Theory of Stochastic Processes
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SIMULATION OF RANDOM PROCESSES WITH KNOWN CORRELATION FUNCTION WITH THE HELP OF KARHUNEN-LOEVE DECOMPOSITION

A theorem is proved that allows to use approximations for construction of the Karhunen-Loeve model of stochastic process with known correlation function.

1. Introduction

While modeling the financial state of the insurance company demand flow is often considered as a Poisson process with the intensity generated by the process $e^{X(t)}$, where $X(t)$ is a centered Gaussian process.

Let $X = \{X(t), t \in T\}$ be a centered $(EX(t) = 0)$ Gaussian stochastic process with correlation function $B(t, s) = EX(t)X(s)$. To construct a model of the process, we will use the method of simulation of stochastic process with accuracy and reliability taken as parameters.

Consider a stochastic process $X(t)$, $t \in [0, T]$ which can be represented as a sum

$$X(t) = \sum_{k=1}^{\infty} \xi_k f_k(t)$$

which converges in the mean square. We will call the sum $X_N = X_N(t)$, $t \in [0, T]$,

$$X_N(t) = \sum_{k=1}^{N} \xi_k f_k(t)$$

the model of the process $X(t)$.

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Let the stochastic process $X(t)$ and all $X_N(t), N = 1, 2, \dots$ belong to a Banach space B with the norm $||\cdot||$. Let two numbers α and δ ($0 < \alpha < 1$, $\delta > 0$) be given. The model $X_N(t)$ approximates the stochastic process $X(t)$ with given reliability $1 - \alpha$ and accuracy δ if for this model the following inequality holds true:

$$P\{||X(t) - X_N|| > \delta\} \leq \alpha \tag{1}$$

As a result, to build a model of a stochastic process we have to find such N that this inequality holds true for given α and δ.

Let us assume that we can establish the inequality:

$$P\{||X(t) - X_N(t)|| > \delta\} \leq W_N(\delta),$$

where $W_N(\delta), \delta > 0$ is a known monotone decreasing by N and δ function. If N is such that $W_N(\delta) \leq \alpha$, then for all models $X_{N'}$ with $N' \geq N$ inequality (1) holds. This means that to build a model X_N which approximates the stochastic process X with given reliability $1 - \alpha$ and accuracy δ in the norm of space B, it is sufficient to find such N, minimal if it is possible, that inequality $W_N(\delta) \leq \alpha$ holds true.

2. The Karhunen-Loeve model

Consider method of modeling of stochastic processes based on the processes decomposition using eigenfunctions of some integral equations.

Let $[0, T]$ be an interval in $\mathbf{R}$, let $X = \{X(t), t \in [0, T], EX(t) = 0\}$ be the mean square continuous Hilbert stochastic process, let $B(t, s) = EX(t)X(s), t, s \in [0, T]$ be the correlation function of this process. The function $B(t, s)$ is positive semidefinite. As $X(t)$ is mean square continuous, then the function $B(t, s)$ is continuous on $[0, T] \times [0, T]$.

Consider the homogenous Fredholm integral equation of the second type

$$\varphi(t) = \lambda \int_0^T B(t, s)\varphi(s)ds.$$

This equation has at most countable set of nonnegative eigenvalues. Let λ_n^2, $n = 1, 2, \dots$ be eigenvalues of the equation, and let $\varphi_n(t)$ be the corresponding eigenfunctions. Let us also put λ_n in order $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \lambda_{n+1} \leq \dots$. It is known that $\varphi_n(t)$ are continuous orthonormal functions

$$\int_0^T \varphi_n(t)\varphi_m(t)dt = \delta_m^n,$$

where δ_m^n is the Kronecker delta.
In the following we will use the next theorem

Theorem. (Karhunen-Loeve decomposition) *Let $X(t) = \{X(t), t \in [0,T]\}$ be a mean square continuous centered Hilbert stochastic process with the correlation function $B = B(t,s)$. Then for all $t \in [0,T]$*

$$X(t) = \sum_{n=1}^{\infty} \xi_n \varphi_n(t),$$

with the mean square convergence, where ξ_n are centered orthonormal Gaussian random variables: $E\xi_n = 0, E\xi_n\xi_m = \delta_m^n \lambda_n^{-2}$.

Definition. We call the Karhunen-Loeve model of stochastic process $X(t) = \{X(t), t \in [0,T]\}$ the stochastic process $X_N(t) = \{X_N(t), t \in [0,T]\}$, where $X_N(t) = \sum_{n=1}^{N} \xi_n \varphi_n(t)$.

3. Approximate solutions of homogenous Fredholm integral equation of the second order

Only some types of integral equations can be solved in explicit forms. This is the main problem while building the Karhunen-Loeve model of stochastic process. That is why we have to use approximate methods to find eigenfunctions and eigenvalues of the second order homogenous Fredholm integral operator.

To construct an approximation we will use the method based on the formula of rectangles:

$$\int_a^b f(x)dx = h \sum_{k=1}^{n} f(x_k),$$

where $h = (b-a)/n$, $x_k = \xi + (k-1)h \quad a \le \xi \le a+h$.

Consider the equation

$$g(x) = \lambda \int_a^b K(x,y) g(y) dy \tag{2}$$

with given a, b and twice differentiable function $K(x,y)$. Let us apply the formula of mean ordinates to the integral $\int_a^b K(x,y)g(y)dy$:

$$\int_a^b K(x,y)g(y)dy = h \sum_{k=1}^{n} K(x,x_k) g(x_k)$$

and use this result in (2):

$$g(x) - \lambda h \sum_{k=1}^{n} K(x, x_k) g(x_k) = 0. \tag{3}$$

We take x_i from the mean ordinates formula, substitute them in the equation (3), and receive the system of equations:

$$\begin{array}{c} g(x_1) - \lambda h \sum_{k=1}^{n} K(x_1, x_k) g(x_k) = 0 \\ g(x_2) - \lambda h \sum_{k=1}^{n} K(x_2, x_k) g(x_k) = 0 \\ \dots \\ g(x_n) - \lambda h \sum_{k=1}^{n} K(x_n, x_k) g(x_k) = 0 \end{array}$$

If we solve this system we will receive trivial solution because this system is homogenous. But we can take the last equation off and get $g(x_k)$, $k = 1, \dots, n-1$ expressed through $g(x_n)$.

Lets find $g(x_n)$ now. We use the property of eigenfunctions which states that $||g(x)|| = 1$. For $L_2[a, b]$ this property if of the form

$$\int_a^b (g(x))^2 \, dx = 1 \tag{4}$$

After applying the mean ordinates formula to (4), we get

$$h \sum_{k=1}^{n} g(x_k)^2 = 1$$

From this relation we can find

$$g(x_n) = \sqrt{\frac{1}{h} - \sum_{k=1}^{n-1} g(x_k)}.$$

Finally, we can obtain $g(x_k)$, $k = 1, \dots, n-1$.

Now we substitute these expressions to (3) and, after some manipulations, we get the approximation of eigenfunction that corresponds to the eigenvalue λ.

$$g(x) = \lambda h \sum_{k=1}^{n} K(x, x_k) g(x_k)$$

Let us estimate the error of the approximation of eigenfunctions. We designate

$$I_m^n(t) = \lambda_m h \sum_{k=1}^{n} K(x, x_k) g_m(x_k)$$

According to the Runge rule, the error of this approximation is

$$\Delta_m = \frac{I_m^{2n} - I_m^n}{3}$$

4. Estimation of Karhunen-Loeve model accuracy in the space $L_p(0,T)$

Let us consider a Karhunen-Loeve model in the space $L_p(0,T)$. We will designate $\tau_N(t)$ as the error expectation.

$$E(X(t) - X_N(t))^2 = \tau_N^2(t)$$

To fulfill the required reliability $1-\alpha$ and accuracy δ in the space $L_p(0,T)$ the model X_N has to fulfill the next inequality:

$$P\{(\int_0^T (X(t) - X_N(t))^p dt)^{\frac{1}{p}} > \delta\} \leq \alpha$$

The following inequality was proved in [3].

$$P\{(\int_0^T (X(t) - X_N(t))^p dt)^{\frac{1}{p}} > \delta\} \leq 2exp\{-\frac{\delta^2}{2\tau_N^2 T^{2/p}}\},$$

if $\delta \geq p^{1/2} T^{1/p} \tau_N$.

Transforming this inequality, we will obtain

$$2exp\{-\frac{\delta^2}{2\tau_N^2 T^{1/p}}\}\} \leq \alpha \quad \Rightarrow \quad \frac{\delta^2}{2\tau_N^2 T^{2/p}} \geq -\ln\frac{\alpha}{2}$$

Hence that, (1) holds true if

$$\tau_N^2 \leq \frac{\delta^2}{2(-\ln\frac{\alpha}{2})T^{2/p}}$$

and

$$\tau_N^2 \leq \frac{\delta^2}{T^{2/p}p}$$

Let us now pay more attention to expectation τ_N

$$\tau_N^2 = \sup_{0\leq t\leq T} E|X(t) - X_N(t)|^2 =$$

$$= \sup_{0\leq t\leq T} E\left(\sum_{k=1}^{\infty} \xi_k \frac{\varphi_k(t)}{\sqrt{\lambda_k}} - \sum_{k=1}^{N} \xi_k \frac{\widehat{\varphi}_k(t)}{\sqrt{\widehat{\lambda}_k}}\right)^2 =$$

$$= \sup_{0\le t\le T}\left(\sum_{k=1}^{N}\left(\frac{\widehat{\varphi}_k(t)}{\sqrt{\widehat{\lambda}_k}}-\frac{\varphi_k(t)}{\sqrt{\lambda_k}}\right)^2+\sum_{k=N+1}^{\infty}\frac{\varphi_k^2(t)}{\lambda_k}\right)\le$$

$$\le\sum_{k=1}^{N}\sup_{0\le t\le T}\left(\frac{\widehat{\varphi}(k)}{\sqrt{\widehat{\lambda}_k}}-\frac{\varphi(k)}{\sqrt{\lambda_k}}\right)^2+\sum_{k=N+1}^{\infty}\frac{\sup_{0\le t\le T}(\varphi_k^2(t))}{\lambda_k}$$

Now we can apply approximations errors for eigenfunctions and eigenvalues mentioned above

$$\tau_N^2<\sum_{k=1}^{N}\frac{(3|\sqrt{\widehat{\lambda}_k+\eta}-\sqrt{\widehat{\lambda}_k}|\sup_{0\le t\le T}\widehat{\varphi}_k(t)+\sqrt{\widehat{\lambda}_k}\sup_{0\le t\le T}|I_k^{2n}(t)-I_k^{n}(t)|)^2}{9\widehat{\lambda}_k(\widehat{\lambda}_k-\eta)}$$

$$+\sum_{k=N+1}^{\infty}\frac{\sup_{0\le t\le T}(\varphi_k^2(t))}{\lambda_k}$$

It is also known that $sup_{t\in(0,T)}\varphi(t)<C$, where C is a constant, and we can build a sequence μ_k such that $\forall k\ 1/\lambda_k<1/\mu_k$, where $\sum_{k=1}^{\infty}\mu_k$ is finite.

The results obtained can be summarized in a theorem

Theorem. *Stochastic process X_N approximates stochastic process X with reliability $1-\alpha$ and accuracy δ in the space $L_p(0,T)$,*

$$P\{(\int_0^T(X(t)-X_N(t))^2dt)^{\frac{1}{p}}>\delta\}\le\alpha,$$

if N fulfills the next condition:

$$\sum_{k=1}^{N}\frac{(3|\sqrt{\widehat{\lambda}_k+\eta}-\sqrt{\widehat{\lambda}_k}|\sup_{0\le t\le T}\widehat{\varphi}_k(t)+\sqrt{\widehat{\lambda}_k}\sup_{0\le t\le T}|I_k^{2n}(t)-I_k^{n}(t)|)^2}{9\widehat{\lambda}_k(\widehat{\lambda}_k-\eta)}+$$

$$+\sum_{k=N+1}^{\infty}\frac{\sup_{0\le t\le T}(\varphi_k^2(t))}{\lambda_k}<\min\left\{\frac{\varepsilon^2}{2(-\ln\frac{\alpha}{2})T^{1/p}},\ \frac{\delta^2}{T^{2/p}p}\right\},$$

where $\widehat{\lambda}_k$ is the approximation of k-th eigenvalue of the equation

$$\varphi_k(t)=\lambda\int_a^b K(t,s)\varphi_k(s)ds,$$

η is the error of approximation of this eigenvalue, $\varphi_k(t)$ is the corresponding eigenfunction, and I_k^n is the n-th approximation of the eigenfunction $\varphi_k(t)$.

This theorem allows us to build a Karhunen-Loeve model with given accuracy and reliability in the space $L_p(0,T)$.

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