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ON ONE STOCHASTIC OPTIMAL CONTROL PROBLEM WITH VARIABLE DELAY

The purpose of this paper is to give necessary conditions for the optimality of non-linear stochastic control systems with variable delay and with constraint on the right end of a trajectory. The necessary optimality conditions in the form of a stochastic analogy of the maximum principle are obtained. These conditions are contained in Theorems 1 and 2.

Introduction

Stochastic differential equations with delay find many applications in authomatic control theory, in the theory of self-oscillating systems, etc., where real systems are subjected to the influence of random disturbances which cannot be ignored [1,2]. Optimal control problems for the systems described by means of such equations have been already investigated in [3]-[5]. This research is devoted to a problem of stochastic optimal control with delay both on control and state, when the cost function contains a variable delay as well.

Statement of the problem

Let (Ω, F, P) be a complete probability space with the filtration $\{F^t : t_0 \le t \le t_1\}$ generated by the Wiener process w_t and $F^t = \sigma(w_s; t_0 \le s \le t)$. $L^2_F(t_0, t_1, R^n)$ – space of predictable processes $x_t(\omega)$ such that: $E\int_{t_0}^{t_1} |x_t|^2 dt < +\infty$. Consider the following stochastic system with delay:

(1) $$dx_t = g(x_t, x_{t-h(t)}, u_t, u_{t-h_1(t)}, t)dt + \sigma(x_t, x_{t-h(t)}, t)dw_t, \quad t \in (t_0, t_1];$$

(2) $$x_t = \Phi(t), \ t \in [t_0 - h(t_0), t_0);$$

(3) $$x_{t_0} = x_0;$$

(4) $$u_t = Q(t), \ t \in [t_0 - h_1(t_0), t_0);$$

(5) $$u_t \in U_\partial \equiv \{u(\cdot,\cdot) \in L^2_F(t_0, t_1; R^m) | u(t,\cdot) \in U \subset R^m, \text{ a.s.}\}$$

where U – non-empty bounded set, $\Phi(t), Q(t)$ – piecewise continuous non-random functions, $h(t) \ge 0$ and $h_1(t) \ge 0$ – continuously differentiable non-random functions, and $\frac{dh(t)}{dt} < 1$, $\frac{dh_1(t)}{dt} < 1$.

It is required to minimize the following functional in a set of admissible controls:

(6) $$J(u) = E\left\{p(x_{t_1}) + \int_{t_0}^{t_1} l(x_t, x_{t-h(t)}, u_t, u_{t-h_1(t)}, t)dt\right\}$$

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under the condition

$$Eq(x_{t_1}) \in G \subset R^k, \tag{7}$$

where G – closed convex set in R^k.

Let assume that the following requirements are satisfied:

I. Functions $l, g,$ and σ are continuous in all arguments.

II. When (t, u) are fixed, then l, g, σ functions are continuously differentiable with respect to (x, y) and satisfy the condition of linear growth:

$$(1 + |x| + |y|)^{-1}(|g(x, y, u, v, t)| + |g_x(x, y, u, v, t)| +$$

$$+|g_y(x, y, u, v, t)| + |\sigma(x, y, t)| + |\sigma_x(x, y, t)| + |\sigma_y(x, y, t)|) \leq N$$

$$(1 + |x|)^{-1}(|l(x, y, u, v, t)| + |l_x(x, y, u, v, t)| + |l_y(x, y, u, v, t)|) \leq N.$$

III. Function $p(x) : R^n \to R^1$ is continuously differentiable, and $|p(x)| + |p_x(x)| \leq N(1 + |x|)$.

IV. Function $q(x) : R^m \to R^k$ is continuously differentiable, and $|q(x)| + |q_x(x)| \leq N(1 + |x|)$.

First, we consider the stochastic optimal control problem (1)-(6).

Problem without constraint

We obtained the following result that is a necessary condition of optimality for problem (1)-(6):

Theorem 1. *Let conditions I-III hold, and let (x_t^0, u_t^0) be a solution of problem (1)–(6). Let there exist the random processes $(\psi_t, \beta_t) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$, which are the solutions of the adjoint equation*

$$\begin{cases} d\psi_t = -[H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + H_y(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)|_{z=s(t)} s'(t)]dt + \\ +\beta_t dw_t, \ t_0 \leq t < t_1 - h(t_1), \\ d\psi_t = -H_x(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + \beta_t dw_t, \ t_1 - h(t_1) \leq t < t_1, \\ \psi_{t_1} = -p_x(x_{t_1}^0). \end{cases} \tag{8}$$

Then, $\forall\ \widetilde{u} \in U$ a.c., the following relations hold:

$$\begin{cases} H(\psi_t, x_t^0, y_t^0, u, v_t^0, t) - H(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) + \\ +[H(\psi_z, x_z^0, y_z^0, u_z^0, u, z) - H(\psi_z, x_z^0, y_z^0, u_z^0, v_z^0, z)]|_{z=r(t)} r'(t) \leq 0, \\ a.e.\ t \in [t_0, t_1 - h_1(t_1)), \\ H(\psi_t, x_t^0, y_t^0, u, v_t^0, t) - H(\psi_t, x_t^0, y_t^0, u_t^0, v_t^0, t) \leq 0, \ \ a.e.\ t \in [t_1 - h_1(t_1), t_1]. \end{cases} \tag{9}$$

Here, $\tau = s(\tau)$ is a solution of the equation $\tau = t - h(t)$, $\tau = r(\tau)$ is a solution of equation $\tau = t - h_1(t)$, $y_t = x_{t-h(t)}$, $v_t = u_{t-h_1(t)}$, and

$$H(\psi_t, x_t, y_t, u_t, v_t, t) = \psi_t^* g(x_t, y_t, u_t, v_t, t) + \beta_t^* \sigma(x_t, y_t, t) - l(x_t, y_t, u_t, v_t, t).$$

Proof. Let $\overline{u}_1 = u_t^0 + \Delta u_t$ be some admissible control, and let $\overline{x}_1 = x_t^0 + \Delta x_t$ be the trajectory of system (1)-(5) corresponding to this control. We use the identities

$$\begin{aligned} & d\Delta x_t = [g(\overline{x}_t, \overline{y}_t, \overline{u}_t, \overline{v}_t, t) - g(x_t^0, y_t^0, u_t^0, v_t^0, t)]dt + [\sigma(\overline{x}_t, \overline{y}_t, t) - \\ & - \sigma(x_t^0, y^0, t)]dw_t = \{\Delta_{\overline{u}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) + \Delta_{\overline{v}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) + \\ & + g_x(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta x_t + g_y(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta y_t\}dt + \\ & + \{\sigma_x(x_t^0, y_t^0, t)\Delta x_t + \sigma_y(x_t^0, y_t^0, t)\Delta y_t\}dw_t + \eta_t^1, \quad t \in (t_0, t_1] \\ & \Delta x_t = 0, \ t \in [t_0 - h(t_0), t_0], \end{aligned} \tag{10}$$

where

$$\eta_t^1 = \Big\{ \int_0^1 [g_x^*(x_t^0 + \mu\Delta x_t, \overline{y}_t, \overline{u}_t, \overline{v}_t, t) - g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t)]\Delta x_t d\mu +$$
$$+ \int_0^1 [g_y^*(x_t^0, y_t^0 + \mu\Delta y_t, \overline{u}_t, \overline{v}_t, t) - g_y^*(x_t^0, y_t^0, \overline{u}_t, \overline{v}_t, t)]\Delta y_t d\mu \Big\} dt +$$
$$+ \Big\{ \int_0^1 [\sigma_x^*(x_t^0 + \mu\Delta x_t, \overline{y}_t, t) - \sigma_x^*(x_t^0, \overline{y}_t, t)]\Delta x_t d\mu +$$
$$+ \int_0^1 [\sigma_y^*(x_t^0, y_t^0 + \mu\Delta y_t, t) - \sigma_y^*(x_t^0, y_t^0, t)]\Delta y_t d\mu \Big\} dw_t$$

and

$$d(\psi_t^* \cdot \Delta x_t) = d\psi_t^* \cdot \Delta x_t + \psi_t^* \cdot d\Delta x_t + \{\beta_t^* \sigma_x(x_t^0, y_t^0, t) \cdot \Delta x_t + \beta_t^* \sigma_y(x_t^0, y_t^0, t) \cdot \Delta y_t +$$
$$+ \beta_t^* \int_0^1 [\sigma_x(x_t^0 + \mu\Delta x_t, \overline{y}, t) - \sigma_x(x_t^0, \overline{y}, t)]\Delta x_t d\mu +$$
$$+ \beta_t^* \int_0^1 [\sigma_y(x_t^0, y_t^0 + \mu\Delta y_t, t) - \sigma_y(x_t^0, y_t^0, t)]\Delta y_t d\mu\} dt. \tag{11}$$

The increment of functional (6) along the admissible control looks like

$$\Delta_{\overline{u}} J(u) = E\Big\{ p\Big(\overline{x}_{t_1} - p(x_{t_1}^0) + \int_{t_0}^{t_1} \Big) [l(\overline{x}_t, \overline{y}_t, \overline{u}_t, \overline{v}_t, t) - l(x_t^0, y_t^0, u_t^0, v_t^0, t)]dt \Big\} =$$
$$= Ep_x(x_{t_1}^0)\Delta x_{t_1} + E\int_{t_0}^{t_1} [\Delta_u l(x_t^0, y_t^0, u_t^0, v_t^0, t) + \Delta_v l(x_t^0, y_t^0, u_t^0, v_t^0, t) +$$
$$+ l_x(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta x_t + l_y(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta y_t] dt + \eta^2, \tag{12}$$

where

$$\eta^2 = E\int_0^1 [p_x^*(x_{t_1}^0 + \mu\Delta x_{t_1}) - p_x^*(x_{t_1}^0)]\Delta v x_{t_1} d\mu$$
$$+ E\int_{t_0}^{t_1} \Big\{ \int_0^1 [l_x^*(x_{t_1}^0 + \mu\Delta x_t, \overline{y}_t, \overline{u}_t, \overline{v}_t, t) - l_x^*(x_t^0, \overline{y}_t, \overline{u}_t, \overline{v}_t, t)\Delta x_t d\mu$$
$$+ \int_0^1 [l_y^*(x_t^0, y_t^0 + \mu\Delta y_t, \overline{u}_t, \overline{v}_t, t) - l_y^*(x_t^0, y_t^0, \overline{u}_t, \overline{v}_t, t)]\Delta y_t d\mu \Big\} dt.$$

Taking (10) and (11) into consideration, expression (12) takes the form

$$\Delta_{\overline{u}} J(u^0) = -E\int_{t_0}^{t_1} d\psi_t^* \Delta x_t - E\int_{t_0}^{t_1} \psi_t^* \{[\Delta_{\overline{u}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) +$$
$$+ \Delta_{\overline{v}} g(x_t^0, y_t^0, u_t^0, v_t^0, t) + g_x(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta x_t +$$
$$+ g_y(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta y_t]dt + [\sigma_x(x_t^0, y_t^0, t)\Delta x_t + \sigma_y(x_t^0, y_t^0, t)\Delta y_t]\} dw_t -$$
$$- E\int_{t_0}^{t_1} \beta_t^* [\sigma_x(x_t^0, y_t^0, t)\Delta x_t + \sigma_y(x_t^0, y_t^0, t)\Delta y_t]dt + E\int_{t_0}^{t_1} [\Delta_{\overline{u}} l(x_t^0, y_t^0, u_t^0, v_t^0, t) +$$
$$+ \Delta_{\overline{v}} l(x_t^0, y_t^0, u_t^0, v_t^0, t) + l_x(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta x_t + l_y(x_t^0, y_t^0, u_t^0, v_t^0, t)\Delta y_t dt] + \eta_{t_0, t_1}, \tag{13}$$

where

$$\eta_{t_0,t_1}=\eta^2+E\int_{t_0}^{t_1}\left\{\int_0^1\psi_t^*(g_x(x_t^0+\mu\Delta x_t,\overline{y}_t,u_t^0,v_t^0,t)-g_x(x_t^0,\overline{y}_t,u_t^0,v_t^0,t))\Delta x_t d\mu+\right.$$
$$\left.+\int_0^1\psi_t^*(g_y(x_t^0,y_t^0+\mu\Delta y_t,u_t^0,v_t^0,t)-g_y(x_t^0,y_t^0,u_t^0,v_t^0,t))\Delta y_t d\mu\right\}+$$
$$+E\int_{t_0}^{t_1}\left\{\int_0^1\beta_t^*(\sigma_x(x_t^0+\mu\Delta x_t,\overline{y}_t,t)-\sigma_x(x_t^0,y_t^0,t))\Delta x_t d\mu+\right.$$
$$\left.+\int_0^1\beta_t^*(\sigma_y(x_t^0,y_t^0+\mu\Delta y_t,t)-\sigma_y(x_t^0,y_t^0,t))\Delta y_t d\mu\right\}dt.$$

Using simple transformations and taking (8) into consideration, expression (13) takes the form

$$\Delta J(u^0)=-E\int_{t_0}^{t_1}[\psi_t^*\Delta_{\overline{u}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\overline{u}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt-$$
$$-E\int_{t_0}^{t_1}[\psi_t^*\Delta_{\overline{v}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\overline{v}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+\eta_{t_0,t_1}. \tag{14}$$

Let's consider the following spike variation:

$$\Delta u_t=\Delta u_{t,\varepsilon}^{\theta}=\begin{cases}0, & t\overline{\in}[\theta,\theta+\varepsilon),\varepsilon>0,\theta\in[t_0,t_1)\\ \widetilde{u}-u_t^0, & t\in[\theta,\theta+\varepsilon),\widetilde{u}\in L^2(\Omega,F^{\theta},P;R^m).\end{cases}$$

Then (14) takes the form

$$\Delta_\theta J(u^0)=-E\int_{\theta}^{\theta+\varepsilon}[\psi_t^*\Delta_{\widetilde{u}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)+\psi_t^*\Delta_{\widetilde{v}}g(x_t^0,y_t^0,u_t^0,v_t^0,t)-$$
$$-\Delta_{\widetilde{u}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)-\Delta_{\widetilde{v}}l(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+\eta_{\theta,\theta+\varepsilon}. \tag{15}$$

We will use the following lemma.

Lemma 1. *Let conditions I-III be satisfied. Then*

$$E|x_{t,\varepsilon}^{\theta}-x_t^0|^2\leq N\varepsilon^2,\ \textit{if}\ \varepsilon\to 0,$$

where $x_{t,\varepsilon}^{\theta}$ *is the trajectory corresponding to the control* $u_{t,\varepsilon}^{\theta}=u_t^{\theta}+\Delta u_{t,\varepsilon}^{\theta}$.

Proof. Let's designate

$$\widetilde{x}_{t,\varepsilon}=\frac{x_{t,\varepsilon}^{\theta}-x_t^0}{\varepsilon},\ \widetilde{y}_{t,\varepsilon}=\widetilde{x}_{t-h(t),\varepsilon}=\frac{x_{t-h(t),\varepsilon}^{\theta}-x_{t-h(t)}^0}{\varepsilon}.$$

It is clear that $\forall\ t\in[t_0,\theta)$ $\widetilde{x}_{t,\varepsilon}=0$. Then, for $\forall\ t\in[\theta,\theta+\varepsilon)$,

$$d\widetilde{x}_{t,\varepsilon}=\frac{1}{\varepsilon}[g(x_t^0+\varepsilon\widetilde{x}_{t,\varepsilon},y_t^0+\varepsilon\widetilde{y}_{t,\varepsilon},\widetilde{u},v_t^0,t)-g(x_t^0,y_t^0,u_t^0,v_t^0,t)]dt+$$
$$+\frac{1}{\varepsilon}[\sigma(x_t^0+\varepsilon\widetilde{x}_{t,\varepsilon},y_t^0+\varepsilon\widetilde{y}_{t,\varepsilon},t)-\sigma(x_t^0,y_t^0,t)]dw_t,\ t\in(\theta,\theta+\varepsilon)$$
$$\widetilde{x}_{\theta,\varepsilon}=-(g(x_\theta^0,y_\theta^0,\widetilde{u},v_\theta^0,\theta)-g(x_\theta^0,y_\theta^0,u_\theta^0,v_\theta^0,\theta)).$$

Therefore, conditions I-II and the Gronwall inequality yield

$$E|\widetilde{x}_{\theta+\varepsilon,\varepsilon}|^2 \le N\Big[E \sup_{\theta\le t\le\theta+\varepsilon} |x^\theta_{t,\varepsilon} - x^0_t|^2 + E \sup_{\theta\le t\le\theta+\varepsilon} |x^0_t - x^0_\theta|^2 + E \sup_{\theta\le t\le\theta+\varepsilon} |y^\theta_{t,\varepsilon} - y^0_t|^2 +$$
$$+ E \sup_{\theta\le t\le\theta+\varepsilon} |y^0_t - y^0_\theta|^2 + \sup_{\theta\le t\le\theta+\varepsilon} E|g(x^0_t, y^0_t, \widetilde{u}, v^0_t, t) - g(x^0_\theta, y^0_\theta, \widetilde{u}, v^0_\theta, \theta)|^2 +$$
$$+ \frac{1}{\varepsilon} ER \int_\theta^{\theta+\varepsilon} |g(x^0_t, y^0_t, u^0_t, v^0_t, t) - g(x^0_\theta, y^0_\theta, u^0_\theta, v^0_\theta, \theta)|^2 dt\Big].$$

Hence: $E|\widetilde{x}_{t+\varepsilon,\varepsilon}|^2 \le N,\ \varepsilon \to 0,\ \forall\ t \in [\theta, \theta+\varepsilon)$. In the same way for $\forall\ t \in [\theta+\varepsilon, t_1]$, we have

$$d\widetilde{x}_{t,\varepsilon} = \frac{1}{\varepsilon}[g(x^0_t + \varepsilon\widetilde{x}_{t,\varepsilon}, y^0_t + \varepsilon\widetilde{y}_{t,\varepsilon}, u^0_t, \widetilde{u}, t) - g(x^0_t, y^0_t, u^0_t, v^0_t, t)]dt +$$
$$+ \frac{1}{\varepsilon}[\sigma(x^0_t + \varepsilon\widetilde{x}_{t,\varepsilon}, y^0_t + \varepsilon\widetilde{y}_{t,\varepsilon}, t) - \sigma(x^0_t, y^0_t, t)]dw_t.$$

Whence we have $E|\widetilde{x}_{t,\varepsilon}|^2 \le N$, for $\forall\ t \in [\theta+\varepsilon, t_1]$, if $\varepsilon \to 0$. Thus, $\sup_{t_0\le t\le t_1} E|\widetilde{x}_{t,\varepsilon}|^2 \le N$. Lemma 1 is proved.

According to Lemma 1 and from expression for η_{t_0,t_1}, we obtain $\eta_{\theta,\theta+\varepsilon} = o(\varepsilon)$.

Then it follows from (15) that

$$\Delta_\theta J(u^0) = -E[\psi^*_\theta \Delta_{\widetilde{u}} g(x^0_\theta, y^0_\theta, x^0_\theta, u^0_\theta, v^0_\theta, \theta) - \Delta_{\widetilde{u}} l(x^0_\theta, y^0_\theta, x^0_\theta, u^0_\theta, v^0_\theta, \theta) +$$
$$+ [\psi^*_z \Delta_{\widetilde{v}} g(x^0_z, y^0_z, x^0_z, u^0_z, v^0_z, \theta) - \Delta_{\widetilde{v}} l(x^0_z, y^0_z, x^0_z, u^0_z, v^0_z, z]|_{z=r(\theta)} r'(\theta)]\varepsilon + o(\varepsilon) \ge 0.$$

Hence, due to the sufficient smallness of ε, relation (9) is fulfilled. Theorem 1 is proved.

Problem with constraint

Using the obtained result and the variation principle of Ekeland [6], we will prove the following theorem for a stochastic optimal control problem with the endpoint constraint (7).

Theorem 2. *Let conditions I-IV hold, and let (x^0_t, u^0_t) be a solution of problem (1)–(7). Let there exist the random processes $(\psi_t, \beta_t) \in L^2_F(t_0, t_1; R^n) \times L^2_F(t_0, t_1; R^{n\times n})$ which are solutions of the adjoint system*

$$(16)\quad \begin{cases} d\psi_t = -[H_x(\psi_t, x^0_t, y^0_t, u^0_t, v^0_t, t) + H_y(\psi_z, x^0_z, y^0_z, u^0_z, v^0_z, z)|_{z=s(t)} s'(t)]dt + \\ +\beta_t dw_t,\ t_0 \le t < t_1 - h(t_1), \\ d\psi_t = -H_x(\psi_t, x^0_t, y^0_t, u^0_t, v^0_t, t)dt + \beta_t dw_t,\ t_1 - h(t_1) \le t < t_1, \\ \psi_{t_1} = -\lambda_0 p_x(x^0_{t_1}) - \lambda_1 q_x(x^0_{t_1}), \end{cases}$$

where $(\lambda_0, \lambda_1) \in R^{k+1}$, $\lambda_0 \ge 0$, λ_1 is the normal to the set G at the point $Eq(x^0_{t_1})$, and $\lambda_0^2 + |\lambda_1|^2 = 1$. Then, $\forall\ \widetilde{u} \in U$ a.c., the following relations hold:

$$(17)\quad \begin{cases} H(\psi_t, x^0_t, y^0_t, u, v^0_t, t) - H(\psi_t, x^0_t, y^0_t, u^0_t, v^0_t, t) + \\ +[H(\psi_z, x^0_z, y^0_z, u^0_z, u, z) - H(\psi_z, x^0_z, y^0_z, u^0_z, v^0_z, z)]|_{z=r(t)} r'(t) \le 0, \\ a.e.\ t \in [t_0, t_1 - h_1(t_1)), \\ H(\psi_t, x^0_t, y^0_t, u, v^0_t, t) - H(\psi_t, x^0_t, y^0_t, u^0_t, v^0_t, t) \le 0,\ t \in [t_1 - h_1(t_1), t_1],\ a.e. \end{cases}$$

Proof. For any natural j, we introduce the approximating functional

$$J_j(u) = S_j(Ep(x_{t_1}) + E\int_{t_0}^{t_1} l(x_t, y_t, u_t, v_t, t)dt, Eq(x_{t_1})) =$$

$$= \min_{(c,y)\in\varepsilon} \sqrt{\left|c - 1/j - Ep(x_{t_1}) - E\int_{t_0}^{t_1} l(x_t, y_t, u_t, v_t, t)dt\right|^2 + \|y - Eq(x_{t_1})\|^2},$$

$\mathcal{E} = \{(c, y) : c \le J^0, y \in G\}$, where J^0 is the minimal value of the functional in (1)-(7). By $V \equiv (U_\partial, d)$, we denote the space of controls obtained by means of introducing the metric

$$d(u, v) = (l \otimes P)\{(t, \omega) \in [t_0, t_1] \times \Omega : v_t \ne u_t\},$$

so that V is a complete metric space. In what follows, we need the following lemma.

Lemma 2. *We assume that conditions I-IV hold, u_t^n – a sequence of admissible controls from V, x_t^n – a sequence of the corresponding trajectories of system (1)-(3). If $d(u_t^n, u_t) \to 0, n \to \infty$, then $\lim_{n\to\infty}\left\{\sup_{t_0\le t\le t_1} E|x_t^n - x_t|^2\right\} = 0$, where x_t is a trajectory corresponding to an admissible control u_t.*

Proof. Let u_t^n be a sequence of admissible controls from V, and let x_t^n be a sequence of the corresponding trajectories. Then, for any $t \in (t_0; t_1]$, we have

$$|x_t^n - x_t| =$$

$$= \left|\int_{t_0}^{t} [g(x_s^n, y_s^n, u_s^n, v_s^n, s) - g(x_s, y_s, u_s, v_s, s)]ds + \int_{t_0}^{t} [\sigma(x_s^n, y_s^n, s) - \sigma(x_s, y_s, s)]dw_s\right|.$$

Let's square and take expectation of both sides of the last expression. Due to assumption II, we have

$$E|x_t^n - x_t|^2 \le$$

$$\le NE\int_{t_0}^{t} |\Delta_{u^n} g(x_s, y_s, u_s, v_s, s)|^2 ds + NE\int_{t_0}^{t} |x_t^n - x_t|^2 dt + N\int_{t_0}^{t} E|y_t^n - y_t|^2 dt.$$

Hence, condition I and the Gronwall inequality yield

$$E|x_t^n - x_t|^2 \le C\exp(C(t - t_0)),$$

where $C = NE\int_{t_0}^{t} |\Delta_{u^n} g(x_s, y_s, u_s, v_s, s)|^2 ds$. Lemma 2 is proved.

Due to continuity of the functional $J_j : V \to R^n$, according to the variation principle of Ekeland, we have that there exists a control $u_t^j : d(u_t^j, u_t^0) \le \sqrt{\varepsilon_j}$ and, $\forall\ u \in V$, the following inequality holds: $J_j(u^j) \le J_j(u) + \sqrt{\varepsilon_j} d(u^j, u),\ \varepsilon_j = \frac{1}{j}$.

This inequality means that (x_t^j, u_t^j) is a solution of the following problem:

$$\begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} E\int_{t_0}^{t_1} \delta(u_t, u_t^j)dt \to \min \\ dx_t = g(x_t, y_t, u_t, v_t, t)dt + \sigma(x_t, y_t, t)dw_t,\ t \in (t_0, t_1] \\ x_t = \Phi(t),\ t \in [t_0 - h(t_0), t_0] \\ u_t = Q(t),\ t \in [t_0 - h_1(t_0), t_0] \\ u_t \in U_\partial. \end{cases} \tag{18}$$

The function $\delta(u, v)$ is determined in the following way: $\delta(u, v) = \begin{cases} 0, & u = v \\ 1, & u \ne v. \end{cases}$

Let (x_t^j, u_t^j) be a solution of problem (18). If there exist the random processes $\psi_t^j \in L_F^2(_0, t_1; R^n)$, $\beta_t^j \in L_F^2(t_0, t_1; R^{n\times n})$, which are solutions of the system

$$(19)\quad \begin{cases} d\psi_t^j = -[H_x(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) + H_y(\psi_z^j, x_z^j, y_z^j, u_z^j, v_z^j, z)|_{z=s(t)} s'(t)]dt + \\ +\beta_t^j dw_t,\ t_0 \le t \le t_1 - h(t_1) \\ d\psi_t^j = -H_x(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t)dt + \beta_t^j dw_t,\ t_1 - h(t_1) \le t < t_1 \\ \psi_{t_1}^j = -\lambda_0^j p_x(x_{t_1}^j) - \lambda_1^j q_x(x_{t_1}^j), \end{cases}$$

where the non-zero $(\lambda_0^j, \lambda_1^j) \in R^{k+1}$ meet the requirement

$$(20)\quad (\lambda_0^j, \lambda_1^j) = (-c_j + 1/j + Ep(x_{t_1}^j) + E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt, -y_j + Eq(x_{t_1}^j))/I_j^0,$$

then, according to Theorem 1,

$$(21)\quad \begin{cases} H(\psi_t^j, x_t^j, y_t^j, u, v_t^j, t) - H(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) + [H(\psi_z^j, x_z^j, y_z^j, u_z^j, u, z) - \\ -H(\psi_z^j, x_z^j, y_z^j, u_z^j, v_z^j, z)]|_{z=r(t)} r'(t) \le 0,\ \text{a.c., a.e. } t \in [t_0, t_1 - h_1(t_1)), \\ H(\psi_t^j, x_t^j, y_t^j, u, v_t^j, t) - H(\psi_t^j, x_t^j, y_t^j, u_t^j, v_t^j, t) \le 0,\ \text{a.c, a.e. } t \in [t_1 - h_1(t_1), t_1]. \end{cases}$$

Here,

$$I_j^0 = \sqrt{\left|c_j - 1/j - Ep(x_{t_1}^j) - E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt\right|^2 + |y_j - Eq(x_{t_1}^j)|^2}.$$

Since $\|(\lambda_0^j, \lambda_1^j)\| = 1$, we can think that $(\lambda_0^j, \lambda_1^j) \to (\lambda_0, \lambda_1)$.

It is known that S_j is a convex function which is Gateaux-differentiable at a point: $(Ep(x_{t_1}^j) + E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt, Eq(x_{t_1}^j)$. Then, for all $(c, y) \in \mathcal{E}$,

$$\left(\lambda_0^j, c - \frac{1}{j} - Ep(x_{t_1}^j) - E\int_{t_0}^{t_1} l(x_t^j, y_t^j, u_t^j, v_t^j, t)dt\right) + (\lambda_1^j, y - Eq(x_{t_1}^j)) \le \frac{1}{j}.$$

Proceeding to the limit in the last inequality, we get that $\lambda_0 \ge 0$ and λ_1 is a normal to the set G at $Eq(x_{t_1}^0)$. Since

$$(22)\qquad \psi_{t_1}^j = -\lambda_0^j p_x(x_{t_1}^j) - \lambda_1^j q_x(x_{t_1}^j),\ \text{ we have } \psi_{t_1}^j \to \psi_{t_1} \text{ in } L_F^2(t_0, t_1; R^n).$$

Lemma 3. *Let ψ_t^j be a solution of system (19), and let ψ_t be a solution of system (16). Then*

$$E\int_{t_0}^{t_1} |\psi_t^j - \psi_t|^2 dt + E\int_{t_0}^{t_1} |\beta_t^j - \beta_t|^2 dt \to 0,\ \textit{if } d(u_t^j, u_t) \to 0,\ j \to \infty.$$

Proof. According to Ito formula $\forall\ s \in [t_1 - h(t), t_1]$,

$$\begin{aligned} &E|\psi_{t_1}^j - \psi_{t_1}|^2 - E|\psi_s^j - \psi_s|^2 = \\ &= 2E\int_s^{t_1} [\psi_t^j - \psi_t][(g_x^*(x_t^j, y_t^j, u_t^j, v_t^j, t) - g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t))\psi_t^j + \\ &+ g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t)(\psi_t^j - \psi_t) + (\sigma_x^*(x_t^j, y_t^j, t) - \sigma_x^*(x_t^0, y_t^0, t) \times \\ &\times (\beta_t^j - \beta_t) - l_x(x_t^j, y_t^j, u_t^j, v_t^j, t) + l_x(x_t^0, y_t^0, u_t^0, v_t^0, t)]dt + E\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt. \end{aligned}$$

Due to assumptions I-II and using simple transformations, we obtain

$$E\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt + E|\psi_t^j - \psi_t|^2 dt + EN\varepsilon\int_s^{t_1} |\beta_t^j - \beta_t|^2 dt + E|\psi_{t_1}^j - \psi_{t_1}|^2.$$

Hence, according to the Gronwall inequality, we have

$$E|\psi_s^j - \psi_s|^2 \le De^{N(t_1-s)} \text{ a.e. in } [t_1 - h(t), t_1], \tag{23}$$

where the constant D is determined in the following way: $D = E|\psi_{t_1}^j - \psi_{t_1}|^2$. According to Ito formula $\forall\ s \in [t_0, t_1 - h(t_1))$,

$$E|\psi_{t_1-h(t_1)}^j - \psi_{t_1-h(t_1)}|^2 - E|\psi_s^j - \psi_s|^2 = 2E\int_s^{t_1-h(t_1)} (\psi_t^j - \psi_t)[(g_x^*(x_t^j, y_t^j, u_t^j, v_t^j, t) -$$
$$- g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t))\psi_t^j + g_x^*(x_t^0, y_t^0, u_t^0, v_t^0, t)(\psi_t^j - \psi_t) + (\sigma_x^*(x_t^j, y_t^j, t) -$$
$$- \sigma_x^*(x_t^0, y_t^0, t))\beta_t^j + \sigma_x^*(x_t^0, y_t^0, t)(\beta_t^j - \beta_t) + (g_y^*(x_z^j, y_z^j, u_z^j, v_z^j, z) -$$
$$- g_y^*(x_z^0, y_z^0, u_z^0, v_z^0, z))\psi_z^j s'(t) + g_y^*(x_z^0, y_z^0, u_z^0, v_z^0, z)(\psi_z^j - \psi_z)s'(t)\sigma_y^*(x_z^j, y_z^j, z) -$$
$$- \sigma_y^*(x_z^0, y_z^0, z))\beta_z^j s'(t) + \sigma_y^*(x_z^0, y_z^0, z)(\beta_z^j - \beta_z)s'(t) + l_x(x_t^0, y_t^0, u_t^0, v_t^0, t) -$$
$$- l_x(x_t^j, y_t^j, u_t^j, v_t^j, t) + l_y(x_t^0, y_t^0, u_t^0, v_t^0, t) -$$
$$- l_y(x_t^j, y_t^j, u_t^j, v_t^j, t)]dt + E\int_s^{t_1-h(t_1)} |\beta_t^j - \beta_t|^2 dt.$$

In view of assumptions I-II and expression (22), we obtain

$$E\int_s^{t_1-h(t_1)} |\beta_t^j - \beta_t|^2 dt + E|\psi_s^j - \psi_s|^2 \le$$
$$\le EN\int_s^{t_1-h(t_1)} |\psi_t^j - \psi_t|^2 dt + EN\varepsilon\int_s^{t_1-h(t_1)} |\psi_z^j - \psi_z|^2 dt +$$
$$+ EN\varepsilon\int_s^{t_1-h(t_1)} |\beta_t^j - \beta_t|^2 dt + E|\psi_{t_1-h(t_1)}^j - \psi_{t_1-h(t_1)}|^2.$$

Hence, using simple transformations, we have

$$E(1-2N\varepsilon)\int_s^{t_1-h(t_1)} |\beta_t^j - \beta_t|^2 dt + E|\psi_s^j - \psi_s|^2 \le E(N+N\varepsilon)\int_s^{t_1-h(t_1)} |\psi_t^j - \psi_t|^2 dt +$$
$$EN\varepsilon\int_{t_1-h(t_1)}^{t_1} |\psi_t^j - \psi_t|^2 dt + EN\varepsilon\int_{t_1-h(t_1)}^{t_1} |\beta_t^j - \beta_t|^2 dt + E|\psi_{t_1-h(t_1)}^j - \psi_{t_1-h(t_1)}|^2.$$

According to the Gronwall inequality,

$$E|\psi_s^j - \psi_s|^2 \le D\exp[-(N+N\varepsilon)(t_1 - h(t_1) - s)], \text{ a.e. in } [t_0, t_1 - h(t_1)),$$

where

$$D = E|\psi_{t_1-h(t_1)}^j - \psi_{t_1-h(t_1)}|^2 + EN\varepsilon\int_{t_1-h(t_1)}^{t_1} |\psi_t^j - \psi_t|^2 dt + EN\varepsilon\int_{t_1-h(t_1)}^{t_1} |\beta_t^j - \beta_t|^2 dt.$$

Due to sufficient smallness of ε and from inequality (23), we get $D \to 0$. Thus, $\psi_t^j - \psi_t$ $L_F^2(t_0, t_1; R^n), \beta_t^j \to \beta_t$ $L_F^2(t_0, t_1; R^{n\times n})$. Lemma 3 is proved.

It follows from Lemma 3 and assumptions I-III that we can proceed to the limit in systems (19), (21) and get the fulfillment of (16) and (17). Theorem 2 is proved.

Corollary. *In the case where $g \equiv g(x_t, y_t, u_t, t)$ and $l \equiv l(x_t, u_t, t)$ we obtain the result proved in [4].*

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