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RANDOM PROCESS FROM THE CLASS $V(\varphi,\psi)$: EXCEEDING A CURVE

Random processes from the class $V(\varphi,\psi)$ which is more general than the class of ψ-sub-Gaussian random process. The upper estimate of the probability that a random process from the class $V(\varphi,\psi)$ exceeds some function is obtained. The results are applied to generalized process of fractional Brownian motion.

1. Introduction

In this paper we consider random process from the class $V(\varphi,\psi)$ defined on compact set and the probability that this process exceeds some function. Recall that random process belongs to class $V(\varphi,\psi)$ if its trajectories belong to the space $\mathrm{Sub}_\psi(\Omega)$ and increments belong to the space $\mathrm{Sub}_\varphi(\Omega)$. Properties of random variables and processes from the spaces $\mathrm{Sub}_\varphi(\Omega)$ and $\mathrm{SSub}_\varphi(\Omega)$ can be found in the book of Buldygin V.V. and Kozachenko Yu.V. [1] and in the papers [2-7]. Here we generalize the results obtained earlier in [6-8].

The paper is organized as follows. Basic definitions and some properties of φ-sub-Gaussian and strictly φ-sub-Gaussian spaces of random variables and processes are given in section 2. In section 3 we obtain general results on estimates of probability that random process from the class $V(\varphi,\psi)$ overruns a level specified by a continuous function. The methods used in the section are the same as in [6]. However for convenience of readers we give here complete proofs. In section 4 we apply results from the previous section to generalized process of fractional Brownian motion from the class $V(\varphi,\psi)$ and obtain the estimate of overcrossing by its trajectories the level defined by function ct, where $c > 0$ is a given constant. Such estimate has applications in the queuing theory as estimate of buffer overflow probability or in the risk theory as estimate of ruin probability.

Invited lecture.

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2. Class $V(\varphi,\psi)$: essential definitions and properties

Let $(\Omega,\mathcal{B},P)$ be a standard probability space and T be some parametrical space.

Definition 2.1.[1] Function $u=\{u(x),x\in\mathbb{R}\}$ is called an Orlicz N-function if u is a continuous even convex function such that $u(0)=0$, $u(x)$ monotonically increases as $x>0$, $\frac{u(x)}{x}\to 0$ as $x\to 0$ and $\frac{u(x)}{x}\to\infty$ as $x\to\infty$.

Definition 2.2.[1] Let φ be such an Orlicz N-function that $\varphi(x)=cx^2$ as $|x|\le x_0$ for some $x_0>0$ and $c>0$. Centered random variable ξ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, the space of φ-sub-Gaussian random variables, if for all $\lambda\in\mathbb{R}$ there exists a constant $r_\xi\ge 0$ which satisfies the following inequality

$$\mathsf{E}\exp(\lambda\xi)\ \le\ \exp\{\varphi(\lambda r_\xi))\}.$$

Theorem 2.1.[1] *The space $Sub_\varphi(\Omega)$ is a Banach space with respect to the norm*

$$\tau_\varphi(\xi)\ =\ \sup_{\lambda>0}\frac{\varphi^{(-1)}\left(\log \mathsf{E}\exp\{\lambda\xi\}\right)}{\lambda},$$

where $\varphi^{(-1)}$ is an inverse function to the function φ, and for all $\lambda\in\mathbb{R}$ the following inequality holds

$$\mathsf{E}\exp(\lambda\xi)\le\exp(\varphi(\lambda\tau_\varphi(\xi)))\,. \tag{1}$$

Moreover, there exist constants $r>0$, $c_r>0$ such that

$$(\mathsf{E}|\xi|^r)^{\frac{1}{r}}\le c_r\tau_\varphi(\xi).$$

Lemma 2.1.[2] *Let $\xi\in Sub_\varphi(\Omega)$. Then for all $\varepsilon>0$ the following inequality holds true*

$$\mathsf{P}\{|\xi|>\varepsilon\}\le 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.$$

Definition 2.3. Random process $X=(X(t),t\in T)$ belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, if for all $t\in T:\ X(t)\in\mathrm{Sub}_\varphi(\Omega)$ and $\sup_{t\in T}\tau_\varphi(X(t))<\infty$.

Let (T,ρ) be a pseudometrical (metrical) compact space with pseudometric (metric) ρ.

Definition 2.4.[3] Metric entropy in relation to pseudometric (metric) ρ, or just metric entropy is a function

$$H_{(T,\rho)}(u)=H(u)=\begin{cases}\log N_{(T,\rho)}(u), & \text{if } N_{(T,\rho)}(u)<+\infty\\ +\infty, & \text{if } N_{(T,\rho)}(u)=+\infty\end{cases},$$

where $N_{(T,\rho)}(u) = N(u)$ denotes the least the least number of closed ρ-balls with radius u.

Definition 2.5. [3] A family of random variables Δ from the space $\mathrm{Sub}_\varphi(\Omega)$ is called strictly $\mathrm{Sub}_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$, and for any $\lambda_i \in \mathsf{R}$ the following inequality takes place

$$\tau_\varphi\left(\sum_{i\in I}\lambda_i\xi_i\right) \le C_\Delta\left(\mathbf{E}\left(\sum_{i\in I}\lambda_i\xi_i\right)^2\right)^{\frac{1}{2}}. \tag{2}$$

If Δ is a family of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables, then linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ also is strictly $\mathrm{Sub}_\varphi(\Omega)$ family of random variables. Linearly closed families of strictly $\mathrm{Sub}_\varphi(\Omega)$ random variables form a space of strictly φ-sub-Gaussian random variables. This space is denoted by $\mathrm{SSub}_\varphi(\Omega)$.

When $\varphi(x) = \frac{x^2}{2}$ the space $\mathrm{SSub}_\varphi(\Omega)$ is called the space of strictly sub-Gaussian random variables and is denoted by $\mathrm{SSub}(\Omega)$. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSub}(\Omega)$ and $\tau^2(\xi) = \mathsf{E}\xi^2$.

Definition 2.6. A random process $X = (X(t), t \in T)$ is a strictly φ-sub-Gaussian process if the corresponding family of random variables belongs to the space $\mathrm{SSub}_\varphi(\Omega)$.

Definition 2.7.[7] φ is subordinated to an Orlizc N-function ψ ($\varphi \prec \psi$) if there are exist such numbers $x_0 > 0$ and $k > 0$ that $\varphi(x) < \psi(kx)$ for $x > x_0$.

Definition 2.8.[7] Let $\varphi \prec \psi$ are two Orlicz N-functions. Random process $X = (X(t), t \in T)$ belongs to class $V(\varphi,\psi)$ if for all $t \in T$ the process $X(t)$ is from $\mathrm{Sub}_\psi(\Omega)$ and for all $s, t \in T$ increments $(X(t) - X(s))$ belong to the space $\mathrm{Sub}_\varphi(\Omega)$.

3. Main Results

Let (T,ρ) be a pseudometrical (metrical) compact space with pseudometric (metric) ρ and $Y = \{Y(t), t \in T\}$ be a separable random process from the class $V(\varphi,\psi)$.

Suppose there exists such continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$, that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality for increments of the process is true

$$\sup_{\rho(t,s)\le h} \tau_\varphi(Y(t) - Y(s)) \le \sigma(h). \tag{3}$$

Let $\beta > 0$ be some number such that

$$\beta \le \sigma\left(\inf_{s\in T}\sup_{t\in T}\rho(t,s)\right) \tag{4}$$

and let $\varepsilon_k = \sigma^{(-1)}(\beta p^k)$, $p \in (0,1)$, $k = 0,1,2,\ldots$, $\gamma(u) = \tau_\psi(Y(u))$.

Lemma 3.1. *Let $f = \{f(t), t \in T\}$ be a continuous function such that $|f(u) - f(v)| \le \delta(\rho(u,v))$, where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function, and $X(t) = Y(t) - f(t)$. Let $\{q_k, k = 1,2,\ldots\}$ be such a sequence that $q_k > 1$ and $\sum_{k=1}^{\infty} q_k^{-1} \le 1$. Then for all $\lambda \in \mathbb{R}$, $p \in (0,1)$ we have*

$$\mathsf{E}\exp\{\lambda\sup_{t\in T}X(t)\} \le \exp\left\{\frac{1}{q_1}\sup_{u\in T}\left(\psi(\lambda q_1\gamma(u)) - \lambda q_1 f(u)\right)\right\}\times \tag{5}$$
$$\times\left(\prod_{k=1}^{\infty}(N(\varepsilon_k))^{\frac{1}{q_k}}\right)\left(\prod_{k=2}^{\infty}\exp\left\{\frac{1}{q_k}\varphi\left(\lambda q_k\beta p^{k-1}\right) + \lambda\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\}\right).$$

Proof. Denote by V_{ε_k} the set of the centers of the closed balls with radius ε_k, which form minimal covering of the space (T,ρ). Number of elements in the set V_{ε_k} is equal to $N_T(\varepsilon_k) = N(\varepsilon_k)$.

The process $Y(t)$ and, therefore, the process $X(t)$ are separable processes.

It follows from lemma 2.1 and condition (3) that for any $\varepsilon > 0$

$$\begin{aligned}\mathsf{P}\{|Y(t) - Y(s)| > \varepsilon\} &\le 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\tau_\varphi(Y(t) - Y(s))}\right)\right\}\\ &\le 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\sigma(\rho(t,s))}\right)\right\}.\end{aligned}$$

Therefore the process Y is continuous on probability and the process X is continuous on probability as well. If a separable random process on (T,ρ) is continuous on probability, then any set, which is countable and everywhere dense with respect to ρ, can be taken as a set of separability of this process. Therefore the set $V = \bigcup_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of separability of the process X and we have that with probability one

$$\sup_{t\in T}X(t) = \sup_{t\in V}X(t). \tag{6}$$

Consider a mapping $\alpha_n = \{\alpha_n(t), n = 0,1\ldots\}$ of the set V in V_{ε_n}, where $\alpha_n(t)$ is such a point from the set V_{ε_n}, that $\rho(t,\alpha_n(t)) < \varepsilon_n$. If $t \in V_{\varepsilon_n}$ then $\alpha_n(t) = t$. If there exist several points from the set V_{ε_n}, such that

$\rho(t,\alpha_n(t)) < \varepsilon_n$, then we choose one of them and denote it by $\alpha_n(t)$. Then it follows from the theorem 2.1 and (3) that

$$\mathsf{P}\left\{|Y(t)-Y(\alpha_n(t))| > p^{\frac{n}{2}}\right\} \le \frac{E(Y(t)-Y(\alpha_n(t)))^2}{p^n} \le \frac{c_2^2\tau_\varphi^2(Y(t)-Y(\alpha_n(t)))}{p^n} \le \frac{c_2^2\sigma^2(\varepsilon_n)}{p^n} = c_2^2\beta^2 p^n.$$

This inequality means that $\sum\limits_{n=1}^{\infty} \mathsf{P}\left\{|Y(t)-Y(\alpha_n(t))| > p^{\frac{n}{2}}\right\} < \infty$.

From the Borell-Kantelli's lemma follows that $Y(t)-Y(\alpha_n(t)) \to 0$ as $n\to\infty$ with probability one. Since the function f is continuous then $X(t)-X(\alpha_n(t)) \to 0$ as $n\to\infty$ with probability one as well. Since the set V is countable, then $X(t)-X(\alpha_n(t)) \to 0$ as $n\to\infty$ for all t simultaneously. Let t be an arbitrary point from the set V. Denote by $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m), \ldots, t_1 = \alpha_1(t_2)$ for any $m \ge 1$. Then for all $m \ge 2$ we have the following inequality

$$\begin{aligned} X(t) &= X(t_1) + \sum_{k=2}^{m}(X(t_k)-X(t_{k-1})) + X(t) - X(\alpha_m(t)) \le \max_{u\in V_{\varepsilon_1}} X(u) + \\ &+ \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u)-X(\alpha_{k-1}(u))) + X(t) - X(\alpha_m(t)). \end{aligned} \tag{7}$$

It follows from (7) and (6) that with probability one

$$\sup_{t\in T} X(t) = \sup_{t\in V} X(t) \le \liminf_{m\to\infty}\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m}\max_{u\in V_{\varepsilon_k}}\left(X(u)-X(\alpha_{k-1}(u))\right)\right). \tag{8}$$

From the Helder's inequality, Fatu's lemma and (8) follows that for all $\lambda > 0$

$$\begin{aligned} &\mathsf{E}\exp\left\{\lambda\sup_{t\in T} X(t)\right\} \\ &\le \mathsf{E}\liminf_{m\to\infty}\exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m}\max_{u\in V_{\varepsilon_k}}\left(X(u)-X(\alpha_{k-1}(u))\right)\right)\right\} \\ &\le \liminf_{m\to\infty}\mathsf{E}\exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m}\max_{u\in V_{\varepsilon_k}}\left(X(u)-X(\alpha_{k-1}(u))\right)\right)\right\} \\ &\le \liminf_{m\to\infty}\left(\left(\mathsf{E}\exp\left\{q_1\lambda\max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{\frac{1}{q_1}} \times\right. \end{aligned}$$

$$\times \prod_{k=2}^{m} \left(\mathsf{E} \exp \left\{ q_k \lambda \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right\} \right)^{\frac{1}{q_k}} \Bigg)$$

$$\leq \left(\mathsf{E} \exp \left\{ q_1 \lambda \max_{u \in V_{\varepsilon_1}} X(u) \right\} \right)^{\frac{1}{q_1}} \times$$

$$\times \prod_{k=2}^{\infty} \left(\mathsf{E} \exp \left\{ q_k \lambda \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right\} \right)^{\frac{1}{q_k}} = I_1 \cdot \prod_{k=2}^{\infty} I_k. \qquad (9)$$

Let's consider each term in (9). It follows from the theorem 2.1 that $\mathsf{E} \exp\{q_1 \lambda Y(u)\} \leq \exp\{\psi(q_1 \lambda \gamma(u))\}$. Therefore

$$\begin{aligned} I_1 &\leq \left(\sum_{u \in V_{\varepsilon_1}} \mathsf{E} \exp \left\{ q_1 \lambda Y(u) \right\} \exp \left\{ -q_1 \lambda f(u) \right\} \right)^{\frac{1}{q_1}} \\ &\leq \left(\sum_{u \in V_{\varepsilon_1}} \exp \left\{ \psi(q_1 \lambda \gamma(u)) - q_1 \lambda f(u) \right\} \right)^{\frac{1}{q_1}} \\ &\leq \left(N(\varepsilon_1) \exp \left\{ \sup_{u \in T} \Big(\psi(q_1 \lambda \gamma(u)) - q_1 \lambda f(u) \Big) \right\} \right)^{\frac{1}{q_1}} \\ &\leq \Big(N(\varepsilon_1) \Big)^{\frac{1}{q_1}} \exp \left\{ \frac{1}{q_1} \sup_{u \in T} \Big(\psi(q_1 \lambda \gamma(u)) - q_1 \lambda f(u) \Big) \right\}. \qquad (10) \end{aligned}$$

It also follows from the theorem 2.1 and assumption (3) that

$$\mathsf{E} \exp\{q_k \lambda (Y(u) - Y(\alpha_{k-1}(u)))\} \leq \exp\{\varphi(q_k \lambda \sigma(\varepsilon_{k-1}))\}.$$

In that way since $|f(u) - f(v)| \leq \delta(\rho(u, v))$ then

$$\begin{aligned} I_k \leq \Bigg(& N(\varepsilon_k) \max_{u \in V_{\varepsilon_k}} \mathsf{E} \exp \left\{ q_k \lambda [Y(u) - Y(\alpha_{k-1}(u))] \right\} \times \\ & \times \exp \left\{ -q_k \lambda [f(u) - f(\alpha_{k-1}(u))] \right\} \Bigg)^{\frac{1}{q_k}} \\ \leq & \Big(N(\varepsilon_k) \Big)^{\frac{1}{q_k}} \left(\max_{u \in V_{\varepsilon_k}} \exp \left\{ \varphi(q_k \lambda \sigma(\varepsilon_{k-1})) - q_k \lambda [f(u) - f(\alpha_{k-1}(u))] \right\} \right)^{\frac{1}{q_k}} \\ \leq & \Big(N(\varepsilon_k) \Big)^{\frac{1}{q_k}} \left(\max_{u \in V_{\varepsilon_k}} \exp \left\{ \varphi(q_k \lambda \sigma(\varepsilon_{k-1})) + q_k \lambda \delta(\rho(u, \alpha_{k-1}(u))) \right\} \right)^{\frac{1}{q_k}} \\ \leq & \Big(N(\varepsilon_k) \Big)^{\frac{1}{q_k}} \exp \left\{ q_k^{-1} \varphi(q_k \lambda \beta p^{k-1}) + \lambda \delta(\sigma^{(-1)}(\beta p^{k-1})) \right\}. \qquad (11) \end{aligned}$$

From inequalities (9), (10) and (11) we have the assertion of the lemma.□

Theorem 3.1. *Let $Y=\{Y(t),t\in T\}$ be a separable random process from the class $V(\varphi,\psi)$ and $f=\{f(t),t\in T\}$ be such a continuous function that $|f(u)-f(v)|\le\delta(\rho(u,v))$, where $\delta=\{\delta(s),s>0\}$ is some monotonically increasing nonnegative function, and $X(t)=Y(t)-f(t)$. Let $r_1=\{r_1(u):u\ge 1\}$ be such a continuous function that $r_1(u)>0$ as $u>1$ and the function $s(t)=r_1(\exp\{t\}),t\ge 0$, is convex. If*

$$\int_0^\beta r_1(N(\sigma^{(-1)}(u)))du<\infty, \tag{12}$$

then for all $p\in(0;1)$ and $x>0$ the following inequality holds true

$$P\left\{\sup_{t\in T}X(t)>x\right\}\le\inf_{\lambda>0}Z_{r_1}(\lambda,p,\beta), \tag{13}$$

where

$$\begin{aligned}Z_{r_1}&(\lambda,p,\beta)\\&=\exp\left\{\theta_\psi(\lambda,p)+p\varphi\left(\frac{\lambda\beta}{1-p}\right)+\lambda\left(\sum_{k=2}^\infty\delta(\sigma^{(-1)}(\beta p^{k-1}))-x\right)\right\}\times\\&\quad\times r_1^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p}r_1(N(\sigma^{(-1)}(u)))du\right),\end{aligned} \tag{14}$$

$$\theta_\psi(\lambda,p)=\sup_{u\in T}\left((1-p)\psi\left(\frac{\lambda\gamma(u)}{1-p}\right)-\lambda f(u)\right). \tag{15}$$

Proof. Let $q_k=((1-p)p^{k-1})^{-1}$ in the inequality (5) then

$$\begin{aligned}&\mathsf{E}\exp\left\{\lambda\sup_{t\in T}X(t)\right\}\\&\quad\le\exp\left\{\theta_\psi(\lambda,p)+\sum_{k=2}^\infty(1-p)p^{k-1}\varphi\left(\frac{\lambda\beta}{1-p}\right)+\lambda\sum_{k=2}^\infty\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right)\right\}\\&\quad\times\exp\left\{\sum_{k=1}^\infty(1-p)p^{k-1}\log N\left(\sigma^{(-1)}(\beta p^k)\right)\right\}.\end{aligned} \tag{16}$$

Since

$$\begin{aligned}&\exp\left\{\sum_{k=1}^\infty(1-p)p^{k-1}\log N\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\\&\quad=r_1^{(-1)}\left(r_1\left(\exp\left\{\sum_{k=1}^\infty(1-p)p^{k-1}\log N\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\right)\right)\end{aligned}$$

$$\le r_1^{(-1)}\left(\sum_{k=1}^{\infty}(1-p)p^{k-1}r_1\left(N\left(\sigma^{(-1)}(\beta p^k)\right)\right)\right)$$

$$\le r_1^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r_1\left(N\left(\sigma^{(-1)}(u)\right)\right)\mathsf{d}u\right) \tag{17}$$

the assertion of the theorem follows from the lemma 3.1, (16) and Chebyshev's inequality. □

Lemma 3.2. *Suppose that all assumptions of lemma 3.1 are satisfied and*

$$\int_0^{\beta}\frac{H(\sigma^{(-1)}(u))}{\varphi^{(-1)}(H(\sigma^{(-1)}(u)))}\mathsf{d}u<\infty, \tag{18}$$

where $H(\varepsilon)=\log N(\varepsilon)$. *Then for all* $p\in(0,1)$ *and* $\lambda>0$ *we have that*

$$\mathsf{E}\exp\left\{\lambda\sup_{t\in T}X(t)\right\}\le Z(\lambda,p,\beta), \tag{19}$$

where

$$Z(\lambda,p,\beta)=\exp\left\{W(\lambda,p,\beta)+p\varphi\left(\frac{\lambda\beta}{1-p}\right)\right\}\times$$

$$\times\exp\left\{\frac{2\lambda}{p(1-p)}\int_0^{\beta p^2}\frac{H(\sigma^{(-1)}(u))}{\varphi^{(-1)}(H(\sigma^{(-1)}(u)))}\mathsf{d}u+\lambda\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right)\right\},$$

$$W(\lambda,p,\beta)=\inf_{v\ge(1-p)^{-1}}\left(\frac{1}{v}H(\sigma^{(-1)}(\beta p))+\sup_{u\in T}\left(\frac{\psi(\lambda\gamma(u)v)}{v}-\lambda f(u)\right)\right).$$

Proof. It follows from lemma 3.1 (see inequality (5)) that for all $q_k>1, k=1,2,\dots$ such that $\sum_{k=1}^{\infty}\frac{1}{q_k}\le 1$, and all $\lambda>0$ the following inequality holds true

$$\mathsf{E}\exp\left\{\lambda\sup_{t\in T}X(t)\right\}$$

$$\le\exp\left\{\lambda\sum_{k=2}^{\infty}\delta\Big(\sigma^{(-1)}(\beta p^{k-1})\Big)\right\}\exp\left\{\sum_{k=2}^{\infty}\frac{H(\varepsilon_k)+\varphi(\lambda q_k\beta p^{k-1})}{q_k}\right\}\times$$

$$\times\exp\left\{\frac{1}{q_1}\left(H(\varepsilon_1)+\sup_{u\in T}(\psi(\lambda q_1\gamma(u))-\lambda q_1 f(u))\right)\right\}. \tag{20}$$

Let $q_1=v$, where v is such a number that $v\ge\frac{1}{1-p}$ and

$$q_k=\frac{1}{\lambda\beta p^{k-1}}\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right)+H(\varepsilon_k)\right),\ k=2,3\dots \tag{21}$$

Since

$$\frac{1}{q_k} \le \frac{\lambda\beta p^{k-1}}{\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right)\right)} = p^{k-1}(1-p)$$

as $k = 2, 3 \ldots$, then

$$\sum_{k=1}^{\infty}\frac{1}{q_k} \le \sum_{k=1}^{\infty} p^{k-1}(1-p) = 1.$$

Consider

$$\tilde{Z} = \sum_{k=2}^{\infty}\frac{H(\varepsilon_k) + \varphi(\lambda q_k \beta p^{k-1})}{q_k}.$$

For the sequence q_k defined in (21) we have

$$\begin{aligned}
\tilde{Z} &= \sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k} + \sum_{k=2}^{\infty}\frac{1}{q_k}\varphi\left(\lambda\beta p^{k-1}\frac{\varphi^{(-1)}\left(\varphi\left(\frac{\lambda\beta}{1-p}\right) + H(\varepsilon_k)\right)}{\lambda\beta p^{k-1}}\right)\\
&= \sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k} + \sum_{k=2}^{\infty}\frac{H(\varepsilon_k)}{q_k} + \varphi\left(\frac{\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty}\frac{1}{q_k}\\
&\le 2\sum_{k=2}^{\infty} H(\varepsilon_k)\frac{\lambda\beta p^{k-1}}{\varphi^{(-1)}(H(\varepsilon_k))} + \varphi\left(\frac{\lambda\beta}{1-p}\right)\sum_{k=2}^{\infty} p^{k-1}(1-p)\\
&= \varphi\left(\frac{\lambda\beta}{1-p}\right)p + 2\lambda\sum_{k=2}^{\infty}\frac{H(\sigma^{(-1)}(\beta p^k))\beta p^{k-1}}{\varphi^{(-1)}(H(\sigma^{(-1)}(\beta p^k)))}. \qquad (22)
\end{aligned}$$

The function $\frac{\varphi(x)}{x}$ increases as $x > 0$ (see, for example, [1]) therefore the function $\frac{x}{\varphi^{(-1)}(x)}$ increases as well. Then

$$\int_{\beta p^{k+1}}^{\beta p^k}\frac{H(\sigma^{(-1)}(u))}{\varphi^{(-1)}(H(\sigma^{(-1)}(u)))}\mathsf{d}u \ge \frac{H(\sigma^{(-1)}(\beta p^k))}{\varphi^{(-1)}(H(\sigma^{(-1)}(\beta p^k)))}\beta p^k(1-p). \qquad (23)$$

And from (22) and (23) it follows that

$$\tilde{Z} \le \varphi\left(\frac{\lambda\beta}{1-p}\right)p + \frac{2\lambda}{p(1-p)}\int_0^{\beta p^2}\frac{H(\sigma^{(-1)}(u))}{\varphi^{(-1)}(H(\sigma^{(-1)}(u)))}\mathsf{d}u. \qquad (24)$$

Therefore the assertion of the lemma follows from (5) and (24). □

Theorem 3.2. *Let $Y = \{Y(t), t \in T\}$ be a separable random process from the class $V(\varphi,\psi)$ and $f = \{f(t), t \in T\}$ be a continuous function such that $|f(u) - f(v)| \le \delta(\rho(u,v))$, where $\delta = \{\delta(s), s > 0\}$ is some* *monotonically increasing nonnegative function, and* $X(t) = Y(t) - f(t)$. *Let* $r_2 = \{r_2(u) : u \geq 1\}$ *be such a continuous function that* $r_2(u) > 0$ *as* $u > 1$, $r_2(1) = 0$ *and the function* $s(t) = r_2(\exp\{t\}), t \geq 0$, *is convex. If*

$$\int_0^\beta \frac{r_2(N(\sigma^{(-1)}(u)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(u)))} du < \infty \tag{25}$$

then for all $p \in (0;1)$ *and* $x > 0$ *the following inequality holds true*

$$P\left\{\sup_{t\in T} X(t) > x\right\} \leq \inf_{\lambda>0} Z_{r_2}(\lambda, p, \beta), \tag{26}$$

where

$$Z_{r_2}(\lambda, p, \beta) = \exp\left\{W(\lambda, p, \beta) + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}\left(\beta p^{k-1}\right)\right) - x\right)\right\} \times \times \left(r_2^{(-1)}\left(\frac{\lambda}{p(1-p)}\int_0^{\beta p^2} \frac{r_2(N(\sigma^{(-1)}(u)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(u)))} du\right)\right)^2, \tag{27}$$

$$W(\lambda, p, \beta) = \inf_{v \geq (1-p)^{-1}}\left(\frac{1}{v}H(\sigma^{(-1)}(\beta p)) + \sup_{u\in T}\left(\frac{\psi(\lambda\gamma(u)v)}{v} - \lambda f(u)\right)\right). \tag{28}$$

Proof. Let q_1 and $q_k, k = 2, 3, \ldots$ be defined as in the proof of the lemma 3.2. It follows from (20) and (22) that for $\lambda > 0$, $p \in (0;1)$ and $v \geq \frac{1}{1-p}$

$$\mathsf{E}\exp\left\{\lambda\sup_{t\in T} X(t)\right\} \leq \exp\left\{\frac{1}{v}H(\sigma^{(-1)}(u)) + \sup_{u\in T}\left(\frac{\psi(\lambda v\gamma(u))}{v} - \lambda f(u)\right) + \lambda\sum_{k=2}^{\infty}\delta(\sigma^{(-1)}(\beta p^{k-1})) + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + 2\sum_{k=2}^{\infty}\frac{H\left(\sigma^{(-1)}(\beta p^k)\right)}{q_k}\right\}. \tag{29}$$

From the convexity of the function $s(t) = r_2(\exp\{t\})$ it follows that for all $\delta_i > 0, i \geq 1$, such that $\sum_{i=1}^{\infty}\delta_i = 1$ and all $x_i \geq 0$

$$s\left(\sum_{i=1}^{\infty}\delta_i x_i\right) \leq \sum_{i=1}^{\infty}\delta_i s(x_i).$$

If $\sum_{i=1}^{\infty}\delta_i < 1$ remembering $s(0) = 0$ we have

$$s\left(\sum_{i=1}^{\infty}\delta_i x_i\right) = s\left(\sum_{i=1}^{\infty}\delta_i x_i + 0(1 - \sum_{i=1}^{\infty}\delta_i)\right)$$

$$\le \sum_{i=1}^{\infty}\delta_i s(x_i) \quad + \quad \left(1-\sum_{i=1}^{\infty}\delta_i\right)s(0)=\sum_{i=1}^{\infty}\delta_i s(x_i). \tag{30}$$

It follows from (30) that

$$\begin{aligned}
&\exp\left\{2\sum_{k=2}^{\infty}\frac{1}{q_k}H\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\\
&=\left(r_2^{(-1)}\left(r\left(\exp\left\{\sum_{k=2}^{\infty}q_k^{-1}\log N\left(\sigma^{(-1)}(\beta p^k)\right)\right\}\right)\right)\right)^2\\
&\le\left(r_2^{(-1)}\left(\sum_{k=2}^{\infty}q_k^{-1}s\left(\log N\left(\sigma^{(-1)}(\beta p^k)\right)\right)\right)\right)^2\\
&\le\left(r_2^{(-1)}\left(\lambda\sum_{k=2}^{\infty}\beta p^{k-1}\frac{r_2(N(\sigma^{(-1)}(\beta p^k)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(\beta p^k)))}\right)\right)^2.
\end{aligned}\tag{31}$$

The function $a(t)=r_2(\exp\{\varphi(t)\}), t\ge 0$, is a convex function and $a(0)=0$, that is $a(t)$ is an Orlicz function and the function $\frac{a(t)}{t}$ increases as $t>0$ [**?**]. Therefore the function $r_2(\exp\{u\})/\varphi^{(-1)}(u)$ increases as well. Consequently we have the following inequality

$$\int_{\beta p^{k+1}}^{\beta p^k}\frac{r_2(N(\sigma^{(-1)}(u)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(u)))}\mathsf{d}u\ge\frac{r_2(N(\sigma^{(-1)}(\beta p^k)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(\beta p^k)))}\beta p^k(1-p)$$

and

$$\begin{aligned}
&\sum_{k=2}^{\infty}\beta p^{k-1}\frac{r_2(N(\sigma^{(-1)}(\beta p^k)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(\beta p^k)))}\\
&\qquad\le\frac{1}{p(1-p)}\sum_{k=2}^{\infty}\int_{\beta p^{k+1}}^{\beta p^k}\frac{r_2(N(\sigma^{(-1)}(u)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(u)))}\mathsf{d}u\\
&\qquad\le\frac{1}{p(1-p)}\int_0^{\beta p^2}\frac{r_2(N(\sigma^{(-1)}(u)))}{\varphi^{(-1)}(\log N(\sigma^{(-1)}(u)))}\mathsf{d}u.
\end{aligned}\tag{32}$$

Using the Chebyshev's inequality the assertion of the theorem follows from the (29), (31), (32). □

4. Examples

Definition 4.1.[4] Let $\varphi \prec \psi$ are two Orlicz N-functions. We call the process $Z^H = (Z^H(t), t \in T)$ generalized fractional Brownian motion from the class $V(\varphi, \psi)$ with Hurst index $H \in (0,1)$ ($V(\varphi,\psi)$-GFBM) if Z^H is strictly ψ-sub-Gaussian process with stationary strictly φ-sub-Gaussian increments and covariance function

$$R_H(t,s) = \mathsf{E} Z^H(s) Z^H(t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right). \tag{33}$$

Theorem 4.1. *Let $Z^H = (Z^H(t), t \in [a,b])$, $0 \le a < b < \infty$ be a generalized fractional Brownian motion from the class $V(\varphi,\psi)$ with Hurst index $H \in (0,1)$ and let $c > 0$ be a constant. Then for all $p \in (0,1)$, $\beta \in \left(0, \left(\frac{b-a}{2}\right)^H\right]$ and $\lambda > 0$ the following inequlity holds true*

$$\begin{aligned} P\left\{\sup_{a\le t\le b}\left(Z^H(t) - ct\right) > x\right\} &\le (b-a)\left(\frac{e}{\beta p}\right)^{\frac{1}{H}} \times \\ &\times \exp\left\{\frac{\lambda c (\beta p)^{\frac{1}{H}}}{C_\Delta (1-p^{\frac{1}{H}})} + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\psi(\lambda,p) - \frac{\lambda x}{C_\Delta}\right\}, \end{aligned} \tag{34}$$

where $\theta_\psi(\lambda,p) = \sup_{a\le u\le b}\left(\psi\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda c u}{C_\Delta(1-p)}\right)$, C_Δ is the constant from definition 2.5 of the space $SSub_\varphi(\Omega)$.

Proof. Let's apply theorem 3.1.

$$\mathsf{P}\left\{\sup_{a\le t\le b}\left(Z^H(t) - ct\right) > x\right\} = \mathsf{P}\left\{\sup_{a\le t\le b}\left(Y(t) - Ct\right) > \varepsilon\right\}, \tag{35}$$

where $\varepsilon = \frac{x}{C_\Delta}$ and $C = \frac{c}{C_\Delta}$. Since

$$\begin{aligned} \tau_\varphi(Y_i(t) - Y_i(s)) &= \frac{1}{C_\Delta}\tau_\varphi(Z_i^H(t) - Z_i^H(s)) \\ &\le \left(\mathsf{E}(Z_i^H(t) - Z_i^H(s))^2\right)^{\frac{1}{2}} = |t-s|^H, \end{aligned}$$

put $\gamma(u) = u^H$ and $\sigma(h) = h^H$, then $0 \le \beta \le \left(\frac{b-a}{2}\right)^H$. Also we have that $|f(u) - f(v)| = |Cu - Cv| = C|u-v|$, i.e. $\delta(h) = Ch$. As function $r_1(u)$ let's choose $r_1(u) = u^\alpha, u \ge 1, 0 < \alpha < H$. Then

$$\begin{aligned} \theta_\psi(\lambda,p) &= \sup_{a\le u\le b}\left(\psi\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda C u}{1-p}\right), \\ \sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) &= \sum_{k=2}^{\infty} C(\beta p^{k-1})^{\frac{1}{H}} = \frac{C(\beta p)^{\frac{1}{H}}}{1-p^{\frac{1}{H}}}. \end{aligned}$$

Since

$$\log\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \le H(u) \le \ln\left(\frac{b-a}{2u}+1\right),$$

then for $u \le \left(\frac{b-a}{2}\right)^H$ the following estimate is fulfilled

$$r_1\left(N_B\left(\sigma^{(-1)}(u)\right)\right) \le r_1\left(\frac{b-a}{2\sigma^{(-1)}(u)}+1\right) = \left(\frac{b-a}{2u^{\frac{1}{H}}}+1\right)^\alpha \le \frac{(b-a)^\alpha}{u^{\frac{\alpha}{H}}}.$$

Since $\beta p < \beta \le \left(\frac{b-a}{2}\right)^H$ then

$$r^{(-1)}\left(\frac{1}{\beta p}\int\limits_0^{\beta p} r\left(N_B\left(\sigma^{(-1)}(u)\right)\right) du\right)$$
$$\le \left(\frac{1}{\beta p}\int\limits_0^{\beta p} \frac{(b-a)^\alpha}{u^{\frac{\alpha}{H}}} du\right)^{\frac{1}{\alpha}} = (b-a)\beta^{-\frac{1}{H}} p^{-\frac{1}{H}}\left(1-\frac{\alpha}{H}\right)^{-\frac{1}{\alpha}}. \quad (36)$$

Infinum of the right of estimate (36) equals to

$$\lim_{\alpha\to 0}(b-a)\beta^{-\frac{1}{H}} p^{-\frac{1}{H}}\left(1-\frac{\alpha}{H}\right)^{-\frac{1}{\alpha}} = (b-a)\left(\frac{e}{\beta p}\right)^{\frac{1}{H}}. \quad (37)$$

Therefore from (35)-(37) we obtain the assertion of the theorem. □

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References

1. Buldygin, V. V. and Kozachenko, Yu. V., *Metric Characterization of Random Variables and Random Processes*, American Mathematical Society, Providence, RI, (2000).
2. R. Giuliano-Antonini, Yu. Kozachenko, T. Nikitina., *Spaces of ϕ-Subgaussian Random Variables*, Rendiconti, Academia Nazionale delle Scienze detta dei XL, *Memorie di Matematica e Applicazioni*, 121^o(2003), **XXVII**, fasc.1, 95–124.
3. Kozachenko Yu.V., Kovalchuk Yu.A.,*Boundary value problems with random initial conditions, and functional series from $Sub_\varphi(\Omega)$*, Ukrain.Mat.Zn. **50**, (4), (1998), 504–515.
4. Kozachenko Yu., Sottinen T. and Vasilik O., *Weakly self-similar stationary increment processes from the space $SSub_\varphi(\Omega)$*, Probab. Theory and Math. Statistics, **65**, (2001), 67–78.

5. Kozachenko Yu., Vasylyk O. and Yamnenko R., *On the probability of exceeding some curve by φ-subgaussian random process*, Theory of Stochastic Processes. **9**(25), no.3-4, (2003), 70–80.
6. Kozachenko Yu., Vasylyk O. and Yamnenko R., *Upper estimate of overrunning by $Sub_\varphi(\Omega)$ random process the level specified by continuous function*, Random Operators and Stochastic Equations, **13**, no.2, (2005), 111–128.
7. Kozachenko Yu., Vasylyk O., *Random processes from classes $V(\varphi, \psi)$*, Probab. Theory and Math. Statistics, **63**, (2000), 100–111.
8. Yamnenko R., *Ruin probability for generalized φ-sub-Gaussian fractional Brownian motion*, Theory of Stochastic Processes, **12**(28), no.3-4, (2009), 261–275.

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