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STRONG INVARIANCE PRINCIPLE FOR RENEWAL AND RANDOMLY STOPPED PROCESSES

The strong invariance principle for renewal process and randomly stopped sums when summands belong to the domain of attraction of an α-stable law is presented

1. Introduction

Let $\{X, X_i, i \geq 1\}$ be independent identically distributed random variables (i.i.d.r.v) with common distribution function (d.f.) $F(x)$ and characteristic function (ch.f.) $\varphi(t)$. Suppose that $EX = m$ if $E|X| < \infty$ and $Var(X) = 1$ if $E|X|^2 < \infty$. Put

$$S(n) = \sum_{i=1}^{n} X_i, \quad S(0) = 0, \quad S(x) = S([x]), \tag{1}$$

where $[a]$ is entire of $a > 0$.

Let $\{z, z_i, i \geq 1\}$ be another sequence of i.i.d.r.v. independent of $\{X_i, i \geq 1\}$ with d.f. $F_1(x)$ and ch.f. $\varphi_1(t)$, $Ez = 1/\lambda$ if $E|z| < \infty$ and $Var(X) = \tau^2$ if $E|Z|^2 < \infty$.

Denote

$$Z(n) = \sum_{i=1}^{n} z_i, \quad Z(0) = 0, \quad Z(x) = Z([x])$$

and define the renewal counting process as

$$N(t) = \inf\{x \geq 0 : Z(x) > t\}. \tag{2}$$

Invited lecture.

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We shall consider also the randomly stopped sum process (i.e. the superposition of random processes $S(n)$ and $N(t)$)

$$D(t) = S(N(t)) = \sum_{i=1}^{N(t)} X_i, \tag{3}$$

where renewal process $N(t)$ is defined by (2).

The main task of this paper is to study the asymptotic behavior of the random processes $D(t)$ and $N(t)$ when $F(x)$ and $F_1(x)$ are heavy tailed. This problem has a deep relation with investigations of risk process $U(T)$ and approximation of ruin probabilities in Sparre Anderssen collective risk model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \tag{4}$$

where: $u \geq 0$ denotes the initial capital; $c > 0$ stands for the premium income rate; i.i.d.r.v $\{X_i, i \geq 1\}$ are interpreted as claim sizes; $N(t)$ describes the claim arrival process and stands for the number of claims until time t; $\{z_i, i \geq 1\}$ being the inter-arrival times.

In such model $S(N(t))$ is interpreted as total claim amount process and is a stochastic part of risk process.

Limit theorems for risk process such as (weak) invariance principle which constitute the weak convergence of $U(t)$ to the Wiener process $W(t)$ with the drift (when $EX^2 < \infty$, $Ez^2 < \infty$) or to the α-stable Lévy process $Y_\alpha(t)$ (when $EX^2 = \infty$, $Ez^2 < \infty$) lead to useful approximations of the ruin probability

$$\psi(u) = P\{\inf_{t>0} U(t) < 0\}. \tag{5}$$

Thus, in the case $EX^2 < \infty$, $Ez^2 < \infty$ one obtains the "diffusion approximation" for $\psi(u)$ as a distribution of infimum of the Wiener process (Iglehart (1969), Grandell (1991), Embrechts et al.(1997)) and in the case $EX^2 = \infty$, $Ez^2 < \infty$ the ruin probability $\psi(u)$ is approximated by the distribution of infimum of the corresponding α-stable process (Furrur, Michna and Weron (1997), Furrur (1998))

2. Strong invariance principle for the partial sums

Strong invariance principle (almost sure approximation) is an umbrella name for the class of limit theorems which ensure the possibility to construct $\{X_i, i \geq 1\}$ and Lévy process $Y(t)$, $t \geq 0$ on the same probability space in such a way that with probability 1

$$|S(t) - mt - Y(t)| = o(r(t)), \text{ as } t \to \infty \tag{6}$$

or

$$|S(t) - mt - Y(t)| = O(r(t)), \text{ as } t \to \infty \tag{7}$$

were approximation error (rate) $r(.)$ is a non-random function depending only on assumptions posed on X.

Additional assumptions on X clear up the type of $Y(t)$ and form of $r(\cdot)$. Note that the complete solution of the problem of a.s. approximation depends not only on the distribution of $\{X_i, i \geq 1\}$ but also on a structure of the probability space, and (possibly) requires a "richer" probability space and equivalent r.v.$\{X_i^{'}, i \geq 1\}$. However, for brevity we do not distinguish between r.v.$\{X_i\}$ and$\{X_i^{'}\}$ as well as between their sums $S(n)$ and $S^{'}(n) = \sum_{i=1}^{n} X_i^{'}$.

We also use the concept of a.s. approximation in a wider sense, and say that a random process $\xi(t)$ admits the a.s. approximation by the random process $\eta(t)$ if $\xi(t)$ (or stochastically equivalent $\{\xi^{'}(t), t \geq 0\}$) can be constructed on the rich enough probability space together with $\eta(t), t \geq 0$ in such a way that a.s.

$$|\xi(t) - \eta(t)| = o(r_1(t)) \vee O(r_1(t)),$$

where $r_1(.)$ is again a non-random function.

The origin of this topic in the theory of limit theorems goes back to the famous "Skorokhod representation" and "Skorokhod embedding scheme" (Skorokhod (1961). Skorokhod representation allows one to study a sequence of values of the Wiener process $W(T_n)$, where T_n, $n \geq 1$, are some stopping times, instead of partial sums $S(n)$. Based on Skorokhod embedding scheme Strassen (1964, 1965) proved the first variant of the strong invariance principle.

In 1970 – 1995 the further investigations were carried out by a number of authors, among them: Kiefer, M.Csörgő, Révész, Komlós, Major, Tusnady, Berkes, Horváth (quantile Hungerian method), Stout, Phillip, Berkes (relationship between the strong invariance principle and convergence in Prokhorov metrics), Horváth (inverse processes).

The wide bibliography which covers the period between 1961 and 1980 is presented in M.Csörgő, P.Révész (1981); more recent results in M.Csörgő, L. Horváth, (1993), see also Zinchenko (2000).

Summarizing all mentioned above results we have

Theorem A1. *It is possible to define partial sum process $S(t), t \geq 0$ and a standard Wiener process $W(t), t \geq 0$ in such a way that a.s.*

$$|S(t) - mt - W(t)| = o(r(t)), \tag{8}$$

with: $r(t) = t^{1/p}$ if and only if $E|X|^p < \infty, p > 2$, $r(t) = (t \log\log(t))^{1/2}$ if *and only if* $E|X|^2 < \infty$*; while (8) can be changed on* $O(\log t)$ *if and only if* $E\exp(tX) < \infty$ *for some* $t > 0$*.*

3. Strong invariance principle for the sums of r.v. attracted to the stable law

Suppose that $EX^2 = \infty$; more precise we assume that $\{X, X_i, i \geq 1\}$ belong to the domain of normal attraction of the stable law $G_{\alpha,\beta}$ (notation $\{X_i\} \in DNA(G_{\alpha,\beta})$).

Here $G_{\alpha,\beta}(.)$ is a d.f. of the stable law with parameters $0 < \alpha < 2$, $|\beta| \leq 1$ and ch.f.

$$g_{\alpha,\beta}(u) = \exp(-K(u)), \tag{10}$$

where

$$K(u) = K_{\alpha,\beta}(u) - |u|(1 - i\beta(u/|u|)\varpi(u,\alpha)), \tag{11}$$

$\varpi(u,\alpha) = \tan(\pi\alpha/2)$ if $1 < \alpha < 2$, $\varpi(u,\alpha) = -(2/\pi)\log|u|$ if $\alpha = 1$.

We recall that i.i.d.r.v. $\{X_i\} \in DNA(G_{\alpha,\beta})$ if for normalized and centered sums S_n^* there is a weak convergence

$$S_n^* = n^{-1/\alpha}(S(n) - a_n) \Rightarrow G_{\alpha,\beta}, \tag{12}$$

where $a_n = nEX = mn$ if $1 < \alpha < 2$, $a_n = O$ if $0 < \alpha < 1$ and $a_n = (2/\pi)\beta\log n$ if $\alpha = 1$.

For the r.v. $X \in DNA(G_{\alpha,\beta})$ (as well as for the α-stable r.v.) $E|X|^p < \infty\ \forall p < \alpha$, but $E|X|^p = \infty\ \forall p > \alpha$.

Denote by $Y(t) = Y_\alpha(t) = Y_{\alpha,\beta}(t)$, $t \geq 0$, the *α-stable Lévy process* with ch.f.

$$g_\alpha(t;u) = g_{\alpha,\beta}(t;u) = \exp(tK_{\alpha,\beta}(u)), \tag{13}$$

where $K_{\alpha,\beta}(u)$ is defined in (11), $Y_\alpha(0) = 0$. In what follows we omit index β if it is not essential.

Strong invariance principle for $\{X_i\} \in DNA(G_{\alpha,\beta})$ when approximating process is α-stable Lévy process (or partial sum process with stable summands) was studied by Stout (1979), Mijnheer (1983, 1995), Zinchenko (1984), Berkes, Dabrowski, Dehling, Philipp (1986), Berkes and Dehling (1989) in the case of symmetric stable law ($\beta = 0$) and Zinchenko (1985, 1989, 1997) without any restriction on parameters α and β.

The fact that $\{X_i\} \in DNA(G_{\alpha,\beta})$ is not enough to obtain "good" error term in (6), thus, certain additional assumptions are needed. We formulate them in terms of ch.f.

Assumption (C) :
there are $a_1 > 0, a_2 > 0$ and $l > \alpha$ such that for $|u| < a_1$

$$|f(u) - g_{\alpha,\beta}(u)| < a_2|u|^l \tag{14}$$

where $f(u) = e^{-itm}\varphi(t)$ is a ch.f. of $(X - EX)$ if $1 < \alpha < 2$ and $f(u) = \varphi(t)$, i.e. ch.f of X if $0 < \alpha \leq 1$.

Put

$$A = [max\{\alpha + 1, 2\alpha(2 + \alpha + 1/\alpha)/(l - \alpha)\}] + 1, \quad \text{if } 0 < \alpha < 1,$$

$$A = [max\{\alpha(\alpha + 1), 2\alpha(2\alpha + 1)/(l - \alpha)\}] + 1, \quad \text{if } 1 < \alpha < 2,$$

$$A = \{[10/(l - 1)] + 1, 6\}], \quad \text{if } \alpha = 1,$$

where $[a]$ is entire of $a > 0$.

Theorem A2. *Put $m = EX$ for $1 < \alpha < 2$ and $m = 0$ for $0 < \alpha \leq 1$. Under assumption (**C**) it is possible to define α-stable process $Y_{\alpha,\beta}(t), t \geq 0$ such that a.s.*

$$\sup_{0 \leq t \leq T} |S(t) - mt - Y_{\alpha,\beta}(t)| = o(T^{1/\alpha - \rho}), \tag{15}$$

for some $\rho = \rho(\alpha, l) \in (0, 1/\alpha(A + 1))$.

In the case $EX = 0$ Theorem A2 was proved by Zinchenko (1987, 1997), obvious centering when $1 < \alpha < 2$ provides (15). If $\alpha \neq 1$ detail analysis of the proof in Zinchenko (1997) shows that it is possible to obtain a shaper estimate for ρ and establish that a.s.

$$\sup_{0 \leq t \leq T} |S([t]) - mt - Y_{\alpha,\beta}(t)| = O(T^{1/\alpha - \rho_1}), \tag{16}$$

where

$$\rho_1 = 1/4\alpha(A + 1). \tag{17}$$

Thus, (15) holds for any $\rho \in (0, 1/4\alpha(A + 1))$.

It worth mentioning that unlike Theorem A1 ($\alpha = 2$) Theorem A2 presents only sufficient condition for strong invariance principle and tells nothing about optimality of the error term.

4. Asymptotic behaviour of the renewal process. Auxiliary results

Let $N(t) = \inf\{x > 0 : Z(x) > t\}$ be renewal(counting) process associated with sum process $Z(n) = \sum_{i=1}^{n} z_i$ with $0 < Ez_1 = 1/\lambda < \infty$. For applications it is often convenient to suppose that z_i are non-negative (non-zero) r.v. It is clear that $N(t)$ is the generalized right-continuous inverse of right-continuous process $Z(t)$.

Following auxiliary results will be useful for further investigations.

Lemma 1 (Csörgő, Horvách (1993))*Let* $0 < \lambda < \infty$*, then a.s.*

$$\limsup_{t\to\infty} N(t)/t \le \lambda. \tag{18}$$

Order of magnitude of $N(t)$ is described by following theorem which includes strong law of large numbers (SLLN), Marcinkiewich-Zygmund SLLN and law of iterated logarithm for renewal process.

Theorem A3.
(i) If $0 < Ez = 1/\lambda < \infty$*, then a.s.*

$$N(t)/t \to \lambda, \tag{19}$$

(ii) if $E|z|^p < \infty$ *for some* $p \in (1,2)$ *then a.s.*

$$t^{-1/p}(N(t) - \lambda t) \to 0, \tag{20}$$

(iii) if $\tau^2 = Var(X) < \infty$ *then*

$$\limsup_{t\to\infty}(2t\log\log t)^{-1/2}|N(t) - \lambda t| = \tau\lambda^{3/2}, \tag{21}$$

while the for the moments we have

$$EN(t) \sim \lambda t, Var(N(t)) \sim \tau\lambda^{3/2}.$$

The sketch of the proof is presented in Embrechts et al. (1977), see also A.Gut (1988). Original Marcinkiewich-Zygmund SLLN for partial sums of i.i.d.r.v. can be find in Loéve (1978).

Weak convergence, particularly, weak invariance principle for renewal process is in details presented in the book by Whitt (2002).

Next two simple lemmas from Csörgő, Horvách (1993) deals with the properties of the inverse step functions.

Here a function $\theta(t), t \in [0,\infty)$, is called a right-continuous step function if there is a decomposition of $[0,\infty) = \bigcup_{i=1}^{\infty}[t_i, t_{i+1})$ such that $0 = t_1 < t_2 < \ldots$ and $\theta(t) = q_i$ for $t \in [t_i, t_{i+1})$, $q_i \in R^1$, $q_1 = 0$. The right-continuous inverse of θ is defined by

$$\psi(x) = \inf\{t \ge 0 : \theta(t) > x\}, 0 \le u < \infty, \inf\emptyset = \infty$$

Lemma 2. *For any* $T \ge 0$

$$\sup_{0\le x\le T}|\psi(x) - x| \le \sup_{0\le t\le\psi(T)}|\theta(t) - t|. \tag{22}$$

Lemma 3. *For any* $T \geq 0$

$$t - \psi(t) = \theta(\psi(t)) - \psi(t) - (\theta(\psi(t)) - t), \tag{23}$$

$$\sup_{0 \leq t \leq T} |\theta(\psi(t)) - t| \leq \sup_{0 \leq t \leq T} |\theta(t) - \theta(t-)|. \tag{24}$$

The growth rate of α-stable Lèvy process $Y_\alpha(t)$ when $t \to \infty$ is described by the following statement.

Lemma 4. *If* $Y_\alpha(t)$ *is an* α*-stable Lèvy process with* $0 < \alpha < 2$ *then a.s.*

$$Y_\alpha(t) = o(t^{1/\alpha+\varepsilon}), \forall \varepsilon > 0.$$

This fact follows immediately from the integral test for upper/lower functions of Lévy process (Gikhman and Skorokhod (1973, ch.4).

Keeping in mind these facts and equivalence in weak convergence for $S(n)$ and associated $N(t)$ it is natural to ask about a.s. approximation of $N(t)$.

5. Strong invariance principle for renewal process

5.a. Assumptions: $Ez^2 < \infty$, $0 < Ez = 1/\lambda < \infty$.

During 1984 - 2000 strong approximation of the counting process $N(t)$ associated with partial sum process $Z(x) = \sum_{i=1}^{[x]} z_i$ in the case $E|z|^p < \infty$ for $p \geq 2$ (or more general moment conditions) was investigated by a number of authors, among them Horvách, M. Csörgő, Steinebach, Aalex, Deheuvels, Mason, van Zwet. They studied a.s. approximation of the type

$$|\lambda t - N(t) - \lambda W(\lambda t)| = o(r(t)) \vee O(r(t)). \tag{25}$$

For instance, M. Csörgő, Horvách and Steinebach (1986, 1987) obtained the best possible approximations of $N(t)$. It turned out that conditions which provide (25) and corresponding optimal errors in the case of nonnegative r.v. $\{z_i\}$ are just the same as for partial sums $Z(n)$ (see Theorem A1).

5.b. Assumptions: $\{z_i\} \in NDA(G_{\alpha,\beta})$ with $1 < \alpha < 2$.

Theorem 1. *Let* $\{z_i\}$ *satisfy* ***(C)*** *with* $1 < \alpha < 2$ *and* $0 < Ez = 1/\lambda < \infty$ *then a.s.*

$$|t\lambda - N(t) - \lambda Y_{\alpha,\beta}(\lambda t)| = o(r(t)), \tag{26}$$

where $r(t) = t^{1/\alpha+\delta}$ *for any* $\delta > 0$.

Proof. We use the idea of M. Csörgő, L.Horvách and Steinebach about the correspondence between a.s.approximation of $Z(n)$ and associated counting process $N(t)$. Consider

$$Z_1(x) = \lambda Z(x), \quad N_1 = \inf\{x : Z_1(x) > t\} = \inf\{x : Z(x) > t/\lambda\}. \tag{27}$$

Thus, $N_1(t) = N(t/\lambda)$ and (18) or (19) yields

$$\limsup_{t\to\infty} N_1(t)/t \le 1. \tag{28}$$

As far as condition **(C)** is concerned, Theorem A2 ensure the possibility to define α-stable process $Y_\alpha(t) = Y_{\alpha,\beta}(t)$ such that a.s.

$$|Z(t) - t\lambda - Y_\alpha(t)| = O(T^{1/\alpha-\rho_1}), \tag{29}$$

for some $\rho_1 = \rho_(\alpha, l) > 0$.

Thus,

$$|Z_1(t) - t - \lambda Y_\alpha(t)| = O(T^{1/\alpha-\rho_1}). \tag{30}$$

By Lemma 3 and definition of $Z_1(t)$, $N(t)$

$$t - N_1(t) = Z_1(N_1(t)) - N_1(t) + A_1(t) \tag{31}$$

where

$$\sup_{0\le t\le T} |A_1(t)| \le \sup_{0\le t\le N_1(T)} |Z_1(t) - Z_1(t-)| \le \lambda \max_{0\le t\le N_1(T)} |z_i| \tag{32}$$

Since r.v.$\{z_i\} \in NDA(G_{\alpha,\beta})$ with $1 < \alpha < 2$ have finite moments $E|Z_i|^p < \infty$ for any $p < \alpha$, Marcinkiewich-Zygmund SLLN for $Z(n)$ yields a.s.

$$\max_{0\le i\le n} |z_i| = o(n^{1/p}), \quad \forall p \in (1, \alpha) \tag{33}$$

From (28) and (33) we conclude that a.s.

$$\sup_{0\le t\le T} |A_1(t)| \le \lambda \max_{0\le t\le N_1(T)} |z_i| = o(T^{1/\alpha+\varepsilon}) \ \ \forall\varepsilon > 0. \tag{34}$$

Therefore, (31) and (34) implies that

$$L(T) = \sup_{0\le t\le T} |t - N_1(t) - \lambda Y_\alpha(t)| \le$$

$$\le \sup_{0\le t\le T} |Z_1(N_1(t)) - N_1(t) - Y_\alpha(N_1(t))| +$$

$$+ \sup_{0\le t\le T} |Y_\alpha(N_1(t)) - \lambda Y_\alpha(t)| + \sup_{0\le t\le T} |A_1(t)|. \tag{35}$$

Next by (30) and (28) a.s.

$$\sup_{0\le t\le T} |Z_1(N_1(t)) - N_1(t) - Y_\alpha(N_1(t))| = O(T^{1/\alpha-\rho_1}), \tag{36}$$

for some $\rho_1 = \rho_1(\alpha, l) > 0$.

Lemma 4, which provides an upper function for $Y_\alpha(t)$, implies

$$\sup_{0\le t\le T} |Y_\alpha(N_1(t)) - Y_\alpha(t)| = o(T^{1/\alpha+\varepsilon}), \quad \forall\varepsilon > 0. \tag{37}$$

Hence, (35)-(37) provide

$$L(T) = o(T^{1/\alpha+\varepsilon}), \quad \forall\varepsilon > 0. \tag{38}$$

Recalling that $N_1(t) = N(t/\lambda)$, we immediately derive (26) from (38). □

6. Strong invariance principle for randomly stopped processes

Let $\{X, X_i, \ge 1\}$, $\{z, z_i, \ge 1\}$, $S(n)$, $Z(n)$,$N(t)$ be as in Introduction, $EX = m$, $Ez = 1/\lambda > 0$. Put

$$D(t) = S(N(t)) = \sum_{i=1}^{N(t)} X_i.$$

Weak invariance principle for $D(t)$ was studied in a lot of works; we mention only fundamental monographs: Billingsley (1968), Gut (1988), Gnedenko and Korolev (1996), Whitt (2002), Silvestrov (1974, 2004), for applications of such topic to risk theory see also Embrechts et al.(1997), Korolev, Bening and Shorgin (2007).

Strong invariance principle for $S(N(t))$ when $EX^2 < \infty$ and $EY_1^2 < \infty$ (and may satisfy stronger moment conditions) was studied by M.Csörgő, Horváth, Steinebach, Deheuvels, for detail bibliography see already cited monograph by Csörgő and Horváth (1994), as well as survey article by Aalex and Steinebach (1994).

In forthcoming we focus on the case $E|X|^2 = \infty$ when $\{X, X_i, \ge 1\}$ belong to $DNA(G_{\alpha_1,\beta})$, $1 < \alpha_1 < 2$, while $\{z, z_i, \ge 1\}$ can be attracted to the normal law ($\alpha = 2, Var(z) = \tau^2 < \infty$) or to the α_2-stable law, $1 < \alpha_2 < 2$.

Our approach is close to the methods presented in Csörgő and Horváth (1993).

Theorem 2. *Let $\{X_i, i \ge 1\}$ satisfy **(C)** with $1 < \alpha < 2$ and $Ez^2 < \infty$ then a.s.*

$$|D(N(t)) - m\lambda t - Y_{\alpha,\beta}(\lambda t)| = o(t^{1/\alpha-\varrho_2}), \quad \rho_2 \in (0, \rho_0), \tag{39}$$

for some $\varrho_0 = \varrho_0(\alpha, l) > 0$.

Proof. The key moment in the proof is an expression

$$\Delta(T) = \sup_{0\le t\le T} |S(N(t)) - m\lambda t - Y_\alpha(\lambda t)|$$

$$\le \sup_{0\le t\le T} |S(N(t)) - mN(t) - Y_\alpha(N(t))|$$

$$+ \sup_{0\le t\le T} |m(N(t) - \lambda t)| + \sup_{0\le t\le T} |Y_\alpha(N(t)) - Y_\alpha(\lambda t)|$$

$$\le \Delta_1(T) + \Delta_2(T) + \Delta_3(T). \tag{40}$$

Now we estimate separately $\Delta_i, i = 1, 2, 3$. Condition **(C)**, Theorem A2 and (28) ensure the possibility to define $Y_\alpha(t)$ such that a.s. for certain ϱ_1

$$\Delta_1 = O((N(T))^{1/\alpha-\rho_1}) = O(T^{1/\alpha-\rho_1}). \tag{41}$$

The LIL for renewal process $N(t)$ (see (21)) yields

$$\Delta_2(T) = O((T \log\log T)^{1/2}). \tag{42}$$

Using the stationary of increments of the stable process, Lemma 4 and (21) we obtain a.s.

$$\Delta_3(T) = o((T \log\log T)^{1/2\alpha+\varepsilon_2}),\ \forall\varepsilon_2 > 0. \tag{43}$$

Thus, $\Delta_3(T)$ can be made $o(T^{1/\alpha-\rho_1})$ by choosing an appropriate ε_2.

Hence, combining (40) – (43) we obtain

$$\Delta(T) = o(T^{1/\alpha-\rho_2}) \qquad \forall\rho_2 \in (0, \rho_0)$$

for $1 < \alpha < 2$ and $\rho_0 = min(\rho_1, (2-\alpha)/2\alpha)$. □

Corollary. *Theorem 2 holds if* $N(t)$ *is a Poisson process.*

In this case $D(t)$ can be interpreted as total claims until moment t in classic risk model.

Developing such approach we proved rather general result concerning a.s. approximation of the randomly stopped process (not obligatory connected with the partial sum processes).

Let $Z^*(t)$, $D^*(t)$ be two real-valued random processes, N^* – the inverse of $Z^*(t)$ is defined by

$$N^*(t) = \inf\{t > 0 : Z^*(x) > t\}, 0 \le t < \infty,$$

Theorem 3. *Suppose that for some constants $m, \gamma, a > 0, \sigma > 0, \tau > 0$ a.s.*

$$\sup_{0\le t\le T} |\sigma^{-1}(Z(t) - at) - W_1(t)| = O(r(T)), \tag{44}$$

where $W_1(t)$ is a Wiener process, $r(t) \uparrow \infty$, $r(t)/t \downarrow 0$ as $t \to \infty$ and

$$\sup_{0\le t\le T} |D^*(t) - mt - Y_\alpha(t)| = O(q(T)), \tag{45}$$

$Y_\alpha(t)$ being α-stable process independent of $W_1(t)$, $q(t) \uparrow \infty$, $q(t)/t \downarrow 0$ as $t \to \infty$, then $\forall \varepsilon > 0$ a.s.

$$|D(N^*(t)) - (m/a)t - (Y_\alpha(t/a) - (m\sigma/a)W_2(t/a))| =$$

$$= O(q(t)) + O(r(T) + \log t)$$
$$+O((r(t) + (t \log\log t)^{1/2})^{1/(\alpha-\varepsilon)}), \tag{46}$$

where $W_2(t)$ is a Wiener process independent of $Y_\alpha(t)$.

Proof. The essential point of the proof is to apply the inequality

$$|D(N^*(t)) - (m/a)t - (Y_\alpha(t/a) - (m\sigma/a)W_2(t/a))|$$
$$\le |D(N^*(t)) - mN^*(t) - Y_\alpha(N^*(t))|$$
$$+|Y_\alpha(N^*(t)) - Y_\alpha(t/a)|$$
$$+|m(N^*(t) - t/a + (\sigma/a)W_2(t/a))|$$
$$\le \Delta_1^*(t) + \Delta_2^*(t) + \Delta_3^*(t)$$

and estimate each $\Delta_i^*(t)$ using a.s.approximation for $D^*(t)$, $N^*(t)$ and growth rate for stable and Wiener processes.□

In the case of partial sum processes $S(t)$ and $Z(t)$ with $Ez^2 < \infty$, X_i satisfying **(C)**, $N^*(t) = N(t)$ is counting renewal process, $q(t) = T^{1/\alpha-\varrho_1}$, $\varrho_1 > 0$, the worst estimate for $r(t)$ is $(t \log\log t)^{1/2}$. These facts lead to statement of the Theorem 2.

The same approach provides

Theorem 4. *Let $\{X_i, i \ge 1\}$ satisfy (C) with $1 < \alpha_1 < 2$, and $\{z_i\}$ satisfy (C) with $1 < \alpha_2 < 2$,*

$$\alpha_1 \le \alpha_2$$

then a.s.

$$|S(N(t)) - m\lambda t - Y_{\alpha_1,\beta}(\lambda t)| = o(t^{1/\alpha_1-\varrho_3})$$

for some $\varrho_3 = \varrho_3(\alpha_1, l) > 0$.

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