4(t+1),$$

$$\Big((\sup_{u\le t} EX_u^2)^{1/2} - C_3/2\Big)^2 \le C_1 + C_4(t+1) + C_3^2/4,$$

whence it follows that $EX_t^2 \le C(t+1)$, where constant C is independent on t. Therefore

$$E\Big(\frac{X_t - X_0}{t}\Big)^2 \le 2\Big(\frac{EX_t^2}{t^2} + \frac{EX_0^2}{t^2}\Big) \to 0,\ t\to\infty.$$

$$\frac{1}{t^2}E\Big(\int_0^t g(X_s)\,dw_s\Big)^2 \le \frac{1}{t^2}\int_0^t b(EX_s^2)^{1/2}\,ds \le$$

$$\le \frac{C}{t^2}\int_0^t (s+1)^{1/2}\,ds = C\frac{(t+1)^{3/2}-1}{t^2} \to 0,\ t\to\infty.$$

$$\frac{1}{t^2}E\Big(\int_0^t\int_{\mathrm{R}} q(X_s,y)\tilde{\nu}(ds,dy)\Big)^2 \le \frac{1}{t^2}\int_0^t b(EX_s^2)^{1/2}\,ds \to 0,\ t\to\infty.$$

Therefore, taking into account (14), we obtain $\frac{1}{t}\int_0^t X_s\,ds \to \frac{\bar{\delta}}{-2\beta}$ in mean square. □

4. A two-factor stochastic interest rate model

Let us consider an application of theorem 2 and theorem 3. We study the two-factor model

$$dr_t = k(\gamma_t - r_t) + \sigma\sqrt{|r_t|}\,dw_t + \int_{\mathrm{R}} q(r_t,y)\tilde{\nu}(dt,dy),$$

$$d\gamma_t = \tilde{k}(\gamma^* - \gamma_t)\,dt + \tilde{\sigma}\sqrt{\gamma_t}\,dw_t,$$

$k,\tilde{k} > 0$; γ^*, σ and $\tilde{\sigma}$ are positive constants; $(w_t)_{t\ge0}$ and $(\tilde{w}_t)_{t\ge0}$ are two Wiener processes; $\tilde{\nu}(dt,dy) = \nu(dt,dy) - \Pi(dy)dt$, $\nu(dt,dy)$ is the Poisson measure independent on w_t, $E\nu(dt,dy)=\Pi(dy)dt$; $\Pi(\cdot)$ is a sigma-finite measure on the σ-algebra of Borel sets in R.

Suppose that the first equation of the model has a solution and that

$$\int_{\mathrm{R}} |q(x_1,y)-q(x_2,y)|^2\,\Pi(dy) \le b|x_1-x_2|\ \forall\, x_1,x_2\in \mathrm{R}, \quad q(0,y)=0\ \forall\, y\in \mathrm{R}.$$

We are interested in the convergence of the long-term return $\frac{1}{t}\int_0^t r_s\,ds$.

Applying theorem 2 to the second equation of the model we have, that $\frac{1}{t}\int_0^t \gamma_s\,ds \to y^*$ in mean square. Really, if we define $Y_t=4\gamma_t/\tilde{\sigma}^2$, then Y_t satisfies stochastic differential equation of the kind

$$dY_t=(\delta_t+2\tilde{\beta}Y_t)dt+\tilde{g}(Y_t)d\tilde{w}_t,$$

with $\tilde{\beta}=-\tilde{k}/2$, $\delta_t=4\tilde{k}\gamma^*/\tilde{\sigma}^2\ \forall\, t\in\mathrm{R}_+$, $\tilde{g}(Y_t)=2\sqrt{Y_t}$.
We can note, that [11] there exists a solution of this equation, it is unique and non-negative. Since conditions of theorem 2 are fulfilled, then $\frac{1}{t}\int_0^t Y_s\,ds \to 4\gamma^*/\tilde{\sigma}^2$ in mean square and accordingly $\frac{1}{t}\int_0^t \gamma_s\,ds \to \gamma^*$ in mean square. Further, taking into consideration remark to the theorem 2, we have $E\gamma_t^2\le C$, where constant C is independent on t.

Now we consider first equation of the model. If we define $X_t=4r_t/\sigma^2$, then X_t satisfies, in the notations of theorem 3, the equation of the following kind

$$dX_t=(2\beta X_t+\delta_t)dt+g(X_t)dw_t+\int_{\mathrm{R}} q_1(X_t,y)\tilde{\nu}(dt,dy),$$

with $\beta=-k/2$, $\delta_t=4k\gamma_t/\sigma^2$, $g(X_t)=2\sqrt{|X_t|}$, $q_1(X_t,y)=\frac{4}{\sigma^2}q(\frac{\sigma^2}{4}X_t,y)$. Since conditions of theorem 3 are fulfilled and $\frac{1}{t}\int_0^t \delta_s\,ds=(\frac{1}{t}\int_0^t \gamma_s\,ds)\frac{4k}{\sigma^2}\to \gamma^*\frac{4k}{\sigma^2}$ in mean square, then $\frac{1}{t}\int_0^t X_s\,ds \to \frac{4\gamma^*}{\sigma^2}$ in mean square and finally $\frac{1}{t}\int_0^t r_s\,ds \to \gamma^*$ in mean square.

We conclude that the long-term return converges in mean square to the long-term constant value γ^*.

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