Theory of Stochastic Processes
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STOCHASTIC PROCESSES IN SOME BESOV SPACES

The norm of increments of stochastic process in space $L_q[a,b]$ is estimated and conditions under which trajectories of process belong to some Besov spaces are found.

1. Introduction

In this paper the stochastic process from $L_p(\Omega)$ is considered. We find conditions under which trajectories of this process belong with probability one to the Besov space $B_q^{\alpha p}[a,b]$ when $0<\alpha<1$ and $p<q$.

The paper consists of 4 sections. In Section 2 the norm of increments of the stochastic process is estimated. In Section 3 the definition of the Besov space is given. Then, it was obtained the estimation for the modulus of continuity of stochastic process. It gives the possibility to find the condition under which the trajectories of stochastic process belong to some Besov space with probability one. Section 4 is the conclusion of this paper.

2. Estimates for increments of the stochastic processes

Let's consider stochastic process $X=\{X(t),t\in[a-\delta,b+\delta]\}$, $a<b$, $\delta>0$ and denote the increments of this process as $\Delta_h X(t)=X(t+h)-X(t)$, $t\in[a,b]$, $|h|<\delta$.

Theorem 1. *Assume that* $1<p<q<\infty$. *Let* $X=\{X(t),t\in[a-\delta,b+\delta]\}$, $a<b$, $\delta>0$ *to be separable measurable stochastic process from space* $L_p(\Omega)$ *for which the following condition on its increments holds true:*

$$\sup_{\substack{t\in[a,b]\\ |h|\le\delta,}} \|\Delta_h X(t)\|_{L_p(\Omega)} = \sup_{\substack{t\in[a,b]\\ |h|\le\delta,}} \left(E|X(t+h)-X(t)|^p\right)^{1/p} \le \sigma(\delta), \quad (1)$$

where $\sigma(\delta)$, $\delta>0$ *is a continuous nondecreasing function such that* $\sigma(\delta)\to 0$ *as* $\delta\to 0$.

Then

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1) there exists $m = m(h) \in \{1, 2, ...\}$:

$$\left\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \right\|_{L_p(\Omega)} = \left(E \left(\int_a^b |X(t+h) - X(t)|^q dt \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq$$

$$\leq 2^{1+\frac{1}{q}-\frac{1}{p}} (b-a)^{\frac{1}{p}} \sum_{k=m-1}^{\infty} (\varepsilon_{k+1})^{\frac{1}{q}-\frac{1}{p}} \left[\sigma(6\varepsilon_{k+1}) + \sigma(6\varepsilon_k) \right] =: B_m,$$

where sequence $\{\varepsilon_k > 0\}_{k\geq 0}$ is such that for all $k \geq 0$ $\varepsilon_k > \varepsilon_{k+1}$, $\varepsilon_k \to 0$, $k \to \infty$ and there exists the sequence $\{0 < \alpha_k < 1\}_{k\geq 0}$, for which the fraction $\frac{\sigma(\varepsilon_k)}{\alpha_k} \to 0$, $k \to \infty$ and the series $\sum_{k=0}^{\infty} \alpha_k < \infty$;

2) if $B_m < \infty$ then $\|\Delta X(\cdot)\|_{L_q[a,b]}$ belongs to $L_p(\Omega)$ and for all $x > 0$

$$P\left\{ \|\Delta X(\cdot)\|_{L_q[a,b]} > x \right\} \leq \left(\frac{B_m}{x} \right)^p.$$

Proof. Let's divide the interval $[a-\delta, b+\delta]$ into measurable sets $\{B_k^r, r = 1, 2, ..., N(\varepsilon_k)\}_{k\geq 0}$ with respect to the sequence $\{\varepsilon_k > 0\}_{k\geq 0}$ according to partition procedure placed in [1]. This partition has the following properties:

1) $B_0^1 = [a-\delta, b+\delta]$, $(\varepsilon_0 = \frac{b-a}{2} + \delta)$;

2) $\forall k \geq 1$ $B_k^u \bigcap B_k^r = \emptyset$ if $u \neq r$ and $\bigcup_{r=1}^{N(\varepsilon_k)} B_k^r = [a-\delta, b+\delta]$;

3) $\forall k \geq 1$ $\forall B_k^r$ $\exists B_{k-1}^l$: $B_k^r \subset B_{k-1}^l$ and $B_k^r \bigcap B_{k-1}^l = B_k^r$;

4) $\forall k \geq 1$ $\forall B_k^r$ $\exists t_k^r$: $\forall t \in B_k^r$ $2\varepsilon_k \leq |t - t_k^r| < 6\varepsilon_k$.

Now we can define the process:

$$X_k(t) = \sum_{r=1}^{N(\varepsilon_k)} X(t_k^r) \chi_{B_k^r}(t), \ k \geq 0, \ t \in [a-\delta, b+\delta], \ \chi_{B_k^R} = \begin{cases} 1, & t \in B_k^r; \\ 0, & t \notin B_k^r. \end{cases}$$

As long as the proof of this theorem is similar to the proof of the lemma 3.5 [1] it was reduced to the statements that are essentially different.

So, if the points t and $t+h$ belong to the same set B_k^r then $X_k(t) = X_k(t+h)$ with probability one and $\Delta_h X_k(t) = 0$ with probability one. Further we will assume that process X is not degenerate. The properties of the sequence ε_k imply that there exists the number $m = m(h) \in \{1, 2, ...\}$ starting from which the points t and $t+h$ will be in different sets of the partition $\{B_m^r\}_{r=1}^{\varepsilon_m}$. Then, taking into consideration that $X_{m-1}(t) = X_{m-1}(t+h)$ we have for some $n > m$:

$$|\Delta_h X(t)| = |X(t+h) - X(t)| = |X(t+h) - X_{m-1}(t+h) + X_{m-1}(t) - X(t)| \leq$$

$$\leq |X(t+h)-X_n(t+h)| + \sum_{k=m-1}^{n-1} |X_{k+1}(t+h)-X_k(t+h)| +$$

$$+|X(t)-X_n(t)| + \sum_{k=m-1}^{n-1} |X_{k+1}(t)-X_k(t)|.$$

So, for all $t \in [a,b]$ as $n \to \infty$:

$$|\Delta_h X(t)| \leq \sum_{k=m-1}^{\infty} \Big[|X_{k+1}(t+h)-X_k(t+h)| + |X_{k+1}(t)-X_k(t)| \Big].$$

The condition (1) yields:

$$\|X_{k+1}(t)-X_k(t)\|_{L_p(\Omega)} \leq \|X_{k+1}(t)-X(t)\|_{L_p(\Omega)} + \|X(t)-X_k(t)\|_{L_p(\Omega)} \leq$$
$$\sigma(6\varepsilon_{k+1}) + \sigma(6\varepsilon_k).$$

Then

$$\|\Delta_h X(\cdot)\|_{L_q[a,b]} = \left(\int_a^b |X(t+h)-X(t)|^q \right)^{1/q} \leq$$

$$\leq \left\| \sum_{k=m-1}^{\infty} \left\{ \frac{|X_{k+1}(t)-X_k(t)|}{\|X_{k+1}(t)-X_k(t)\|_{L_p(\Omega)}} \|X_{k+1}(t)-X_k(t)\|_{L_p(\Omega)} + \right.\right.$$
$$\left.\left. \frac{|X_{k+1}(t+h)-X_k(t+h)|}{\|X_{k+1}(t+h)-X_k(t+h)\|_{L_p(\Omega)}} \|X_{k+1}(t+h)-X_k(t+h)\|_{L_p(\Omega)} \right\} \right\|_{L_q[a,b]} \leq$$

$$\leq \sum_{k=m-1}^{\infty} \left\{ \left\| \frac{|X_{k+1}(t+h)-X_k(t+h)|}{\|X_{k+1}(t+h)-X_k(t+h)\|_{L_p(\Omega)}} \right\|_{L_q[a,b]} + \right.$$
$$\left. + \left\| \frac{|X_{k+1}(t)-X_k(t)|}{\|X_{k+1}(t)-X_k(t)\|_{L_p(\Omega)}} \right\|_{L_q[a,b]} \right\} \Big[\sigma(6\varepsilon_{k+1}) + \sigma(6\varepsilon_k) \Big] \leq$$

$$\leq \sum_{k=m-1}^{\infty} \left\{ \left\| \frac{|X_{k+1}(t+h)-X_k(t+h)|}{\|X_{k+1}(t+h)-X_k(t+h)\|_{L_p(\Omega)}} \right\|_{L_p[a,b]} + \right.$$
$$\left. + \left\| \frac{|X_{k+1}(t)-X_k(t)|}{\|X_{k+1}(t)-X_k(t)\|_{L_p(\Omega)}} \right\|_{L_p[a,b]} \right\} (2\varepsilon_{k+1})^{\frac{1}{q}-\frac{1}{p}} \Big[\sigma(6\varepsilon_{k+1}) + \sigma(6\varepsilon_k) \Big] = B_m.$$

The expression for B_m we will get from the fact that Condition M (see [1]) is fulfilled for the space L_p with constant $(b-a)^{1/p}$.

The second statement follows from lemma 3.1 [2,p.66].

Corollary 1. *If the stochastic process* $X = \{X(t), t \in [a-\delta, b+\delta]\}$ *fatisfies the conditions of theorem 1, then for all* $0 < \theta_1 < 1$ *the following inequality takes place:*

$$\Big\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \Big\|_{L_p(\Omega)} = \left(E \left(\int_a^b |X(t+h)-X(t)|^q dt \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq$$

$$\leq \frac{2(b-a)^{\frac{1}{p}}}{3^{\frac{1}{q}-\frac{1}{p}}} \cdot \frac{1+\theta_1}{\theta(1-\theta)} \int_0^{\theta_1 \sigma\left(\frac{b-a}{2}+\delta\right)} (\sigma^{(-1)}(u))^{\frac{1}{q}-\frac{1}{p}} du =: \tilde{B}_m \tag{2}$$

Proof. Lets choose the sequence $\{\varepsilon_k\}_{k\geq 0}$ in such way

$$\varepsilon_0 = \frac{b-a}{2} + \delta, \quad \gamma_0 = \sigma(\varepsilon_0), \quad \varepsilon_k = \frac{\sigma^{(-1)}(\theta_1^k \gamma_0)}{6}.$$

This sequence has appropriable for theorem 1 properties. So,

$$B_m = 2(b-a)^{\frac{1}{p}} \sum_{k=m-1}^{\infty} \left(\frac{\sigma^{(-1)}(\theta_1^{k+1}\gamma_0)}{3} \right)^{\frac{1}{q}-\frac{1}{p}} [\theta_1^{k+1}\gamma_0 + \theta_1^k \gamma_0] \leq$$

$$\leq 2(b-a)^{\frac{1}{p}} \sum_{k=m-1}^{\infty} \frac{\theta_1^{k+1}\gamma_0 + \theta_1^k\gamma_0}{\theta_1^{k+1}\gamma_0 - \theta_1^{k+2}\gamma_0} \int_{\theta_1^{k+2}\gamma_0}^{\theta_1^{k+1}\gamma_0} \left(\frac{\sigma^{(-1)}(u)}{3} \right)^{\frac{1}{q}-\frac{1}{p}} du \leq$$

$$\leq \frac{2(b-a)^{\frac{1}{p}}}{3^{\frac{1}{q}-\frac{1}{p}}} \cdot \frac{1+\theta_1}{\theta_1(1-\theta_1)} \int_0^{\theta_1\gamma_0} (\sigma^{(-1)}(u))^{\frac{1}{q}-\frac{1}{p}} du = \tilde{B}_m.$$

Corollary 2. *Assume that in the theorem 1 function $\sigma(\delta) = C\delta^\tau$, where $C > 0$ is some constant, $\tau > \frac{1}{p} - \frac{1}{q}$. Then the increments of the process $X = \{X(t), t \in [a-\delta, b+\delta]\}$ belong to the space $L_q[a,b]$ with probability one and for all $0 < \theta_1 < 1$ the inequality follows:*

$$\left\| \|\Delta_h X(\cdot)\|_{L_q[a,b]} \right\|_{L_p(\Omega)} = \left(E\left(\int_a^b |X(t+h) - X(t)|^q dt \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \leq$$

$$\leq \frac{2 \cdot 3^{\frac{1}{p}-\frac{1}{q}} C(b-a+2\delta)^{\frac{1}{q}+\tau}}{\frac{1}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)+1} \cdot \frac{1+\theta_1}{1-\theta_1} \cdot \theta_1^{\frac{m}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)+m-1} =: B_m^*.$$

Proof. Indeed, in this case

$$\tilde{B}_m = \frac{2(b-a)^{\frac{1}{p}}}{3^{\frac{1}{q}-\frac{1}{p}}} \cdot \frac{1+\theta_1}{\theta_1(1-\theta_1)} \int_0^{\theta_1^m C\left(\frac{b-a}{2}+\delta\right)^\tau} \left(\frac{u}{C}\right)^{\frac{1}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)} du =$$

$$= \frac{2 \cdot 3^{\frac{1}{p}-\frac{1}{q}}(b-a)^{\frac{1}{p}}}{C^{\frac{1}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)}} \cdot \frac{1+\theta_1}{\theta_1(1-\theta_1)} \cdot \frac{\left(\theta_1^m C\left(\frac{b-a}{2}+\delta\right)^\tau\right)^{\frac{1}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)+1}}{\frac{1}{\tau}\left(\frac{1}{q}-\frac{1}{p}\right)+1} \leq B_m^*.$$

Remark 1. Similar results hold for stochastic processes in more general case when stochastic process belongs to the Orlicz space of random variables. The estimation for the norm of increments in various functional Orlicz space could be obtained by the same way. More information can be found in [1,2].

3. Conditions under which trajectories of the process belong to some Besov spaces

Let's remind the definition of the Besov space at first. We start from introducing the moduli of continuity of the first and the second order and some of their properties.

Definition 2. Let f be a function in $L_q(T)$, $1 \le q \le \infty$, $T \subseteq R$. Let's denote $\Delta_h f = f(t+h) - f(t)$ and $\Delta_h^2 f = \Delta_h \Delta_h f$. For $\delta > 0$ the moduli of continuity are determined as

$$\omega_q^1(f,\delta) = \sup_{|h|\le\delta} \|\Delta_h f\|_{L_q(T)}, \quad \omega_q^2(f,\delta) = \sup_{|h|\le\delta} \|\Delta_h^2 f\|_{L_q(T)}.$$

Remark 2. For any function f from the space $L_q(R)$ $\omega_q^1(f,\delta)$ and $\omega_q^2(f,\delta)$ are non-decreasing functions of δ and

$$\omega_q^2(f,\delta) \le 2\omega_q^1(f,\delta) \le 4\|f\|_{L_q(T)}.$$

Let $1 \le p \le \infty$ be given, and let the function $y(\delta)$ on $[0,\infty)$ be such that $\|y(\delta)\|_p^* < \infty$, where

$$\|y(\delta)\|_p^* = \begin{cases} \left(\int_0^\infty |y(\delta)|^p \frac{d\delta}{\delta}\right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty; \\ ess\sup_\delta |y(\delta)|, & \text{if } p = \infty. \end{cases}$$

Clearly, $\|\cdot\|_p^*$ is a norm in the weighted L_p-space $L_p\left([0,\infty), \frac{d\delta}{\delta}\right)$, if $p < \infty$.

Definition 2. Let $1 \le q, p \le \infty$ and $s = n + \alpha$, with $n \in \{0, 1, ...\}$ and $0 < \alpha \le 1$. The Besov space $B_q^{sp}(T)$ is the space of all functions f such that

$$f \in W_q^n(T) \quad \text{and} \quad \omega_q^2(f^{(n)},\delta) = y(\delta)\delta^\alpha,$$

where $W_q^n(T)$ is the Sobolev space and $\|y(\delta)\|_p^* < \infty$.

The space $B_q^{sp}(T)$ is equipped with the norm

$$\|f\|_{B_q^{sp}(T)} = \|f\|_{W_q^n(T)} + \left\|\frac{\omega_q^2(f^{(n)},\delta)}{\delta^\alpha}\right\|_p^*.$$

Remark 3. If $0 < \alpha < 1$ we can use ω_q^1 instead of ω_q^2 in the definition of Besov spaces. But this is not true in the case if $\alpha = 1$ (See [3]).

Definition 3. The stochastic process $X = \{X(t), t \in T\}$ belongs to Besov space $B_q^{sp}(T)$ with probability one if all its trajectories belong to this functional space with probability one.

Remark 4. Further, when convergence of the integral $\int_0^c f(t)dt$ doesn't depend on value c we will use the sign $\int_{0+}$.

Theorem 2. *Assume that $1 < p < q < \infty$ and separable measurable stochastic process $X = \{X(t), t \in [a-\delta, b+\delta]\}$, $a < b$, $0 < \delta < \infty$ belongs to the space $L_p(\Omega)$. Besides, let its increments satisfy two conditions:*

a) $\sup\limits_{\substack{t \in [a,b] \\ |h| \le \delta,}} \|\Delta_h X(t)\|_{L_p(\Omega)} = \sup\limits_{\substack{t \in [a,b] \\ |h| \le \delta,}} (E|X(t+h) - X(t)|^p)^{1/p} \le C_p \delta^{\tau_p},$

b) $\sup\limits_{\substack{t \in [a,b] \\ |h| \le \delta,}} \|\Delta_h X(t)\|_{L_q(\Omega)} = \sup\limits_{\substack{t \in [a,b] \\ |h| \le \delta,}} (E|X(t+h) - X(t)|^q)^{1/q} \le C_q \delta^{\tau_q},$

where $C_p, C_q > 0$ are some constants, $\tau_p > \frac{1}{p} - \frac{1}{q}$ and $\tau_q > 0$.
Then

1) $\omega_q^1(X,\delta) = \sup_{|h|\le\delta} \left(\int_a^b |X(t+h) - X(t)|^q dt\right)^{1/q} \in L_p(\Omega)$

2) $\|\omega_q^1(X,\delta)\|_{L_p(\Omega)} = \left(E\left(\sup_{|h|\le\delta}\left(\int_a^b |X(t+h)-X(t)|^q dt\right)^{1/q}\right)^p\right)^{1/p} \le \dfrac{2^{1/p+2\tau_q} C_q^2 (b-a)^{2/q} \delta^{2\tau_q}}{\frac{2\delta}{p-1} + 2^{\tau_q} C_q (b-a)^{1/q}\delta^{\tau_q} - 2\sqrt{\frac{\delta^2}{(p-1)^2} + \frac{2^{\tau_q} C_q (b-a)^{1/q} \delta^{1+\tau_q}}{p-1}}}.$

Proof. Let's determine the stochastic process $\xi(h) = \|\Delta_h X(\cdot)\|_{L_q[a,b]}$, $|h| < \delta$ (so, $\omega_q^1(X,\delta) = \sup_{|h|\le\delta} \xi(h)$). We can conclude from the corollary 2 that $\xi \in L_p(\Omega)$. Then, Lyapunov inequality, Fubini theorem and condition b) yield

$$\beta := \sup_{h_1,h_2 \in [-\delta,\delta]} \|\xi(h_1) - \xi(h_2)\|_{L_p(\Omega)} =$$

$$= \sup_{h_1,h_2\in[-\delta,\delta]} \Big\| \|\Delta_{h_1} X(\cdot)\|_{L_q[a,b]} - \|\Delta_{h_2} X(\cdot)\|_{L_q[a,b]} \Big\|_{L_p(\Omega)} \le$$

$$\le \sup_{h_1,h_2\in[-\delta,\delta]} \Big\| \|\Delta_{h_1} X(\cdot) - \Delta_{h_2} X(\cdot)\|_{L_q[a,b]} \Big\|_{L_p(\Omega)} =$$

$$= \sup_{h_1,h_2\in[-\delta,\delta]} \left(E\left(\int_a^b |X(t+h_1) - X(t+h_2)|^q dt\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \le$$

$$\le \sup_{h_1,h_2\in[-\delta,\delta]} \left(E\int_a^b |X(t+h_1) - X(t+h_2)|^q dt\right)^{\frac{1}{q}} =$$

$$= \sup_{h_1,h_2\in[-\delta,\delta]} \left(\int_a^b E|X(t+h_1) - X(t+h_2)|^q dt\right)^{\frac{1}{q}} \le$$

$$\le \left(\int_a^b (C_q (2\delta)^{\tau_q})^q \, dt\right)^{\frac{1}{q}} \le C_q (2\delta)^{\tau_q} (b-a)^{1/q} < \infty.$$

Since, the entropy integral

$$\int_{0+} \left(\frac{\delta}{\varepsilon} + 1\right)^{1/p} d\varepsilon \sim \int_{0+} \frac{d\varepsilon}{\varepsilon^{1/p}} < \infty, \quad \text{as} \quad p > 1,$$

then the theorem 3.3 [2, p.120] implies that

1) $\sup_{|h|\le\delta}\xi(h)=\omega_q^1(X,\delta)\in L_p(\Omega)$

2) $\left\|\sup_{|h|\le\delta}\xi(h)\right\|_{L_p(\Omega)}=\left\|\omega_q^1(X,\delta)\right\|_{L_p(\Omega)}\le$
$\le\inf_{|h|\le\delta}\|\xi(h)\|_{L_p(\Omega)}+\inf_{0<\theta_2<1}\frac{1}{\theta_2(1-\theta_2)}\int\limits_0^{\theta_2\beta}\left(\frac{\delta}{\varepsilon}+1\right)^{1/p}d\varepsilon.$

Let's calculate each term of latter inequality.

$$\inf_{|h|\le\delta}\|\xi(h)\|_{L_p(\Omega)}=\inf_{|h|\le\delta}\left\|\|X(\cdot+h)-X(\cdot)\|_{L_q[a,b]}\right\|_{L_p(\Omega)}=0,\text{ when } h=0.$$

$$\int_0^{\theta_2\beta}\left(\frac{\delta}{\varepsilon}+1\right)^{1/p}d\varepsilon=\int_0^{\delta}\left(\frac{2\delta}{\varepsilon}\right)^{1/p}d\varepsilon+\int_\delta^{\theta_2C_q(2\delta)^{\tau_q}(b-a)^{1/q}}2^{1/p}d\varepsilon=$$
$$=(2\delta)^{1/p}\frac{p}{p-1}\delta^{1-1/p}+2^{1/p}\left(\theta_2C_q(2\delta)^{\tau_q}(b-a)^{1/q}-\delta\right)=$$
$$=\frac{2^{1/p}}{p-1}\delta+2^{1/p+\tau_q}C_q(b-a)^{1/q}\delta^{\tau_q}\theta_2=A+B\theta_2.$$

Function $\frac{A+B\theta_2}{\theta_2(1-\theta_2)}$ possesses its minimum value $\frac{B^2}{(2A+B)-2\sqrt{A(A+B)}}$ at point $\theta_2=\frac{\sqrt{A(A+B)}-A}{B}$. So,

$$\left\|\omega_q^1(X,\delta)\right\|_{L_p(\Omega)}\le\frac{2^{1/p+2\tau_q}C_q^2(b-a)^{2/q}\delta^{2\tau_q}}{\frac{2\delta}{p-1}+2^{\tau_q}C_q(b-a)^{1/q}\delta^{\tau_q}-2\sqrt{\frac{\delta^2}{(p-1)^2}+\frac{2^{\tau_q}C_q(b-a)^{1/q}\delta^{1+\tau_q}}{p-1}}}.$$

And now let's find the conditions under which stochastic process $X=\{X(t),t\in[a-\delta,b+\delta]\}$ belongs to the Besov space when $1<p<q<\infty$, $n=0$, $0<\alpha<1$. Under this assumptions we can use ω_q^1 instead of ω_q^2 in definition of the Besov space and norm in this particular Besov space $B_q^{\alpha p}[a,b]$ is following:

$$\|X\|_{B_q^{\alpha p}[a,b]}=\|X\|_{L_q[a,b]}+\left\|\frac{\omega_q^1(X,\delta)}{\delta^\alpha}\right\|_p^*=$$

$$=\left(\int_a^b|X(t)|^qdt\right)^{1/q}+\left(\int_0^\infty\left|\frac{\sup_{|h|\le\delta}\left(\int_a^b|X(t+h)-X(t)|^qdt\right)^{1/q}}{\delta^\alpha}\right|^p\frac{d\delta}{\delta}\right)^{1/p}.$$

Theorem 3. *If for stochastic process X $\sup_{t\in[a,b]}\left(E|X(t)|^p\right)^{1/p}<\infty$ and all the assumptions of the theorem 2 hold true, then:*

1) for $0<\tau_q\le1$ the stochastic process X belongs to the Besov space $B_q^{\alpha p}[a,b]$, $0<\alpha<\tau_q$ with probability one;

2) for $\tau_q > 1$ the stochastic process X belongs to the Besov space $B_q^{\alpha p}[a,b]$ with probability one for all $0 < \alpha < 1$.

Proof. Applying the theorem 4.1 [1] to the process X we will get that it belongs to the space $L_q[a,b]$ with probability one. Then:
for $0 < \tau_q \le 1$ the estimation for $\left\|\omega_q^1(X,\delta)\right\|_{L_p(\Omega)}$ from the theorem 2

$$\frac{2^{1/p+2\tau_q} C_q^2 (b-a)^{2/q} \delta^{2\tau_q}}{\frac{2\delta}{p-1} + 2^{\tau_q} C_q (b-a)^{1/q} \delta^{\tau_q} - 2\sqrt{\frac{\delta^2}{(p-1)^2} + \frac{2^{\tau_q} C_q (b-a)^{1/q} \delta^{1+\tau_q}}{p-1}}} \sim \delta^{\tau_q} \text{ as } \delta \to 0$$

and

$$E\int_{0+} \left|\frac{\omega_q^1(X,\delta)}{\delta^\alpha}\right|^p \frac{d\delta}{\delta} = \int_{0+} E\left|\frac{\omega_q^1(X,\delta)}{\delta^\alpha}\right|^p \frac{d\delta}{\delta} = \int_{0+} E\left|\omega_q^1(X,\delta)\right|^p \frac{d\delta}{\delta^{\alpha p+1}} =$$

$$= \int_{0+} \left(\|\omega_q^1(X,\delta)\|_{L_p(\Omega)}\right)^p \frac{d\delta}{\delta^{\alpha p+1}} \sim \int_{0+} \frac{\delta^{\tau_q p}}{\delta^{\alpha p+1}} d\delta < \infty, \quad \text{as} \quad 0 < \alpha < \tau_q;$$

for $\tau_q > 1$

$$\frac{2^{1/p+2\tau_q} C_q^2 (b-a)^{2/q} \delta^{2\tau_q}}{\frac{2\delta}{p-1} + 2^{\tau_q} C_q (b-a)^{1/q} \delta^{\tau_q} - 2\sqrt{\frac{\delta^2}{(p-1)^2} + \frac{2^{\tau_q} C_q (b-a)^{1/q} \delta^{1+\tau_q}}{p-1}}} \sim \delta^{2\tau_q - 1} \text{ as } \delta \to 0$$

and

$$E\int_{0+} \left|\frac{\omega_q^1(X,\delta)}{\delta^\alpha}\right|^p \frac{d\delta}{\delta} \sim \int_{0+} \left(\frac{\delta^{2\tau_q-1}}{\delta^\alpha}\right)^p \frac{d\delta}{\delta} < \infty, \quad \text{as} \quad 0 < \alpha < 2\tau_q - 1.$$

Since, $2\tau_q - 1 \ge 1$ the last statement is true for all $0 < \alpha < 1$.

4. Conclusion

In this paper the processes in the Besov space are investigated. It was found the conditions under which the trajectories of the stochastic process from space $L_p(\Omega)$ belong to certain Besov space $B_q^{sp}[a,b]$ when $s = \alpha$ and $p < q$.

Bibliography

1. Kozachenko, Yu.V. and Yakovenko, T.O., *Conditions under which stochastic processes belong to some function Orlicz spaces.* Bulletin of Kiyv University **5** , Kiev, (2002), 64–74.
2. Buldygin, V.V. and Kozachenko, Yu.V., *Metric characterization of random variables and random processes*, Amer. Math. Soc., Providence, RI (2000).
3. Härdle, W., Kerkyacharian, G., Picard, D., Tsybakov, A., *Wavelets, Approximation, and Statistical Applications,* Springer-Varlag, New York, (1998).

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