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PREDICTION PROBLEM FOR RANDOM FIELDS ON GROUPS

The problem considered is the problem of optimal linear estimation of the functional $A\xi=\sum_{j=0}^{\infty}\int_G a(g,j)\xi(g,j)dg$ which depends on the unknown values of a homogeneous random field $\xi(g,j)$ on the group $G\times\mathbb{Z}$ from observations of the field $\xi(g,j)+\eta(g,j)$ for $(g,j)\in G\times\{-1,-2,\ldots\}$, where $\eta(g,j)$ is an uncorrelated with $\xi(g,j)$ homogeneous random field $\xi(g,j)$ on the group $G\times\mathbb{Z}$. Formulas are proposed for calculation the mean square error and spectral characteristics of the optimal linear estimate in the case where spectral densities of the fields are known. The least favorable spectral densities and the minimax spectral characteristics of the optimal estimate of the functional are found for some classes of spectral densities.

1. Introduction

Traditional methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes and homogeneous random fields may be employed under the condition that spectral densities of processes and fields are known exactly (see, for example, selected works of A. N. Kolmogorov (1992), survey by T. Kailath (1974), Yu. A. Rozanov (1990), N. Wiener (1966); A. M. Yaglom (1987),M. I. Yadrenko (1983)). In practice, however, complete information on the spectral densities is impossible in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies the classical estimation method provided that the estimated or selected densities are the true one. This procedure can result in a significant increasing of the value of error as K. S. Vastola and H. V. Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities

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Invited lecture.

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from a certain class of the admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor (1985). The paper by Ulf Grenander (1957) should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. J. Franke (1984, 1985, 1991), J. Franke and H. V. Poor (1984) investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of densities. In the papers by M. P. Moklyachuk (1998, 2000, 2001, 2002) the minimax approach to extrapolation, interpolation and filtering problems are investigated for functionals which depend on the unknown values of stationary processes and random fields on a sphere.

In this article we considered the problem of estimation of the unknown value of the functional $A\xi = \sum_{j=0}^{\infty}\int_G a(g,j)\xi(g,j)dg$ which depends on the unknown values of a homogeneous random field $\xi(g,j)$ on the group $G\times\mathbb{Z}$, where G is a compact Abelian group, from observations of the field $\xi(g,j)+\eta(g,j)$ for $(g,j)\in G\times\mathbb{Z}^- = G\times\{-1,-2,\ldots\}$, where $\eta(g,j)$ is an uncorrelated with $\xi(g,j)$ homogeneous random field on the group $G\times\mathbb{Z}$. Formulas are proposed for calculation the mean square error and spectral characteristics of the optimal linear estimate of the unknown value of the functional $A\xi$ in the case where spectral densities $f(\lambda)=\{f^{(n)}(\lambda): n=0,1\ldots\}$ and $g(\lambda)=\{g^{(n)}(\lambda): n=0,1\ldots\}$ of the fields are known. Formulas are proposed that determine the least favorable spectral densities and the minimax-robust spectral characteristic of the optimal estimate of the functional $A\xi$ for concrete classes $D=D_F\times D_G$ of spectral densities under the condition that spectral densities $f(\lambda)$, $g(\lambda)$ are not known, but classes $D=D_f\times D_g$ of admissible spectral densities are given.

2. Homogeneous random fields on groups

Let G be a compact Abelian group and let $\mathbb{Z}$ be the group of all integers. We consider random field $\xi(g,j)$ on $G\times\mathbb{Z}$ as a function on $G\times\mathbb{Z}$ with values in the Hilbert space $L_2(\Omega,\mathcal{F},\mathbf{P})$ of complex random variables with finite second moment and zero mean. We will call a random field $\xi(g,j)$ homogeneous if

$$\mathbf{E}\{\xi(g,j)\overline{\xi(h,k)}\} = \mathbf{E}\{\xi(g^{-1}g,j)\overline{\xi(g^{-1}h,k)}\} = \mathbf{E}\{\xi(gg^{-1},j)\overline{\xi(hg^{-1},k)}\} =$$
$$= B(g^{-1}h, j-k) = B(hg^{-1}, j-k),\ \forall h,g\in G, \forall j\in\mathbb{Z}.$$

In the case of commutative group G the correlation function has the property:

$$B(h,j) = B(g^{-1}hg, j),\ \forall h,g\in G, \forall j\in\mathbb{Z}.$$

If the group G is compact it has countable nonequivalent finite dimensional unitary representations. As a representation of the group G consider homomorphism of the group G into the group of unitary matrices of finite order

$$g \to T^{(n)}(g) = \|T^{(n)}_{uv}(g)\|, \ 1 \le u,v \le d_n, \ n = 0,1\dots$$
$$T^{(n)}(gh) = T^{(n)}(g)T^{(n)}(h), \ n = 0,1\dots$$
$$T^{(n)}(g^{-1}) = [T^{(n)}(g)]^{-1} = [T^{(n)}(g)]^{*}, \ \ n = 0,1\dots.$$

Elements $T^{(n)}_{uv}(g)$, $1 \le u,v \le d_n, n = 0,1\dots$ of matrices of such representation satisfy conditions

$$d_n \int_G T^{(n)}_{uv}(g)T^{(m)}_{u_1v_1}(g)dg = \delta^n_m \delta^{u_1}_u \delta^{v_1}_v,$$

where δ^n_m is the Kronecker symbol, and dg is an invariant unit measure on G. The set of all matrix elements $T^{(n)}_{uv}(g)$, $1 \le u,v \le d_n$, $n = 0,1\dots$, form an orthogonal system in the space $L_2(G)$.

Let $\xi(g,j)$ be a homogeneous random field on $G \times \mathbb{Z}$. Such a field may be represented in the form of series

$$\xi(g,j) = \sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n} \xi^{(n)}_{uv}(j)T^{(n)}_{uv}(g), \quad \xi^{(n)}_{uv}(j) = d_n \int_G \xi(g,j)\overline{T^{(n)}_{uv}(g)}dg,$$

that converges in the mean square. Integral is also considered in the mean square sense. It follows from the definition of homogeneous random field $\xi(g,j)$ that

$$\mathbf{E}\xi^{(n)}_{uv}(j)\overline{\xi^{(m)}_{u_1v_1}(k)} = \delta^n_m \delta^{u_1}_u \delta^{v_1}_v R^{(n)}(j-k), \ 1 \le u,v \le d_n, n = 0,1\dots,$$

where $R^{(n)}(j)$ are nonnegative numbers that satisfy the condition

$$\sum_{n=0}^{\infty} d_n R^{(n)}(j) < \infty.$$

The correlation function of the homogeneous random field $\xi(g,j)$ on $G \times \mathbb{Z}$ may be represented in the form

$$B(g,j) = \sum_{n=0}^{\infty} R^{(n)}(j)\chi^{(n)}(g),$$

where $\chi^{(n)}(g) := \mathrm{Tr}[T^{(n)}(g)]$ are characters of the group G.

3. Hilbert space projection method of estimation

Consider the problem of the mean square optimal linear estimation of the functional

$$A\xi = \sum_{j=0}^{\infty} \int_G a(g,j)\xi(g,j)dg$$

which depends on the unknown values of a homogeneous random field $\xi(g,j)$ on the group $G\times\mathbb{Z}$ from observations of the field $\xi(g,j)+\eta(g,j)$ for $(g,j)\in G\times\{-1,-2,\ldots\}$, where $\eta(g,j)$ is an uncorrelated with $\xi(g,j)$ homogeneous random field $\xi(g,j)$ on the group $G\times\mathbb{Z}$. It follows from the matrix representation of the group G that such homogeneous random fields can be represented in the form

$$\xi(g,j)=\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\xi_{uv}^{(n)}(j)T_{uv}^{(n)}(g),\quad \xi_{uv}^{(n)}(j)=d_n\int_G\xi(g,j)\overline{T_{uv}^{(n)}(g)}dg,$$

$$\eta(g,j)=\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\eta_{uv}^{(n)}(j)T_{uv}^{(n)}(g),\quad \eta_{uv}^{(n)}(j)=d_n\int_G\eta(g,j)\overline{T_{uv}^{(n)}(g)}dg,$$

where $\|T_{uv}^{(n)}(g)\|$ are matrix unitray representations of the group G, and $\xi_{uv}^{(n)}(j)$, $\eta_{uv}^{(n)}(j)$, $1\leq u,v\leq d_n, n=0,1\ldots,j\in\mathbb{Z}$, are mutually orthogonal stationary stochastic sequences. Homogeneous random fields $\xi(g,j)$, $\eta(g,j)$ admit the spectral representations

$$\xi(g,j)=\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\int_{-\pi}^{\pi}e^{ij\lambda}(Z_\xi)_{uv}^{(n)}(d\lambda)T_{uv}^{(n)}(g),$$

$$\eta(g,j)=\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\int_{-\pi}^{\pi}e^{ij\lambda}(Z_\eta)_{uv}^{(n)}(d\lambda)T_{uv}^{(n)}(g).$$

Random measures $(Z_\xi)_{uv}^{(n)}(\Delta)$, $(Z_\eta)_{uv}^{(n)}(\Delta)$ satisfy conditions

$$\mathbf{E}(Z_\xi)_{uv}^{(n)}(\Delta_1)\overline{(Z_\xi)_{u_1v_1}^{(m)}(\Delta_2)}=\delta_m^n\delta_u^{u_1}\delta_v^{v_1}F_\xi^{(n)}(\Delta_1\cap\Delta_2),$$

$$\mathbf{E}(Z_\eta)_{uv}^{(n)}(\Delta_1)\overline{(Z_\eta)_{u_1v_1}^{(m)}(\Delta_2)}=\delta_m^n\delta_u^{u_1}\delta_v^{v_1}F_\eta^{(n)}(\Delta_1\cap\Delta_2),$$

$$1\leq u,v\leq d_n,\ n=0,1\ldots,\ \Delta_i\in\mathcal{B}\left([-\pi,\pi]\right),i=1,2.$$

Let measures $F_\xi^{(n)}(\Delta)$, $n=0,1\ldots$ have spectral densities $f^{(n)}(\lambda)$, $n=0,1\ldots$, and measures $F_\eta^{(n)}(\Delta)$, $n=0,1\ldots$ have spectral densities $g^{(n)}(\lambda)$, $n=0,1\ldots$.

If the spectral densities $f^{(n)}(\lambda)$, $n=0,1\ldots$, of the field $\xi(g,j)$ on $G\times\mathbb{Z}$ admit the canonical factorizations [8,15]

$$f^{(n)}(\lambda)=|d^{(n)}(\lambda)|^2,\quad d^{(n)}(\lambda)=\sum_{j=0}^{\infty}d^{(n)}(j)e^{-ij\lambda},\tag{1}$$

the field $\xi(g,j)$ can be represented as one sided moving average random field

$$\xi(g,j)=\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\sum_{k=-\infty}^{j}d^{(n)}(j-k)\zeta_{uv}^{(n)}(k)T_{uv}^{(n)}(g),\tag{2}$$

where $\zeta_{uv}^{(n)}(k), 1 \leq u, v \leq d_n, n = 0, 1 \ldots, k \in \mathbb{Z}$, are mutually uncorrelated stochastic sequences with orthonormal values (white noise).

Let $M(f+g)$ be a set of n such that the minimality condition holds true [1,2?]:

$$\int_{-\pi}^{\pi} (f^{(n)}(\lambda) + g^{(n)}(\lambda))^{-1} d\lambda < \infty.$$

A sequence $\xi_{uv}^{(n)}(j) + \eta_{uv}^{(n)}(j)$ that has the spectral density $f^{(n)}(\lambda) + g^{(n)}(\lambda)$ which do not satisfies the minimality condition can be estimated with zero mean square error. For fixed $u, v \colon 1 \leq u, v \leq d_n$; $n \in M(f+g)$ consider the functional

$$A_{uv}^{(n)}\xi = \sum_{j=0}^{\infty} a_{uv}^{(n)}(j) \xi_{uv}^{(n)}(j)$$

from the sequence $\xi_{uv}^{(n)}(j)$, where

$$a_{uv}^{(n)}(j) = d_n \int_G a(g, j) \overline{T_{uv}^{(n)}(g)} dg.$$

We will suppose that the following conditions hold true

$$\sum_{n=0}^{\infty} \sum_{u,v=1}^{d_n} \sum_{j=0}^{\infty} |a_{uv}^{(n)}(j)| < \infty, \quad \sum_{n=0}^{\infty} \sum_{u,v=1}^{d_n} \sum_{j=0}^{\infty} (j+1)|a_{uv}^{(n)}(j)|^2 < \infty. \tag{3}$$

Under these conditions the functional $A\xi$ has the second moment and operators $\mathbf{A}_{uv}^{(n)}$, defined below are compact.

Let $\xi_{uv}^{(n)}(j) + \eta_{uv}^{(n)}(j)$, $1 \leq u, v \leq d_n$; $n \in M(f+g)$ be stationary sequence from observations of which we find an estimate of the functional $A_{uv}^{(n)}\xi$. For every $u, v \colon 1 \leq u, v \leq d_n; n \in M(f+g)$ denote by $L_2\left(f^{(n)} + g^{(n)}\right)$ the Hilbert space of complex-valued functions on $[-\pi, \pi]$, which are integrable in square with respect to the measure with the density $f^{(n)}(\lambda) + g^{(n)}(\lambda)$, denote by $H^-\left(f^{(n)} + g^{(n)}\right)$ closed subspace of $L_2\left(f^{(n)} + g^{(n)}\right)$ generated by functions $\{e^{ik\lambda}, k = -1, -2, \ldots\}$. Let $h\left(e^{i\lambda}\right) = \{h_{uv}^{(n)}\left(e^{i\lambda}\right) : 1 \leq u, v \leq d_n$; $n \in M(f+g)\}$ be the spectral characteristic of the linear estimate $\widehat{A\xi}$ of the functional $A\xi$

$$\widehat{A\xi} = \sum_{n \in M(f+g)} \sum_{u,v=1}^{d_n} \int_{-\pi}^{\pi} h_{uv}^{(n)}\left(e^{i\lambda}\right) \left((Z_\xi)_{uv}^{(n)}(d\lambda) + (Z_\eta)_{uv}^{(n)}(d\lambda)\right).$$

the mean square error of the estimate $\widehat{A\xi}$ of the functional $A\xi$

$$\Delta(h; f, g) = \mathbf{E}|A\xi - \widehat{A\xi}|^2 =$$

$$\sum_{n \in M(f+g)} \sum_{u,v=1}^{d_n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[|A_{uv}^{(n)}\left(e^{i\lambda}\right) - h_{uv}^{(n)}\left(e^{i\lambda}\right)|^2 f^{(n)}(\lambda) + |h_{uv}^{(n)}\left(e^{i\lambda}\right)|^2 g^{(n)}(\lambda)\right] d\lambda.$$

The spectral characteristic $h(f,g) = \{h_{uv}^{(n)}\big(f^{(n)},g^{(n)}\big) : 1\le u,v\le d_n; n\in M(f+g)\}$ of the mean square optimal linear estimate minimizes the value of the mean square error

$$\Delta(f,g) = \Delta(h(f,g);f,g) =$$

$$= \sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n} \min_{h_{uv}^{(n)}\in H^-\left(f^{(n)}+g^{(n)}\right)} \Delta(h_{uv}^{(n)};f^{(n)},g^{(n)}).$$

With the help of the Hilbert space projection method proposed by A. N. Kolmogorov [8] we can find the following formulas for calculation the mean square error $\Delta(f,g)=\Delta(h(f,g);f,g)$ the spectral characteristic $h(f,g) = h_{uv}^{(n)}\big(f^{(n)},g^{(n)}\big) : 1\le u,v\le d_n; n\in M(f+g)\}$ of the optimal linear estimate of the functional $A\xi$

$$\Delta(h(f,g);f,g) =$$

$$= \sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|A_{uv}^{(n)}\big(e^{i\lambda}\big)g^{(n)}(\lambda)+C_{uv}^{(n)}\big(e^{i\lambda}\big)|^2}{\left(g^{(n)}(\lambda)+f^{(n)}(\lambda)\right)^2}f^{(n)}(\lambda)d\lambda+\right.$$

$$\left.+\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|A_{uv}^{(n)}\big(e^{i\lambda}\big)f^{(n)}(\lambda)-C_{uv}^{(n)}\big(e^{i\lambda}\big)|^2}{\left(g^{(n)}(\lambda)+f^{(n)}(\lambda)\right)^2}g^{(n)}(\lambda)d\lambda\right\} =$$

$$= \sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left[\langle \mathbf{B}^{(n)}\mathbf{c}_{uv}^{(n)},\mathbf{c}_{uv}^{(n)}\rangle + \langle \mathbf{R}^{(n)}\mathbf{a}_{uv}^{(n)},\mathbf{a}_{uv}^{(n)}\rangle\right]; \tag{4}$$

$$h_{uv}^{(n)}\big(f^{(n)},g^{(n)}\big) = \frac{A_{uv}^{(n)}\big(e^{i\lambda}\big)f^{(n)}(\lambda)-C_{uv}^{(n)}\big(e^{i\lambda}\big)}{f^{(n)}(\lambda)+g^{(n)}(\lambda)} =$$

$$= A_{uv}^{(n)}\big(e^{i\lambda}\big) - \frac{A_{uv}^{(n)}\big(e^{i\lambda}\big)g^{(n)}(\lambda)+C_{uv}^{(n)}\big(e^{i\lambda}\big)}{f^{(n)}(\lambda)+g^{(n)}(\lambda)}, \tag{5}$$

$$1\le u,v\le d_n, n\in M(f+g).$$

$$A_{uv}^{(n)}\left(e^{i\lambda}\right) = \sum_{j=0}^{\infty} a_{uv}^{(n)}(j)e^{ij\lambda},\ C_{uv}^{(n)}\left(e^{i\lambda}\right) = \sum_{j=0}^{\infty} c_{uv}^{(n)}(j)e^{ij\lambda},$$

$$\mathbf{a}_{uv}^{(n)} = \left(a_{uv}^{(n)}(0),a_{uv}^{(n)}(1),\ldots\right),\ \mathbf{c}_{uv}^{(n)} = \left(c_{uv}^{(n)}(0),c_{uv}^{(n)}(1),\ldots\right),$$

$$\mathbf{c}_{uv}^{(n)} = (\mathbf{B}^{(n)})^{-1}\mathbf{D}^{(n)}\mathbf{a}_{uv}^{(n)},\ \langle \mathbf{c}_{uv}^{(n)},\mathbf{a}_{uv}^{(n)}\rangle = \sum_{j=0}^{\infty} c_{uv}^{(n)}(j)a_{uv}^{(n)}(j),$$

Here $\mathbf{B}^{(n)},\mathbf{D}^{(n)},\mathbf{R}^{(n)}$ are operators in the space ℓ_2 generated by matrices that are determined by the Fourier coefficients of the functions $(f^{(n)}(\lambda)+$

$g^{(n)}(\lambda))^{-1},\ f^{(n)}(\lambda)(f^{(n)}(\lambda)+g^{(n)}(\lambda))^{-1},\ f^{(n)}(\lambda)g^{(n)}(\lambda)(f^{(n)}(\lambda)+g^{(n)}(\lambda))^{-1}$ correspondingly

$$\mathbf{B}^{(n)}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\lambda}\frac{1}{(f^{(n)}(\lambda)+g^{(n)}(\lambda)}d\lambda,$$

$$\mathbf{D}^{(n)}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\lambda}\frac{f^{(n)}(\lambda)}{f^{(n)}(\lambda)+g^{(n)}(\lambda)}d\lambda,$$

$$\mathbf{R}^{(n)}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(j-k)\lambda}\frac{f^{(n)}(\lambda)g^{(n)}(\lambda)}{f^{(n)}(\lambda)+g^{(n)}(\lambda)}d\lambda,\ k,j=0,1,\dots$$

The following statements holds true.

Theorem 1. *Let $\xi(g,j),\eta(g,j)$ be uncorrelated homogeneous random fields on $G\times\mathbb{Z}$, which have spectral densities $f(\lambda)=\{f^{(n)}(\lambda),n=0,1\dots\}$, $g(\lambda)=\{g^{(n)}(\lambda),n=0,1\dots\}$. If $M(f+g)\neq\emptyset$ and conditions (3) are satisfied, then the value of mean square error $\Delta(f,g)=\Delta(h(f,g);f,g)$ and the spectral characteristic $h(f,g)=h_{uv}^{(n)}\big(f^{(n)},g^{(n)}\big):1\le u,v\le d_n;n\in M(f+g)\}$ of the optimal linear estimate of the functional $A\xi$ from observations of the field $\xi(g,j)+\eta(g,j)$ for $(g,j)\in G\times\{-1,-2,\dots\}$ can be calculated by formulas (4), (5).*

Corollary 1. *Let $\xi(g,j),g\in G,j\in\mathbb{Z}$, be a homogeneous random field on $G\times\mathbb{Z}$, which has spectral density $f(\lambda)=\{f^{(n)}(\lambda),n=0,1\dots\}$. If $M(f)\neq\emptyset$ and conditions (3) are satisfied, then the value of mean square error $\Delta(f)=\Delta(h(f);f)$ and the spectral characteristic $h(f)=\{h_{uv}^{(n)}\left(f^{(n)}\right):1\le u,v\le d_n;n\in M(f)\}$ of the optimal linear estimate of the functional $A\xi$ from observations of the field $\xi(g,j)$ for $(g,j)\in G\times\{-1,-2,\dots\}$ can be calculated by formulas*

$$\Delta(h(f);f)=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\frac{1}{2\pi}\int_{-\pi}^{\pi}|C_{uv}^{(n)}(e^{i\lambda})|^2(f^{(n)}(\lambda))^{-1}d\lambda=$$

$$=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\langle\left(\mathbf{B}^{(n)}\right)^{-1}\mathbf{a}_{uv}^{(n)},\mathbf{a}_{uv}^{(n)}\rangle=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left\|\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right\|^2, \tag{6}$$

$$\begin{aligned}h_{uv}^{(n)}\left(f^{(n)}\right)&=A_{uv}^{(n)}\left(e^{i\lambda}\right)-C_{uv}^{(n)}\left(e^{i\lambda}\right)\left(f^{(n)}(\lambda)\right)^{-1}=\\&=A_{uv}^{(n)}\left(e^{i\lambda}\right)-\left(\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right)(\lambda)\left(d^{(n)}(\lambda)\right)^{-1},\end{aligned} \tag{7}$$

where

$$C_{uv}^{(n)}\left(e^{i\lambda}\right)=\sum_{j=0}^{\infty}\left(\left(\mathbf{B}^{(n)}\right)^{-1}\mathbf{a}_{uv}^{(n)}\right)(j)e^{ij\lambda},$$

$$\left(\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right)(\lambda)=\sum_{j=0}^{\infty}\left(\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right)(j)e^{ij\lambda},$$

$\mathbf{B}^{(n)}, \mathbf{A}_{uv}^{(n)}$ *are operators in the space* ℓ_2 *generated by matrices*

$$\mathbf{B}^{(n)}(k,j)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i(j-k)\lambda}}{f^{(n)}(\lambda)}\,d\lambda,\quad \mathbf{A}_{uv}^{(n)}(k,j)=a_{uv}^{(n)}(k+j),\ 0\leq k,j<\infty.$$

4. Minimax-robust method of estimation

The proposed formulas may be employed under the condition that spectral densities $f(\lambda)=\{f^{(n)}(\lambda),n=0,1\ldots\}$, $g(\lambda)=\{g^{(n)}(\lambda),n=0,1\ldots\}$ of the field $\xi(g,j)$ and the field $\eta(g,j)$ are known. In the case where the densities are not known exactly, but a set $D=D_f\times D_g$ of possible spectral densities is given, the minimax (robust) approach to estimation of functionals of the unknown values of homogeneous random fields is reasonable. Instead of searching an estimate that is optimal for a given spectral densities we find an estimate that minimizes the mean square error for all spectral densities $f(\lambda), g(\lambda)$ from a given class $D_f\times D_g$ simultaneously.

Definition 1. For a given class of spectral densities $D=D_f\times D_g$ spectral densities $f_0(\lambda)\in D_f$, $g_0(\lambda)\in D_g$ are called least favorable for the optimal linear estimate of the functional $A\xi$ if the following relation holds true

$$\Delta(f_0,g_0)=\Delta(h(f_0,g_0);f_0,g_0)=\max_{(f,g)\in D_f\times D_g}\Delta(h(f,g);f,g).$$

Definition 2. For a given class of spectral densities $D=D_f\times D_g$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A\xi$ is called minimax-robust if there are satisfied conditions

$$h^0(\lambda)\in H_D=\cap_{(f,g)\in D_f\times D_g}L_2^-(f+g),$$

$$\min_{h\in H_D}\max_{(f,g)\in D_f\times D_g}\Delta(h;f,g)=\max_{(f,g)\in D_f\times D_g}\Delta(h^0;f,g).$$

Taking into account relations (3)–(7), we can conclude that the following statements hold true.

Lemma 1. *Spectral densities* $f_0(\lambda)\in D_f$, $g_0(\lambda)\in D_g$ *are least favorable in* $D_f\times D_g$ *for the optimal linear estimation of the functional* $A\xi$ *from observations of the field* $\xi(g,j)+\eta(g,j)$ *for* $(g,j)\in G\times\{-1,-2,\ldots\}$ *if*

$M(f_0+g_0)\neq\emptyset$ and the Fourier coefficients of functions $(f_0^{(n)}(\lambda)+g_0^{(n)}(\lambda))^{-1}$, $f_0^{(n)}(\lambda)(f_0^{(n)}(\lambda)+g_0^{(n)}(\lambda))^{-1}$, $f_0^{(n)}(\lambda)g_0^{(n)}(\lambda)(f_0^{(n)}(\lambda)+g_0^{(n)}(\lambda))^{-1}$ determine operators $\mathbf{B}_0^{(n)},\mathbf{D}_0^{(n)},\mathbf{R}_0^{(n)}$, which give a solution to the extremum problem

$$\max_{f,g\in D_f\times D_g}\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left[\langle\mathbf{D}^{(n)}\mathbf{a}_{uv}^{(n)},\left(\mathbf{B}^{(n)}\right)^{-1}\mathbf{D}^{(n)}\mathbf{a}_{uv}^{(n)}\rangle+\langle\mathbf{R}^{(n)}\mathbf{a}_{uv}^{(n)},\mathbf{a}_{uv}^{(n)}\rangle\right]$$

$$=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left[\langle\mathbf{D}_0^{(n)}\mathbf{a}_{uv}^{(n)},\left(\mathbf{B}_0^{(n)}\right)^{-1}\mathbf{D}_0^{(n)}\mathbf{a}_{uv}^{(n)}\rangle+\langle\mathbf{R}_0^{(n)}\mathbf{a}_{uv}^{(n)},\mathbf{a}_{uv}^{(n)}\rangle\right].\qquad(8)$$

Minimax spectral characteristic $h_0=h(f_0,g_0)$ is calculated by formula (5) if the condition $h_0=h(f_0,g_0)\in H_D$ holds true.

Lemma 2. *Spectral density $f_0(\lambda)\in D_f$, $M(f_0)\neq\emptyset$, is the least favorable in D_f for the optimal linear estimation of the functional $A\xi$ from observations of the field $\xi(g,j)$ for $(g,j)\in G\times\{-1,-2,\ldots\}$ if the Fourier coefficients of functions $(f_0^{(n)}(\lambda))^{-1}, n=0,1,\ldots,$ determine operators $\mathbf{B}_0^{(n)}$, which give a solution to the extremum problem*

$$\max_{f\in D_f}\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\langle\mathbf{a}_{uv}^{(n)},\left(\mathbf{B}^{(n)}\right)^{-1}\mathbf{a}_{uv}^{(n)}\rangle=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\langle\mathbf{a}_{uv}^{(n)},\left(\mathbf{B}_0^{(n)}\right)^{-1}\mathbf{a}_{uv}^{(n)}\rangle.\qquad(9)$$

Minimax spectral characteristic $h_0=h(f_0)$ is calculated by formula (7) if the condition $h(f_0)\in H_{D_f}$ holds true.

Lemma 3. *Spectral density $f_0(\lambda)=\{f_0^{(n)}(\lambda), n=0,1\ldots\}$, is the least favorable in D_f for the optimal linear estimation of the functional $A\xi$ from observations of the field $\xi(g,j)$ for $(g,j)\in G\times\{-1,-2,\ldots\}$ if sequences $\mathbf{d}_0^{(n)}=\{d_0^{(n)}(j):j=0,1,\ldots\}$, that determine the canonical factorization (1) of the density $f_0(\lambda)$, give a solution to the extremum problem*

$$\Delta(f)=\sum_{n\in M(f+g)}\sum_{u,v=1}^{d_n}\left\|\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right\|^2\to\sup,\qquad(10)$$

$$f(\lambda)=\left\{f^{(n)}(\lambda)=\left|\sum_{j=0}^{\infty}d^{(n)}(j)e^{-ij\lambda}\right|^2, n=0,1\ldots\right\}.\qquad(11)$$

The least favorable spectral densities $f_0(\lambda)\in D_f$, $g_0(\lambda)\in D_g$ and the minimax (robust) spectral characteristic $h(f_0,g_0)\in H_D$ form a saddle point of the function $\Delta(h;f,g)$ on the set $H_D\times D$. The saddle point inequalities hold when $h_0 = h(f_0, g_0)$, $h(f_0, g_0) \in H_D$, and (f_0, g_0) is a solution to the conditional extremum problem

$$\Delta(h(f_0, g_0); f_0, g_0) = \max_{(f,g) \in D_f \times D_g} \Delta(h(f_0, g_0); f, g), \tag{12}$$

$$\Delta(h(f_0, g_0); f, g) =$$

$$= \sum_{n \in M(f+g)} \sum_{u,v=1}^{d_n} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|A_{uv}^{(n)}(e^{i\lambda}) g_0^{(n)}(\lambda) + C_{uv}^{(n)}(e^{i\lambda})|^2}{\left(g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)\right)^2} f^{(n)}(\lambda) d\lambda + \right.$$

$$\left. + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|A_{uv}^{(n)}(e^{i\lambda}) f_0^{(n)}(\lambda) - C_{uv}^{(n)}(e^{i\lambda})|^2}{\left(g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)\right)^2} g^{(n)}(\lambda) d\lambda \right\}.$$

Conditional extremum problem (12) is equivalent to the unconditional extremum problem

$$\Delta_D(f, g) = -\Delta(h(f_0, g_0); f, g) + \delta((f, g)|D_f \times D_g) \to \inf,$$

where $\delta((f, g)|D_f \times D_g)$ is the indicator function of the set $D_f \times D_g$. A solution to this unconditional extremum problem is characterized by the condition $0 \in \partial\Delta_D(f_0, g_0)$, where $\partial\Delta_D(f, g)$ is the subdifferential of the convex functional $\Delta_D(f, g)$ [14].

5. Least favorable spectral densities in the class $D_f^0 \times D_v^u$.

Consider the problem for the set of spectral densities $D_f^0 \times D_v^u$, where

$$D_f^0 = \left\{ f(\lambda) : \frac{1}{2\pi} \sum_{n=0}^{\infty} d_n \int_{-\pi}^{\pi} f^{(n)}(\lambda) d\lambda \le P_1 \right\},$$

$$D_v^u = \left\{ g(\lambda) : v^{(n)}(\lambda) \le g^{(n)}(\lambda) \le u^{(n)}(\lambda); \frac{1}{2\pi} \sum_{n=0}^{\infty} d_n \int_{-\pi}^{\pi} g^{(n)}(\lambda) d\lambda \le P_2 \right\},$$

spectral densities $v(\lambda) = \{v^{(n)}(\lambda), n = 0, 1 \ldots\}$, $u(\lambda) = \{u^{(n)}(\lambda), n = 0, 1 \ldots\}$ are known and fixed and densities $u^{(n)}(\lambda), n = 0, 1 \ldots$ are bounded. Let the spectral densities $f_0(\lambda) \in D_f^0$, $g_0(\lambda) \in D_v^u$, $M(f_0 + g_0) \neq \emptyset$ and let the functions

$$h_f^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right) = \sum_{u,v=1}^{d_n} \frac{|A_{uv}^{(n)}(e^{i\lambda}) g_0^{(n)}(\lambda) + C_{uv}^{(n)}(e^{i\lambda})|}{g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)} \tag{13}$$

$$h_g^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right) = \sum_{u,v=1}^{d_n} \frac{|A_{uv}^{(n)}\left(e^{i\lambda}\right) f_0^{(n)}(\lambda) - C_{uv}^{(n)}\left(e^{i\lambda}\right)|}{g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)} \tag{14}$$

be bounded. The condition $0 \in \partial\Delta_D(f_0, g_0)$ is satisfied for $D = D_f^0 \times D_v^u$ if components of the spectral densities $f_0(\lambda) = \{f_0^{(n)}(\lambda), n = 0, 1 \ldots\}$, $g_0(\lambda) = \{g_0^{(n)}(\lambda), n = 0, 1 \ldots\}$ satisfy equations

$$\alpha_1^{(n)} \sum_{u,v=1}^{d_n} |A_{uv}^{(n)}\left(e^{i\lambda}\right) g_0^{(n)}(\lambda) + C_{uv}^{(n)}\left(e^{i\lambda}\right)| = \left(g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)\right), \tag{15}$$

$$\sum_{u,v=1}^{d_n} |A_{uv}^{(n)}\left(e^{i\lambda}\right) f_0^{(n)}(\lambda) - C_{uv}^{(n)}\left(e^{i\lambda}\right)| =$$

$$= \left(g_0^{(n)}(\lambda) + f_0^{(n)}(\lambda)\right)\left(\gamma_1^{(n)}(\lambda) + \gamma_2^{(n)}(\lambda) + \alpha_2^{(n)}\right), \tag{16}$$

$$n \in M(f_0 + g_0),$$

where $\alpha_1^{(n)} \geq 0$, $\alpha_2^{(n)} \geq 0$; $\gamma_1^{(n)}(\lambda) \leq 0$ and $\gamma_1^{(n)}(\lambda) = 0$ if $g_0^{(n)}(\lambda) \geq v^{(n)}(\lambda)$; $\gamma_2^{(n)}(\lambda) \geq 0$ and $\gamma_2^{(n)}(\lambda) = 0$ if $g_0^{(n)}(\lambda) \leq u^{(n)}(\lambda)$, and conditions

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} d_n \int_{-\pi}^{\pi} f_0^{(n)}(\lambda) d\lambda = P_1, \tag{17}$$

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} d_n \int_{-\pi}^{\pi} g_0^{(n)}(\lambda) d\lambda = P_2. \tag{18}$$

The following statements hold true.

Theorem 2 *Let spectral densities $f_0(\lambda) = \{f_0^{(n)}(\lambda), n = 0, 1 \ldots\}$, $g_0(\lambda) = \{g_0^{(n)}(\lambda), n = 0, 1 \ldots\}$ are from the set $D_f^0 \times D_v^u$ and let the functions $h_f^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right)$, $h_g^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right)$, $m \in M(f_0 + g_0)$, determined by formulas (13), (14) be bounded. The spectral densities $f_0(\lambda) \in D_f^0$, $g_0(\lambda) \in D_v^u$ are the least favorable in the class $D_f^0 \times D_v^u$ for the optimal linear estimation of the functional $A\xi$, if they satisfy equations (15), (16), conditions (17), (18), and determine solution to the extremum problem (8). The minimax spectral characteristic $h_0 = h(f_0, g_0)$ is calculated by formula (5).*

Corollary 2. *Let the spectral density $f(\lambda) = \{f^{(n)}(\lambda), n = 0, 1 \ldots\}$ be known, the spectral density $g_0(\lambda) = \{g_0^{(n)}(\lambda), n = 0, 1 \ldots\}$ is from the class D_v^u, and let functions $h_g^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right)$, $n \in M(f_0+g_0)$, determined by formulas (14) be bounded. The spectral density $g_0(\lambda) \in D_v^u$ is the least favorable* *in the class D_v^u for the optimal linear estimation of the functional $A\xi$, if its components satisfy equation*

$$g_0^{(n)}(\lambda) = \max\left\{v^{(n)}(\lambda), \min\left\{u^{(n)}(\lambda), \alpha_2^{(n)} \sum_{u,v=1}^{d_n} |C_{uv}^{(n)}(e^{i\lambda})|\right\}\right\},$$

condition (18), and densities $(f(\lambda), g_0(\lambda))$ determine solution to the extremum problem (8). The minimax spectral characteristic $h_0 = h(f, g_0)$ is calculated by formula (5).

Corollary 3. *Let the spectral density $g(\lambda) = \{g^{(n)}(\lambda), n = 0, 1\ldots\}$ be known, the spectral density $f_0(\lambda) = \{f_0^{(n)}(\lambda), n = 0, 1\ldots\}$ is from the class D_f^0 and let the functions $h_f^{(n)}\left(f_0^{(n)}, g_0^{(n)}\right)$, $n \in M(f_0 + g_0)$, determined by formulas (13) be bounded. The spectral density $f_0(\lambda) \in D_f^0$ is the least favorable in the class D_f^0 for the optimal linear estimation of the functional $A\xi$, if its components satisfy equation*

$$f_0^{(n)}(\lambda) = \max\left\{0, \alpha_1^{(n)} \sum_{u,v=1}^{d_n} |A_{uv}^{(n)}(e^{i\lambda}) g_0^{(n)}(\lambda) + C_{uv}^{(n)}(e^{i\lambda})| - g_0^{(n)}(\lambda)\right\},$$

condition (17), and densities $(f_0(\lambda), g(\lambda))$ determine solution to the extremum problem (8). The minimax spectral characteristic $h_0 = h(f_0, g)$ is calculated by formula (5).

Consider the problem of the optimal linear estimate of the functional $A\xi$ from observations of the field $\xi(g, j)$ for $(g, j) \in G \times \{-1, -2, \ldots\}$. From the condition $0 \in \partial\Delta_D(f_0)$ for $D = D_f^0$ we find the following relations that determine the least favorable spectral density $f_0(\lambda) = \{f_0^{(n)}(\lambda), n = 0, 1\ldots\}$ from the class D_f^0:

$$f_0^{(n)}(\lambda) = \alpha_1^{(n)} \sum_{u,v=1}^{d_n} \left|\mathbf{A}_{uv}^{(n)} \mathbf{d}^{(n)}(\lambda)\right|^2. \tag{19}$$

To find the unknown $\alpha_1^{(n)}$, $d_0^{(n)}(j), n = 0, 1\ldots, j = 0, 1, 2, \ldots$ we use the factorization equations (1), extremum conditions (10), (11), and the condition

$$\|\mathbf{d}\|^2 = \sum_{n=0}^{\infty} d_n \sum_{j=0}^{\infty} |d_0^{(n)}(j)|^2 = P_1. \tag{20}$$

For all solutions $d^{(n)} = \{d^{(n)}(j), j = 0, 1\ldots\}$, $n = 0, 1, 2, \ldots$ of the system of equations

$$\mathbf{A}_{uv}^{(n)} \mathbf{d}^{(n)} = \mu_{uv}^{(n)} \overline{\mathbf{d}^{(n)}}, \quad 1 \le u, v \le d_n \tag{21}$$

the following equality holds true

$$\sum_{u,v=1}^{d_n}\left|\sum_{j=0}^{\infty}\left(\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right)(j)e^{ij\lambda}\right|^2=\alpha_1^{(n)}\left|\sum_{j=0}^{\infty}d^{(n)}(j)\,e^{-ij\lambda}\right|^2.$$

Denote by νP_1 the maximum value of

$$\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\left\|\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right\|^2=\sum_{n=0}^{\infty}\mu^{(n)}\left\|\mathbf{d}^{(n)}\right\|^2,\quad \mu^{(n)}=\sum_{u,v=1}^{d_n}\mu_{uv}^{(n)},$$

where $d^{(n)}=\{d^{(n)}(j),j=0,1\ldots\}$, $n=0,1,2,\ldots$ solutions to the system of equations (21) that satisfy condition (20).

Denote by ν^+P_1 the maximum value of

$$\sum_{n=0}^{\infty}\sum_{u,v=1}^{d_n}\left\|\mathbf{A}_{uv}^{(n)}\mathbf{d}^{(n)}\right\|^2$$

under the condition that $d^{(n)}=\{d^{(n)}(j),j=0,1\ldots\}$, $n=0,1,2,\ldots$, gives the canonical factorization (1) of the density (19) and satisfy condition (20).

If there exists a solution $d^{(n_0)}=\{d^{(n_0)}(j),j=0,1\ldots\}$ to the system of equations (21) for $n=n_0$ such that $d_{n_0}\left\|\mathbf{d}^{(n_0)}\right\|^2=P_1$ and $\nu=\nu^+$, then the spectral density $f_0(\lambda)=\{f_0^{(n)}(\lambda),n=0,1\ldots\}$ with components

$$f_0^{(n)}(\lambda)=\left|\sum_{j=0}^{\infty}d^{(n_0)}(j)\,e^{-ij\lambda}\right|^2\delta_n^{n_0},\quad n=0,1\ldots \tag{22}$$

of the moving average random field

$$\xi(g,j)=\sum_{u,v=1}^{d_{n_0}}\sum_{k=-\infty}^{j}d^{(n_0)}(j-k)\zeta_{uv}^{(n_0)}(k)T_{uv}^{(n_0)}(g), \tag{23}$$

where $\zeta_{uv}^{(n_0)}(k), 1\le u,v\le d_n$, are mutually uncorrelated stochastic sequences with orthonormal values (white noise), is the least favorable in the class D_f^0 for the optimal linear estimation of the functional $A\xi$.

The following statement holds true.

Theorem 3. *Spectral density $f_0(\lambda)=\{f_0^{(n)}(\lambda),n=0,1\ldots\}$ with components (22) of the moving average random field (23) is the least favorable in the class D_f^0 for the optimal linear estimation of the functional $A\xi$ if there exists a solution $d^{(n_0)}=\{d^{(n_0)}(j),j=0,1\ldots\}$ to the system of equations*

(21) for $n = n_0$ such that $d_{n_0} \left\| \mathbf{d}^{(n_0)} \right\|^2 = P_1$ and $\nu = \nu^+$. If $\nu < \nu^+$, then the least favorable in the class D_f^0 for the optimal linear estimation of the functional $A\xi$ is determined by conditions (1), (10), (19), (20). The minimax spectral characteristic $h_0 = h(f_0)$ is calculated by formula (7).

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