Theory of Stochastic Processes
Vol.13 (29), no.4, 2007, pp.1–18

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ANOTHER APPROACH TO THE PROBLEM OF THE RUIN PROBABILITY ESTIMATE FOR RISK PROCESS WITH INVESTMENTS

An exponential estimate of ruin probability for an insurance company which invests all its capital in risk assets is found. The process which describes the risky assets is assumed to follow a geometrical Brownian motion. Insurance premium flow depends on the value of reserves of the insurance company. The problem is solved by reduction of the generalized risk process to the classical risk process without investments.

1. Introduction

A generalized risk model for an insurance company which invests all its reserves into risky assets is considered. We let the value of the insurance premium flow depend on the current value of the insurer's capital.

The risky asset is assumed to follow a geometrical Brownian motion

$$dS_t = S_t(a\,dt + b\,dW_t), \quad S_0 > 0, \tag{1}$$

where S_0 denotes the initial value of the risky asset, $a > 0$, $b > 0$ are some fixed constants, and $\{W_t,\ t \geq 0\}$ is a standard Brownian motion.

Let $\sigma_c := \inf\{t \geq 0 :\ S_t \leq c\}$ denote a stopping time of the investing activity, where $c > 0$ is some fixed constant. Thus we let the insurance company terminate their risky asset investments if the price drops below c. We assume further that $S_0 > c$, otherwise investment problem has no sense.

Let us now consider a risk process described by the equation

$$R_t(u) = u - \sum_{k=1}^{N_t} U_k + \int_0^t p(R_s)\,ds + \int_0^t \frac{R_s I\{s \leq \sigma_c\}}{S_s}\,dS_s, \quad t \geq 0, \tag{2}$$

where

2000 Mathematics Subject Classifications 91B30

Key words and phrases. Ruin process, ruin probability, geometrical Brownian motion, supermartingale approach.

R_t is the value of insurer's capital at a moment of time $t \geq 0$;

$u > 0$ is the initial capital of the insurance company;

$\{N_t,\, t \geq 0\}$ is a Poisson process modeling the number of insurance claims, and it is assumed to have a constant intensity $\beta > 0$;

$\{T_k,\, k \geq 1\}$ are jump times of the Poisson process;

$\{U_k,\, k \geq 1\}$ is a sequence of i.i.d. positive random variables which models the claim sizes incurred by the insurer at times $\{T_k,\, k \geq 1\}$. In this paper we will only consider light tailed distributions, with existing moment generating functions (see below);

$p(R_s)$ is a premium rate process which depends on a value of current insurer's capital at time s.

We will assume that the sequence $\{U_k,\, k \geq 1\}$, the processes $\{N_t,\, t \geq 0\}$, and the standard Brownian motion $\{W_t,\, t \geq 0\}$ are all independent. Also, for $N_t = 0$ we put $\sum_{k=1}^{N_t} U_k = 0$.

Note that equation (2) can be rewritten as

$$R_t(u) = u - \sum_{k=1}^{N_t} U_k + \int_0^t p(R_s)\, ds + a \int_0^{t\wedge\sigma_c} R_s\, ds + b \int_0^{t\wedge\sigma_c} R_s\, dW_s, \quad t \geq 0. \tag{3}$$

Thus, our model is similar to the model described in paper [1]. An essential difference between the models is that the existence of the moment generating function for claim distribution is not demanded in [1]. On the other hand, independence of some processes is demanded in [1], but they are not necessarily independent in our model (3) (look section 4).

Comparisons of the results obtained for model (3) with known results for the classical risk model, and for the risk model with reinvestments in [1] are presented in section 3 and section 4.

2. Modification of Risk Process

Denote by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by a compound Poisson process $\{\sum_{k=1}^{N_t} U_k, t \geq 0\}$ and the process $\{W_t,\ t \geq 0, \}$. Let $E_t(\cdot)$ denote the conditional expectation $E(\cdot|\mathcal{F}_t)$.

Furthermore let us denote by $\tau(u) := \inf\{t \geq 0 :\ R_t(u) \leq 0\}$ the ruin time of the insurance company with an initial capital u. Put $\tau(u) = +\infty$ if $R_t > 0$ for all $t > 0$.

Let $h : \mathbf{R}_+ \to \mathbf{R}_+$ be the shifted moment generating function, defined as

$$h(r) = E[e^{rU_1}] - 1, \quad h(0) = 0.$$

We will use the classical assumption, as in [2], about existence of $r_\infty \in (0, +\infty]$ such that $h(r) < \infty$ on $[0, r_\infty)$, and $h(r) \to \infty$ as $r \uparrow r_\infty$. The function $h(r)$ is an increasing, convex, and continuous function on $[0, r_\infty)$. In other words, we will consider light tailed distributions of random variables to

Distribution, parameters	Probability density function f(x)	Shifted moment generating function h(r)	r_∞
Exponential $Exp(\lambda)$	$\lambda e^{-\lambda x}$	$\frac{r}{\lambda - r}$	λ
Gamma $\Gamma(\alpha,\,\lambda)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$	$\left(\frac{\lambda}{\lambda - r}\right)^\alpha - 1$	λ
Uniform $U(a,\,b)$	$\frac{1}{b-a},\ \ 0 \le a < x < b$	$\frac{e^{br} - e^{ar}}{r(b-a)} - 1$	$+\infty$

Table 1: Examples of shifted moment generating functions

describe claims arriving to the insurance company. Some examples of such distributions with corresponding moment generating functions are given in table 1.

Lemma 1. *Suppose that the following conditions (P1) and (P2) are satisfied:*

(**P1**) $p(\cdot) : \mathbf{R} \to \mathbf{R}_+$ *is a measurable bounded from above on* $\mathbf{R}$ *nonnegative function, i.e.* $\exists C > 0 :\ p(x) \le C,\ \ \forall x \ge 0$;

(**P2**) $p(\cdot)$ *is a Lipschitz continuous function, i.e.* $\exists K > 0$ *such that for all* $x, y \in \mathbf{R} : |p(x) - p(y)| \le K|x - y|$.

Then equation (3) has a unique, up to the stochastic equivalence, $\mathcal{F}_t$*-adapted solution. It can be written in following form*

$$R_t = S_{t\wedge\sigma_c}\left(\frac{u}{S_0} + \int_0^t S^{-1}_{t\wedge\sigma_c}\left(p(R_s)\,ds - U_s\,dZ_s\right)\right), \tag{4}$$

where $\sum_{k=1}^{N_t} U_k = \int_0^t U_s\,dZ_s$*, and* Z_s *is the jump measure of the Poisson process with intensity* β*.*

Proof. Let $Z^i_t,\ i = 1, 2$ be semimartingales such that $Z^i_0 = 0$ a.s., and H_t is an adapted *cádlág* process, i.e. which is right-hand continuous process and has left-hand limits. Then we use the vector form of the Theorem 14.6 [3], p.183. Assume the following conditions are satisfied:

1) $f_i(s, \omega, X_\cdot(\omega))$ are locally bounded predicted processes;

2)$\exists M_i > 0 : |f_i(s, \omega, X_\cdot(\omega)) - f_i(s, \omega, Y_\cdot(\omega))| \le M_i \sup_{0\le v\le s} |X_v(\omega) - Y_v(\omega)|,\ i = 1, 2$ for any *cádlág* adapted processes $X,\ Y$.

Then the stochastic differential equation

$$X_t(\omega) = H_t(\omega) + \int_0^t f_1(s, \omega, X_\cdot(\omega))\,dZ^1_s(\omega) + \int_0^t f_2(s, \omega, X_\cdot(\omega))\,dZ^2_s(\omega) \tag{5}$$

has a unique strong solution.

Let us consider the following modification of equation (3):

$$R_t(u) = u - \sum_{k=1}^{N_t} U_k + \int_0^t p(R_{s-})\, ds + a \int_0^{t\wedge\sigma_c} R_{s-}\, ds + b \int_0^{t\wedge\sigma_c} R_{s-}\, dW_s. \quad (6)$$

In terms of equation (5) we have that $X_t := R_t$; $Z_s^1 := s$, $Z_s^2 := W_s$; $f_1(s,\omega,X_s) := p(X_{s-}) + aX_{s-}$, $f_2(s,\omega,X_s) := bX_{s-}$.

Obviously, condition 2) of Theorem 14.6 [3] holds for the process (6) as a consequence of (P2). Besides, functions $f_i(s,\cdot,x)$, $i = 1,2$ are measurable for any fixed s and x; and also functions $f_i(\cdot,\omega,x)$, $i = 1,2$ are constant for any fixed ω and x. Then, using Lemma 14.14 [3], we obtain that the processes $f_i(s,\omega,X_{s-}(\omega))$, $i = 1,2$ are predicted and locally bounded. This yields the condition 1) of Theorem 14.6 [3] is fulfilled. Therefore, equation (6) has a unique strong solution.

By construction, the process R_t has a finite number of jumps on any trajectory, and the values of these jumps are also finite. Hence, a solution of equation (6) may be written in the form (3).

Equation (3) is a semilinear stochastic differential equation.

Solving the following linear stochastic differential equation

$$d\eta(t) = [\alpha(t) + \gamma(t)\eta(t)]\, dt + \delta(t)\eta(t)\, dW_t$$

we get

$$\eta(t) = e^{\int_0^t \left(\gamma(s) - \frac{\delta^2(s)}{2}\right) ds + \int_0^t \delta(s)\, dW_s} \times$$

$$\times \left(\eta(0) + \int_0^t e^{-\int_0^s \left(\gamma(v) - \frac{\delta^2(v)}{2}\right) dv - \int_0^s \delta(v)\, dW_v} \alpha(s)\, ds\right).$$

(See example 3 [4], p.37-38). Analogously, solution of equation (3) can be given in a form

$$R_t = e^{\int_0^t \left(a - \frac{b^2}{2}\right) I\{s \le \sigma_c\}\, ds + \int_0^t bI\{s\le\sigma_c\}\, dW_s} \times$$

$$\times \left(u + \int_0^t e^{-\int_0^s \left(a - \frac{b^2}{2}\right) I\{v\le\sigma_c\}\, dv - \int_0^s bI\{v\le\sigma_c\}\, dW_v} \left(p(R_s)\, ds - U_s\, dZ_s\right)\right),$$

or

$$R_t = e^{\left(a - \frac{b^2}{2}\right)\cdot(t\wedge\sigma_c) + b\cdot W_{t\wedge\sigma_c}} \times$$

$$\times \left(u + \int_0^t e^{-\left(a - \frac{b^2}{2}\right)\cdot(s\wedge\sigma_c) - b\cdot W_{s\wedge\sigma_c}} \left(p(R_s)\, ds - U_s\, dZ_s\right)\right) =$$

$$= S_{t\wedge\sigma_c}\left(\frac{u}{S_0} + \int_0^t S_{t\wedge\sigma_c}^{-1} \left(p(R_s)\, ds - U_s\, dZ_s\right)\right).$$

The last equality is obtained from the equation $\frac{S_t}{S_0} = e^{\left(a-\frac{b^2}{2}\right)t+bW_t}$, which follows from (1). Thus, we have obtained representation (4).

Validity of formula (4) can also be easily checked by simple substitution of the process R_t from (4) into equation (3). □

Now by using the representation of risk process (4), we can define the ruin time differently: $\tau(u) = \inf\{t \geq 0 | \, G_t(u) < 0\}$, where

$$G_t(u) = \frac{u}{S_0} + \int_0^t S^{-1}_{s\wedge\sigma_c} p(R_s)\, ds - \int_0^t S^{-1}_{s\wedge\sigma_c} U_s\, dZ_s. \qquad (7)$$

Thus we have reduced the original problem of ruin probability estimation for the risk model with investments to the problem of ruin probability estimation in the model without investments, but with another premium income process and claims process.

3. Ruin Probability Estimation

Theorem 1. *Let the risk model be described by equation (7), where R_t is a solution of equation (3). Assume that a premium income function $p(x)$ satisfies conditions (P1), (P2) of Lemma 1, and also that there exist $r \in [0, r_\infty)$, such that $\max\{\frac{r}{c}+1, \frac{2r}{c}\} < r_\infty$, that the following condition (P3) is satisfied:*

(**P3**) $p(x) \geq \beta\left(h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)\right), \quad \forall x \geq 0.$

Then the process $X_t(u,r) := e^{-rG_t(u)}$, $t \geq 0$ is an $\mathcal{F}_t$-supermartingale.

Proof. Note that the process $X_t(u,r) = e^{-rG_t(u)}$ is a semimartingale because the process G_t is semimartingale as a sum of a martingale and a process of a finite variation. (The process $\int_0^t S^{-1}_{s\wedge\sigma_c} U_s\, d(Z_s - EZ_s)$ is a martingale, and the processes $\int_0^t S^{-1}_{s\wedge\sigma_c} p(R_s)\, ds$, $\int_0^t S^{-1}_{s\wedge\sigma_c} U_s\, d(EZ_s) = \beta \int_0^t S^{-1}_{s\wedge\sigma_c} U_s\, ds$ are increasing processes, that is why they have a finite variation).

By Ito formula for semimartingale processes we have

$$F(Y_t) - F(Y_0) = \int_{0_+}^t F'(Y_{s-})\, dY_s + \frac{1}{2}\int_{0_+}^t F''(Y_{s-})\, d\langle Y, Y\rangle^c_s +$$
$$+ \sum_{0<s\leq t} \left(F(Y_s) - F(Y_{s-}) - F'(Y_{s-})(Y_s - Y_{s-})\right), \qquad (8)$$

where $\{Y_t,\, t \geq 0\}$ is a semimartingale, and $F \in \mathcal{C}^2(\mathbf{R})$ (see [5], p.78-79).

If $Y_t(u,r) := -rG_t(u) = -r\left(\frac{u}{S_0} + \int_0^t S^{-1}_{s\wedge\sigma_c} p(R_s)\, ds - \int_0^t S^{-1}_{s\wedge\sigma_c} U_s\, dZ_s\right)$, and $F(Y_t) = e^{Y_t} = F'(Y_t) = F''(Y_t)$, then the formula (8) can be rewritten as

$$e^{Y_t} = e^{Y_0} + \int_{0_+}^t e^{Y_{s-}}\, dY_s + \frac{1}{2}\int_{0_+}^t e^{Y_{s-}}\, d\langle Y, Y\rangle^c_s +$$

$$+\sum_{0<s\leq t}\left(e^{Y_s}-e^{Y_{s-}}-e^{Y_{s-}}(Y_s-Y_{s-})\right), \tag{9}$$

if all integrals on the right-hand side of the equality (9) exist.

$$Y_{t-}(u,r)=-r\left(\frac{u}{S_0}+\int_0^t S^{-1}_{s\wedge\sigma_c}p(R_s)\,ds-\int_0^{t-} S^{-1}_{s\wedge\sigma_c}U_s\,dZ_s\right), \tag{10}$$

$$dY_s=-rS^{-1}_{s\wedge\sigma_c}p(R_s)\,ds+rS^{-1}_{s\wedge\sigma_c}U_s\,dZ_s, \tag{11}$$

$$Y_s-Y_{s-}=rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}, \tag{12}$$

$$e^{Y_s}-e^{Y_{s-}}=e^{Y_{s-}}\left(e^{rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}}-1\right), \tag{13}$$

$$d\langle Y,Y\rangle^c_s=0. \tag{14}$$

Thus, in view of (10)-(14), we can formally rewrite formula (9) as

$$e^{Y_t}=e^{-r\frac{u}{S_0}}-r\int_{0_+}^t e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}p(R_s)\,ds+r\int_{0_+}^t e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}U_s\,dZ_s+$$

$$+\sum_{0<s\leq t}e^{Y_{s-}}\left(e^{rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}}-1-rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}\right). \tag{15}$$

The summands $r\int_{0_+}^t e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}U_s\,dZ_s$ and $-r\sum_{0<s\leq t}e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}$ in (9) cancel each other.

Consequently, (15) can be rewritten as

$$e^{Y_t}=e^{-r\frac{u}{S_0}}-r\int_{0_+}^t e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}p(R_s)\,ds+\sum_{0<s\leq t}e^{Y_{s-}}\left(e^{rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}}-1\right).$$

The process $X_t=e^{Y_t}$ is a supermartingale if it is integrable, and the inequality

$$E_t(X_T-X_t)\leq 0,\quad \forall T>t\geq 0 \tag{16}$$

holds true almost surely.

Let us prove the integrability:

$$E|X_t|\leq e^{-r\frac{u}{S_0}}+rE\left|\int_{0_+}^t e^{Y_{s-}}S^{-1}_{s\wedge\sigma_c}p(R_s)\,ds\right|+$$

$$+E\left|\sum_{0<s\leq t}e^{Y_{s-}}\left(e^{rS^{-1}_{s\wedge\sigma_c}U_{N_s}I\{\triangle N_s\neq 0\}}-1\right)\right|\leq$$

$$\leq e^{-r\frac{u}{S_0}}+\frac{r}{c}E\left|\int_{0_+}^t e^{-rG_{s-}}p(R_s)\,ds\right|+E\left|\sum_{k=1}^{N_t}e^{-rG_{T_k-}}\cdot e^{\frac{r}{c}U_k}\right|.$$

Here we used the definition of the stopping moment σ_c. According to it $S^{-1}_{s\wedge\sigma_c} \le 1/c$. Further, we find an estimate from above for the process $-G_t$:

$$-G_t = -\frac{u}{S_0} - \int_0^t S^{-1}_{s\wedge\sigma_c} p(R_s)\,ds + \int_0^t S^{-1}_{s\wedge\sigma_c} U_s\,dZ_s \le \int_0^t \frac{U_s}{c}\,dZ_s = \frac{1}{c}\sum_{k=1}^{N_t} U_k.$$

Thus,

$$E|X_t| \le e^{-r\frac{u}{S_0}} + \frac{r}{c}E\left|\int_{0_+}^t e^{\frac{r}{c}\sum_{k=1}^{N_{s-}} U_k} p(R_s)\,ds\right| + E\left|\sum_{k=1}^{N_t} e^{\frac{r}{c}\sum_{m=1}^{k} U_m}\cdot e^{\frac{r}{c}U_k}\right| \le$$

$$\le e^{-r\frac{u}{S_0}} + \frac{r}{c}E\left|\int_{0_+}^t e^{\frac{r}{c}\sum_{k=1}^{N_{s-}} U_k} p(R_s)\,ds\right| + E\left|\sum_{k=1}^{N_t} e^{\frac{2r}{c}\sum_{m=1}^{k} U_m}\right|.$$

Here we estimate the second and the third summands.

$$E\left|\int_{0_+}^t e^{\frac{r}{c}\sum_{k=1}^{N_{s-}} U_k} p(R_s)\,ds\right| \le C\int_{0_+}^t E\, e^{\frac{r}{c}\sum_{k=1}^{N_{s-}} U_k}\,ds =$$

$$= C\int_{0_+}^t \left(\sum_{K=0}^{+\infty} e^{\frac{r}{c}\sum_{k=1}^{K} U_k}\cdot P\{N_s = K\}\right) ds =$$

$$= C\int_{0_+}^t \left(\sum_{K=0}^{+\infty} \left(h\left(\frac{r}{c}\right)+1\right)^K \cdot \frac{(\beta s)^K}{K!}e^{-\beta s}\right) ds =$$

$$= C\int_{0_+}^t \left(e^{\left(h\left(\frac{r}{c}\right)+1\right)\cdot\beta s}\cdot e^{-\beta s}\right) ds = C\int_{0_+}^t e^{h\left(\frac{r}{c}\right)\cdot\beta s}\,ds < \infty.$$

Further,

$$E\left|\sum_{k=1}^{N_t} e^{\frac{2r}{c}\sum_{m=1}^{k} U_m}\right| \le$$

$$\le \sum_{K=1}^{+\infty}\left|E\left(e^{\frac{2r}{c}U_1} + e^{\frac{2r}{c}(U_1+U_2)} + \dots + e^{\frac{2r}{c}(U_1+U_2+\dots+U_K)}\right)\right|\cdot P\{N_t = K\} =$$

$$= \sum_{K=1}^{+\infty}\left(\left(h\left(\frac{2r}{c}\right)+1\right) + \left(h\left(\frac{2r}{c}\right)+1\right)^2 + \dots + \left(h\left(\frac{2r}{c}\right)+1\right)^K\right)\times$$

$$\times\frac{(\beta t)^K}{K!}e^{-\beta t} = \sum_{K=1}^{+\infty}\left(h\left(\frac{2r}{c}\right)+1\right)\cdot\frac{\left(h\left(\frac{2r}{c}\right)+1\right)^K - 1}{h\left(\frac{2r}{c}\right)}\cdot\frac{(\beta t)^K}{K!}e^{-\beta t} =$$

$$= \frac{h\left(\frac{2r}{c}\right)+1}{h\left(\frac{2r}{c}\right)}\cdot e^{-\beta t}\cdot\left(\sum_{K=1}^{+\infty}\frac{\left(\left(h\left(\frac{2r}{c}\right)+1\right)\beta t\right)^K}{K!} - \sum_{K=1}^{+\infty}\frac{(\beta t)^K}{K!}\right) =$$

$$= \frac{h\left(\frac{2r}{c}\right)+1}{h\left(\frac{2r}{c}\right)} \cdot e^{-\beta t} \cdot \left(e^{\left(h\left(\frac{2r}{c}\right)+1\right)\beta t} - e^{\beta t}\right) = \frac{h\left(\frac{2r}{c}\right)+1}{h\left(\frac{2r}{c}\right)} \cdot \left(e^{h\left(\frac{2r}{c}\right)\cdot\beta t} - 1\right) < \infty.$$

Thus we have show that $E|X_t| < \infty$.

The left-hand side of inequality (16) can be written as

$$E_t(X_T - X_t) = E_t\left(-r\int_{t_+}^{T} e^{Y_{s-}}\frac{p(R_s)}{S_{s\wedge\sigma_c}}\,ds + \right.$$

$$\left. + \sum_{t<s\le T} e^{Y_{s-}}\left(e^{r\frac{U_{N_s}}{S_{s\wedge\sigma_c}}I\{\triangle N_s\neq 0\}} - 1\right)\right). \tag{17}$$

According to the mean value theorem we have

$$e^{r\frac{U_{N_s}}{S_{s\wedge\sigma_c}}I\{\triangle N_s\neq 0\}} - 1 \le e^{r\frac{U_{N_s}}{S_{s\wedge\sigma_c}}} \cdot r\frac{U_{N_s}}{S_{s\wedge\sigma_c}} \cdot I\{\triangle N_s \neq 0\}.$$

Hence inequality (16) holds true if

$$E_t\left(\sum_{t<s\le T} e^{Y_{s-}} \cdot e^{r\frac{U_{N_s}}{S_{s\wedge\sigma_c}}} \cdot \frac{U_{N_s}}{S_{s\wedge\sigma_c}} \cdot I\{\triangle N_s \neq 0\}\right) -$$

$$-E_t\left(\int_{t_+}^{T} e^{Y_{s-}}\frac{p(R_s)}{S_{s\wedge\sigma_c}}\,ds\right) \le 0.$$

Using the fact that $S^{-1}_{s\wedge\sigma_c} \le 1/c$, we see that inequality (16) holds if the following inequality holds true

$$E_t\left(\sum_{t<s\le T} e^{Y_{s-}}e^{r\frac{U_{N_s}}{c}}\frac{U_{N_s}}{S_{s\wedge\sigma_c}} \cdot I\{\triangle N_s \neq 0\}\right) \le E_t\left(\int_{t_+}^{T} e^{Y_{s-}} \cdot \frac{p(R_s)}{S_{s\wedge\sigma_c}}\,ds\right).$$

Now we transfer the right-hand term of the last inequality to the left, and estimate the result from above.

$$E_t\left(\sum_{t<s\le T}\frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \cdot e^{r\frac{U_{N_s}}{c}} \cdot U_{N_s} \cdot I\{\triangle N_s \neq 0\}\right) - E_t\left(\int_{t_+}^{T}\frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \cdot p(R_s)\,ds\right) \le$$

$$\le E_t\left(-\int_{t_+}^{T}\frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}}p(R_s)\,ds + \sum_{t<s\le T}\frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}}e^{\frac{r}{c}U_{N_s}}\left(e^{U_{N_s}}-1\right)I\{\triangle N_s \neq 0\}\right) \tag{18}$$

To get inequality (18) we used the fact that every $x \in \mathbf{R}$ satisfies inequality $x+1 \le e^x$. For the validity of (17) it is enough to prove that the right-hand side of inequality (18) is not positive. The process

$$P_t := \sum_{0<s\le t}\frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}}\left(e^{\left(\frac{r}{c}+1\right)U_{N_s}} - e^{\frac{r}{c}U_{N_s}}\right) \cdot I\{\triangle N_s \neq 0\}$$

is nondecreasing process almost surely. Write it in the integral form:

$$P_t = \int_0^t \frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \, dO_s,$$

$O_t := \sum_{0<s\leq t} \left(e^{\left(\frac{r}{c}+1\right)U_{N_s}} - e^{\frac{r}{c}U_{N_s}}\right) \cdot I\{\triangle N_s \neq 0\}$.

According to Wald identity ([6], p. 32, we can use it as random variables $\{U_k\}_{k\geq 1}$ and process $\{N_t\}_{t\geq 0}$ are independent), and as a consequence the definition of function $h(r)$, we have

$$EO_t = E\sum_{k=1}^{N_t} \left(e^{\left(\frac{r}{c}+1\right)U_k} - e^{\frac{r}{c}U_k}\right) = \beta t \left(h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)\right).$$

The process $\{O_t\}_{t\geq 0}$ is a process with independent increments, therefore the corresponding compensated process $\{O_t - EO_t\}_{t\geq 0}$ is a martingale.

We have proved above that the process $X_t = e^{Y_t}$ is integrable. Hence the process $\frac{e^{Y_{t-}}}{S_{t\wedge\sigma_c}}$ is integrable too, as $E|\frac{e^{Y_{t-}}}{S_{t\wedge\sigma_c}}| \leq E|\frac{X_{t-}}{c}| < \infty$.

Consequently, the process

$$\int_0^t \frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \, d(O_s - EO_s) = P_t - \beta\left(h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)\right)\int_0^t \frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \, ds$$

is a martingale.

Therefore, it is obvious that the right-hand side of inequality (18) is not positive if

$$E_t\left(-\int_{t_+}^T \frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \cdot p(R_s)\, ds + \beta\left(h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)\right)\int_{t_+}^T \frac{e^{Y_{s-}}}{S_{s\wedge\sigma_c}} \, ds\right) \leq 0. \tag{19}$$

It follows from condition (P3) that inequality (19) holds true.

Thus the process $\{X_t\}_{t\geq 0}$ is a nonnegative $\mathcal{F}_t$-supermartingale. □

Theorem 2. *Let conditions (P1) and (P2) hold true for the risk process with investments described by equation (2). If the equation*

$$\inf_{x\geq 0} p(x) = \beta\left(h\left(\frac{\widehat{r}}{c}+1\right) - h\left(\frac{\widehat{r}}{c}\right)\right). \tag{20}$$

has a solution $\widehat{r}$ *which satisfies* $0 < \max\{\frac{\widehat{r}}{c}+1, \frac{2\widehat{r}}{c}\} < r_\infty$, *then the ruin probability can be bounded from above by*

$$\mathbf{P}\{\tau(u) < \infty\} \leq e^{-\widehat{r}\frac{u}{S_0}}. \tag{21}$$

for all $u \in \mathbf{R}_+$.

Proof. Evidently, for the constant $\widehat{r}$ defined in (20) condition (P3) of Theorem 1 holds true. Then, as all conditions of Theorem 1 hold true, the process $\{G_t\}_{t\geq 0}$ is a supermartingale with respect to the flow $\mathcal{F}_t$.

The process $\{G_{t\wedge\tau(u)}\}_{t\geq 0}$ is a supermartingale as well (it can be shown easily by replacing t by $t\wedge\tau(u)$ in the proof of Theorem 1).

Then a ruin probability estimate of an insurance company can be found in ordinary way.

$$e^{-\widehat{r}\frac{u}{S_0}} = X_0(u,\widehat{r}) \geq EX_{t\wedge\tau(u)}(u,\widehat{r}) =$$

$$= X_{\tau(u)}(u,\widehat{r})\cdot I_{\tau(u)<t} + X_t(u,\widehat{r})\cdot I_{\tau(u)\geq t} \geq X_{\tau(u)}(u,\widehat{r})\cdot I_{\tau(u)<t}.$$

$$\lim_{t\to\infty} EX_{\tau(u)}(u,\widehat{r})\cdot I_{\tau(u)<t} = EX_{\tau(u)}(u,\widehat{r})\cdot I_{\tau(u)<\infty}.$$

Hence

$$e^{-\widehat{r}\frac{u}{S_0}} \geq E(X_{\tau(u)}(u,\widehat{r})\,/\,\tau(u)<\infty)\cdot \mathbf{P}\{\tau(u)<\infty\}$$

$$\Rightarrow\ \mathbf{P}\{\tau(u)<\infty\} \leq \frac{e^{-\widehat{r}\frac{u}{S_0}}}{EX_{\tau(u)}(u,\widehat{r})} \leq e^{-\widehat{r}\frac{u}{S_0}}.$$

The last inequality holds true as a result of the fact that $G_{\tau(u)}<0$, and $X_{\tau(u)}>1$. □

Remark 1. Due to the estimations made by means of the mean value theorem and the inequality $x+1\leq e^x$ in the proof of Theorem 1, we got equation (20) for an adjustment coefficient $\widehat{r}$, which is of a "non-classical type".

Let us remind of the classical results in risk theory (see for example [2], p. 11). If the risk process is described by equation

$$R_t(u) = u + c_1 t - \sum_{k=1}^{N_t} U_k, \quad t\geq 0,$$

where $c_1>0$ is a uniform process of insurance premiums income, then in the case of positive safety loading $\rho := c_1 - \beta EU_1 > 0$ we can estimate ruin probability as

$$\mathbf{P}\{\tau(u)<\infty\} \leq e^{-Ru}, \tag{22}$$

where R is a unique positive solution of the equation

$$\beta h(R) = c_1 R. \tag{23}$$

If we set in the model (3) $S_0 := 1$ and $c_1 := \inf_{x\geq 0} p(x)$ then Theorem 2 will give us ruin probability estimation

$$\mathbf{P}\{\tau(u)<\infty\} \leq e^{-\widehat{r}u}, \tag{24}$$

where $\widehat{r}$ is a solution of equation

$$c_1 = \beta \left(h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)\right). \tag{25}$$

We are interested if the estimation obtained in (24) can be better than the estimation (22), that is whether $\widehat{r} \geq R$.

According to (23), constant R is a solution of the equation $\frac{c_1}{\beta} = \frac{h(R)}{R}$; according to (25) constant $\widehat{r}$ is a solution of $\frac{c_1}{\beta} = h\left(\frac{\widehat{r}}{c}+1\right) - h\left(\frac{\widehat{r}}{c}\right)$.

Let us consider the functions $f_1(r) := \frac{h(r)}{r}$ and $f_2(r) := h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right)$, and study their interrelation.

Since $f_1(0) = EU_1$, $f_2(0) = Ee^{U_1} - 1$, we have that $f_2(0) > f_1(0)$.

Further, $f_1'(r) = \frac{h'(r)r - h(r)}{r^2}$, $f_2'(r) = \frac{1}{c}\left(h'\left(\frac{r}{c}+1\right) - h'\left(\frac{r}{c}\right)\right)$.

Note that $h'(r) = EU_1 e^{rU_1}$, $h''(r) = EU_1^2 e^{rU_1}$, and these derivatives exist at some neighborhood of r for $r < r_\infty$.

With the help of Taylor formula we have

$$h(0) = h(r) - h'(r)r + \frac{1}{2}h''(\theta_1)r^2. \tag{26}$$

Substitution of $h(0) = 0$ in formula (26) gives us

$$f_1'(r) = \frac{h'(r)r - h(r)}{r^2} = \frac{1}{2}h''(\theta_1),$$

where $\theta_1 \in (0, r)$.

On the other hand, according to the mean value theorem

$$f_2'(r) = h\left(\frac{r}{c}+1\right) - h\left(\frac{r}{c}\right) = h''(\theta_2),$$

where $\theta_2 \in \left(\frac{r}{c}, \frac{r}{c}+1\right)$.

Evidently $h''(r) = EU_1^2 e^{rU_1}$ is an increasing function of r. Since $c < R_0 = 1$ we have $\frac{r}{c} > r$ which means that $\theta_1 < \theta_2$. This way we proved that

$$f_1'(r) = \frac{1}{2}h''(\theta_1) \leq h''(\theta_2) = f_2'(r).$$

Thus, the function $f_2(r)$ crosses Y-axis higher then $f_1(r)$, and has faster growth than the function $f_1(r)$. This means that the function $f_2(r)$ will reach level $\frac{c_1}{\beta}$ earlier, and hence $\widehat{r} < R$. Consequently, we can not get as good ruin probability estimate by means of inequality (21) as the Cramer-Lundberg estimate gives.

Remark 2. Let us consider a modification of the risk model with investments by changing the stopping time of the investment activity. Let an insurance company invests into risky assets only when the price of the assets is lying in the range of $c_* < S_t < c^*$. Assume that S_0 belongs to this range. The price of the risky assets is modelled by a geometrical Brownian motion (1).

Denote by $\widetilde{\sigma}_c := \inf\{t \geq 0 : \ S_t \notin (c_*, c^*)\}$ the stopping time, where $0 < c_* < c^*$ are some constants. So an insurer will terminate his investments into the risk asset if the asset price drops below c_* or rises above c^*.

Then, by analogy to Lemma 1, if conditions (P1) and (P2) hold true the equation

$$R_t(u) = u - \sum_{k=1}^{N_t} U_k + \int_0^t p(R_s)\,ds + a\int_0^{t\wedge\widetilde{\sigma}_c} R_s\,ds + b\int_0^{t\wedge\widetilde{\sigma}_c} R_s\,dW_s, \quad t \geq 0 \tag{27}$$

has a unique, accurate within stochastic equivalence, $\mathcal{F}_t$-measured solution, which can be represented as

$$R_t = S_{t\wedge\widetilde{\sigma}_c}\left(\frac{u}{S_0} + \int_0^t S^{-1}_{t\wedge\widetilde{\sigma}_c}\left(p(R_s)\,ds - U_s\,dZ_s\right)\right),$$

where $\sum_{k=1}^{N_t} U_k = \int_0^t U_s\,dZ_s$, Z_s is a measure of jumps of a Poisson process with intensity β.

Then we can reformulate Theorem 1.

Theorem 1*. *Let the risk model be described by an equation*

$$\widetilde{G}_t(u) = \frac{u}{S_0} + \int_0^t S^{-1}_{s\wedge\widetilde{\sigma}_c} p(R_s)\,ds - \int_0^t S^{-1}_{s\wedge\widetilde{\sigma}_c} U_s\,dZ_s,$$

where R_t is risk a process which is a solution of the equation (27). If for some $r \in [0, r_\infty)$ such that $\frac{r}{c_*} < r_\infty$, and premium income function $p(x)$ conditions (P1), (P2) and (P3) hold true,*

(P3)* $\frac{r}{c^*}p(x) \geq \beta h\left(\frac{r}{c_*}\right), \quad \forall x \geq 0,$

then the process $\widetilde{X}_t(u,r) := e^{-r\widetilde{G}_t(u)}$, $t \geq 0$ is $\mathcal{F}_t$-supermartingale.

The proof of the Theorem is similar with the one of Theorem 1. In the same way we show that $E|\widetilde{X}_t| < \infty$.

Denote $\widetilde{Y}_t(u,r) := -r\widetilde{G}_t(u)$, and find an estimation for the expectation of exponential process increment: $E_t(\widetilde{X}_T - \widetilde{X}_t)$.

$$E_t(\widetilde{X}_T - \widetilde{X}_t) =$$

$$= E_t\left(-r\int_{t_+}^T e^{\widetilde{Y}_{s-}}\frac{p(R_s)}{S_{s\wedge\widetilde{\sigma}_c}}\,ds + \sum_{t<s\leq T} e^{\widetilde{Y}_{s-}}\left(e^{r\frac{U_{N_s}}{S_{s\wedge\widetilde{\sigma}_c}}I\{\triangle N_s\neq 0\}} - 1\right)\right) \leq$$

$$\leq E_t \left(-r \int\limits_{t_+}^{T} e^{\widetilde{Y}_{s-}} \frac{p(R_s)}{c^*}\, ds + \sum_{t<s\leq T} e^{\widetilde{Y}_{s-}} \left(e^{r\frac{U_{N_s}}{c_*}} - 1 \right) I\{\triangle N_s \neq 0\} \right). \quad (28)$$

The process $\widetilde{P}_t := \sum_{0<s\leq t} e^{\widetilde{Y}_{s-}} \left(e^{r\frac{U_{N_s}}{c_*}} - 1 \right) I\{\triangle N_s \neq 0\}$ is almost surely nondecreasing. It can be presented in an integral form

$$\widetilde{P}_t = \int_0^t e^{\widetilde{Y}_{s-}}\, d\widetilde{O}_s,$$

where $\widetilde{O}_t := \sum_{0<s\leq t} \left(e^{r\frac{U_{N_s}}{c_*}} - 1 \right) I\{\triangle N_s \neq 0\}$.

According to Wald identity, we have

$$E\widetilde{O}_t = E \sum_{k=1}^{N_t} \left(e^{\frac{r}{c_*}U_k} - 1 \right) = \beta t \cdot h\left(\frac{r}{c_*}\right).$$

The process $\{\widetilde{O}_t\}_{t\geq 0}$ is a process with independent increments. Thus the corresponding compensated process $\{\widetilde{O}_t - E\widetilde{O}_t\}_{t\geq 0}$ is a martingale. Then the process

$$\int_0^t e^{\widetilde{Y}_{s-}}\, d(\widetilde{O}_s - E\widetilde{O}_s) = \widetilde{P}_t - \beta h\left(\frac{r}{c_*}\right) \int_0^t e^{\widetilde{Y}_{s-}}\, ds$$

is a martingale too.

Hence the right-hand side of inequality (28) is not positive if

$$E_t \left(-\frac{r}{c^*} \int_{t_+}^{T} e^{\widetilde{Y}_{s-}} p(R_s)\, ds + \beta h\left(\frac{r}{c_*}\right) \int_{t_+}^{T} e^{\widetilde{Y}_{s-}}\, ds \right) \leq 0. \quad (29)$$

The inequality (29) holds true as a result of condition (P3)*. This proves that the process $\{\widetilde{X}_t\}_{t\geq 0}$ is a nonnegative $\mathcal{F}_t$-supermartingale. □

Theorem 2 can be reformulated then in the following way.

Theorem 2*. *Let conditions (P1) and (P2) hold true for the risk process described by equation (27). If the equation*

$$\frac{\widehat{R}}{c^*} \inf_{x\geq 0} p(x) = \beta h\left(\frac{\widehat{R}}{c_*}\right)$$

has a solution $\widehat{R} > 0$, *then for the risk model described, the ruin probability can be bounded from above by*

$$\mathbf{P}\{\tau(u) < \infty\} \leq e^{-\widehat{R}\frac{u}{S_0}}.$$

The proof of the Theorem repeats completely the proof of Theorem 2.

Thus we can easily see that by reducing risk model (27) to the classical risk model putting $p(x) := c_1$, $\forall x \geq 0$; $S_t = c_* = c^* := 1$, $\forall t \geq 0$, we will receive classical ruin probability estimate in Theorem 2*:

$$\mathbf{P}\{\tau(u) < \infty\} \leq e^{-\widehat{R}u},$$

where $\widehat{R}$ is a solution of the equation $\widehat{R}c_1 = \beta h(\widehat{R})$.

4. Analysis of Results in the Case of Exponential Claims Distribution

Suppose that the distribution of the random variables modeling claim sizes is exponential, i.e. $\{U_k,\, k \geq 1\} \sim Exp(\lambda)$, and $h(r) = \frac{r}{\lambda - r}$, $r_\infty = \lambda$.

Let us denote $\widetilde{c}_1 := \inf_{x \geq 0} p(x)$. Then the equation (20) can be written as

$$\widetilde{c}_1 = \beta \left(\frac{\frac{\widehat{r}}{c} + 1}{\lambda - \left(\frac{\widehat{r}}{c} + 1\right)} - \frac{\frac{\widehat{r}}{c}}{\lambda - \frac{\widehat{r}}{c}} \right) = \frac{\beta\lambda}{\left(\lambda - \frac{\widehat{r}}{c} - 1\right)\left(\lambda - \frac{\widehat{r}}{c}\right)},$$

whence

$$\widehat{r}^2 + c(1 - 2\lambda)\widehat{r} + \lambda c^2 \left(\lambda - 1 - \frac{\beta}{\widetilde{c}_1}\right) = 0. \tag{30}$$

Solving equation (30) gives us

$$D = c^2 \left(1 + \frac{4\lambda\beta}{\widetilde{c}_1}\right) > 0$$

$$\widehat{r}_{1,2} = \frac{c}{2} \left(-1 + 2\lambda \pm \sqrt{1 + \frac{4\lambda\beta}{\widetilde{c}_1}}\right).$$

We are interested in solutions of equation (30), which satisfy the condition

$$0 < \max\left\{\frac{\widehat{r}}{c} + 1,\, \frac{2\widehat{r}}{c}\right\} < r_\infty. \tag{31}$$

of the Theorem 2.

As $\frac{\widehat{r}_2}{c} + 1 = \lambda + \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\lambda\beta}{\widetilde{c}_1}} > \lambda$, then the larger solution does not satisfy (31).

Let us find sufficient conditions for characteristic λ, c, $\widetilde{c}_1$ such that the solution $\widehat{r}_1 = \frac{c}{2}\left(-1 + 2\lambda - \sqrt{1 + \frac{4\lambda\beta}{\widetilde{c}_1}}\right)$ satisfies conditions (31).

Write a minimum capital income into insurance company per unit of time $\widetilde{c}_1$ as a function of average payoffs

$$\widetilde{c}_1 := (1 + K) \cdot \beta E U_1 = (1 + K) \cdot \frac{\beta}{\lambda},$$

where K is some constant. (The necessary condition of existence of non-trivial ruin probability estimation in the classical risk model is positiveness of a safety loading $K > 0$).

Inequalities (31) for $\widehat{r}_1$ can be rewritten as

$$0 < \max\left\{\frac{1}{2} + \lambda - \frac{1}{2}\sqrt{1 + \frac{4\lambda^2}{1+K}},\ -1 + 2\lambda - \sqrt{1 + \frac{4\lambda^2}{1+K}}\right\} < \lambda$$

or as a system of inequalities

$$\begin{cases} \sqrt{1 + \frac{4\lambda^2}{1+K}} < 2\lambda + 1, \\ \sqrt{1 + \frac{4\lambda^2}{1+K}} < 2\lambda - 1, \\ 1 < \sqrt{1 + \frac{4\lambda^2}{1+K}}, \\ \lambda - 1 < \sqrt{1 + \frac{4\lambda^2}{1+K}}. \end{cases} \tag{32}$$

The first and the third inequalities in the system (32) are trivial. From the second inequality we get that $\lambda > 1$, and $K > \frac{1}{\lambda-1}$. The consequence of the forth inequality is if $\lambda > 2$ then we need one more restriction for K: $K < \frac{3\lambda+2}{\lambda-2}$. Note that the last superior bound for K is adjusted with inferior one written above, as inequality $\frac{1}{\lambda-1} < \frac{3\lambda+2}{\lambda-2}$ holds true for all $\lambda > 2$.

Thus Theorem 2 provides us by ruin probability estimate $\mathbf{P}\{\tau(u) < \infty\} \leq e^{-\widehat{r}\frac{u}{S_0}}$, where $\widehat{r} = \frac{c}{2}\left(-1 + 2\lambda - \sqrt{1 + \frac{4\lambda\beta}{\widetilde{c}_1}}\right)$, if characteristic λ and K are in bounds

$$\left[\begin{array}{l} \lambda \in (1, 2],\ \frac{1}{\lambda-1} < K, \\ \lambda > 2,\ \frac{1}{\lambda-1} < K < \frac{3\lambda+2}{\lambda-2}. \end{array}\right. \tag{33}$$

1. Comparison with the classical results. Equation (23) for an adjustment coefficient in the case of claims distributed exponentially, i.e. when $h(r) = \frac{r}{\lambda-r}$ can be written as

$$\frac{\beta}{\lambda - R} = c_1. \tag{34}$$

To compare ruin probability estimates (21) and (22) we will assume that $p(x) := \widetilde{c}_1 = c_1 = (1 + K) \cdot \frac{\beta}{\lambda},\ \forall x \geq 0$, where K is some positive safety loading coeficient, $S_t := c = 1, \forall t \geq 0$.

Then we get R as a solution of equation (34)

$$R = \frac{\lambda K}{1 + K}. \tag{35}$$

Recall that

$$\widehat{r} = \lambda - \frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{4\lambda^2}{1 + K}}. \tag{36}$$

Example 1. For $\lambda = 5$ restrictions of system (33) for the value of safety loading K can be written as $K \in (0.25, 5.66667)$. Set $K := 0.3$. Then according to formulas (35) and (36), $R = 1.15385$, $\widehat{r} = 0.0862976$. In fact, as we see, in the case of exponential distribution of claims, adjustment coefficients do not depend on the frequency of claims arrivals β. However in order to find the value of premiums per unit of time and ruin probability estimates, we set $\beta := 20$, and $u := 50$.

Then we get we get $c_1 = 5.2$, and ruin probability estimates: $8.80135 \cdot 10^{-26}$ using classical formula of Cramer and Lundberg (22), and 0.0133682 using formula (21) from Theorem 2.

2. Comparison with the results of paper [1]. A model similar to (3) has been examined in the paper [1]

$$R_t = u - \sum_{k=1}^{N_t} U_k + \int_0^t p_s\, ds + a \int_0^t R_s\, ds + b \int_0^t R_s\, dW_s, \quad t \geq 0, \tag{37}$$

where

a, b are some positive constants,

p_s is a nonnegative integrable predicted process,

N_t is Poisson process with intensity β and moments of successive jumps $\{T_n,\, n \geq 1\}$,

U_k, $k \geq 1$ are i.i.d. random variables with probability distribution function F.

Processes W_t, N_t, and random variables $\{U_k,\, k \in \mathbf{N}\}$ are assumed to be independent. Denote $\theta_n := T_n - T_{n-1}$, $T_0 := 0$.

An assumption used in the paper [1] is

(**F**) The sequence (λ_n, η_n) is a sequence of two-dimensional i.i.d. random variables, where

$$\lambda_n := e^{bW^n_{\theta_n} + k\theta_n},$$

$$\eta_n := \int_0^{\theta_n} p_{v+T_{n-1}} e^{b(W^n_{\theta_n} - W^n_v) + k(\theta_n - v)}\, dv,$$

$$k = a - \frac{b^2}{2},\ W^n_t = W_{t+T_{n-1}} - W_{T_{n-1}}.$$

Remark 3. The model (3) does not require anything similar to the condition (F). Moreover, such condition would not be true for the model (3). To show this rewrite λ_n and η_n as

$$\lambda_n := e^{b(W_{T_n} - W_{T_{n-1}}) + k(T_n - T_{n-1})},$$

$$\eta_n := \int_{T_{n-1}}^{T_n} p_v e^{b(W_{T_n} - W_v) + k(T_n - v)}\, dv.$$

Whereas in the model (3) we have $p_v = p(R_v)$, p_v depends on realization of the process $\{W_t,\, t \in [T_{n-1},\, v]\}$, as well as λ_n. That is why λ_n and η_n are not necessarily independent.

The main result of the paper [1] is

Theorem 2.1 [1]: *For the risk model (37) in the case of exponential claim size distribution* $F(x) = 1 - e^{-\lambda x}$, $\lambda > 0$ *we can estimate the ruin probability in such a way*

(i) *if* $\varrho := \frac{2a}{b^2} > 1$, *then for some* $M > 0$

$$\mathbf{P}\{\tau(u) < \infty\} = M u^{1-\varrho}(1 + o(1)),\ u \to \infty;$$

(ii) *if* $\varrho < 1$ *then* $\mathbf{P}\{\tau(u) < \infty\} = 1,\ \forall\, u$.

According to the Theorem 2 of the present work, we got exponential ruin probability estimate $\mathbf{P}\{\tau(u) < \infty\} \le e^{-\widehat{r}\frac{u}{S_0}}$. In the case of exponentially distributed claims we had $\widehat{r} = \frac{c}{2}\left(-1 + 2\lambda - \sqrt{1 + \frac{4\lambda\beta}{\inf_{x\ge 0} p(x)}}\right)$. Thus as we got exponential estimate, it improves result of [1] as $u \to \infty$. Moreover, our result does not depend on the volatility of the underlying asset price (on the value of ϱ). Hence for $\varrho > 1$ we can get by means of Theorem 2 non-trivial ruin probability estimate.

5. Conclusions

We have considered a risk model with investment of all insurer's capital into the risky asset. The price of the asset is assumed to follow a geometrical Brownian motion. We let the insurance company invest all its capital into one type of risky assets, but if the price of the risky assets drops below some predetermined fixed level the company is assumed to stop their investing. For such model we have found exponential ruin probability estimate. It is turned out, that the estimate found do not improve the Cramer-Lundberg estimate. Hence we can conclude that, from the point of view of riskiness, investing all the capital into risky assets is worse than not investing at all for the insurance company.

On the other hand, the estimate obtained can be used for risk models which have variable premium income process depending on the current value of the company's capital. Classical estimates can not be used for such processes.

Besides, method introduced in this paper can give better results than estimates of paper [1], where similar problem have been considered. In the case of highly volatile assets inequality (21) will give us exponential estimate, while [1] can give only trivial estimate.

Note that by diversifying investments into risky and non-risky assets, we can get improved ruin probability estimates, sharper than the classical one, see, for example, [7] and [8]. This yields to the conjecture that if minimization of ruin probability is the criteria of insurer's investment behavior, then we should search for an optimal investment strategy in between diversified portfolios.

References

1. Frolova, A. G., Kabanov, Yu. M., Pergamenshchikov, S. M., *In the Insurance Business Risky Investments are Dangerous,* Finance and Stochastics **6**, no. 2, (2002), 227–235.
2. Grandell, J., *Aspects of Risk Theory,* Springer-Verlag, New York, (1991), 175 p.
3. Elliott, R., J., *Stochastic Calculus and Applications,* Springer, New York, (1982), 305 p.
4. Gikhman, I. I., Skorokhod, A. V., *Stochastic Differential Equations,* Naukova Dumka, Kiev, (1968), 354 p. (Rus.)
5. Protter, P. E., *Stochastic Integration and Differential Equations,* Springer, Berlin, (2004), 410 p.
6. Skorokhod, A. V., *Random Processes with Independent Increments,* Nauka, Moscow, (1986), 320 p. (Rus.)
7. Gaier, J., Grandits, P., Schachermayer, W., *Asymptotic Ruin Probabilities and Optimal Investment,* The Annals of Applied Probability, **13**, (2003), 1054–1076.
8. Androshchuk, M. O., Mishura, Yu. S., *An Estimate of Ruin Probability for an Insurance Company which is Functioning on BS-market,* Ukrainian Mathematical Journal, **11**, (2007) (Ukr.)

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