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BIAS CONTROL IN THE ESTIMATION OF SPECTRAL FUNCTIONALS

We consider estimators for integrals of a spectrum and a bispectrum for random fields $X(t)$, $t \in \mathbb{R}^d$, and present conditions guaranteeing the rate of convergence of bias to zero appropriate for dimensions $d = 1, 2, 3$.

1. Introduction

Let $X(t)$, $t \in \mathbb{R}^d$, be a real-valued measurable sixth-order weakly stationary random field with zero mean and spectral densities of second and third orders $f_2(\lambda)$, $f_3(\lambda_1, \lambda_2)$, λ, $\lambda_1, \lambda_2 \in \mathbb{R}^d$, that is the functions $f_2(\lambda) \in L_1(\mathbb{R}^d)$, $f_3(\lambda_1, \lambda_2) \in L_1(\mathbb{R}^{2d})$ such that for the cumulants of the second and third orders we have:

$$c_2(t) = \int_{\mathbb{R}^d} f_2(\lambda) e^{i(\lambda, t)} d\lambda, \quad c_3(t_1, t_2) = \int_{\mathbb{R}^{2d}} f_3(\lambda_1, \lambda_2) e^{i(\lambda_1, t_1) + i(\lambda_2, t_2)} d\lambda_1 d\lambda_2.$$

We will consider the problem of estimation of the spectral functionals

$$J_2(\varphi) = \int_{\mathbb{R}^d} \varphi(\lambda) f_2(\lambda) \, d\lambda \quad \text{and} \quad J_3(\psi) = \int_{\mathbb{R}^{2d}} \psi(\lambda_1, \lambda_2) f_3(\lambda_1, \lambda_2) \, d\lambda_1 d\lambda_2$$

for appropriate functions $\varphi(\lambda)$ and $\psi(\lambda_1, \lambda_2)$ (we suppose $\varphi(\lambda) f_2(\lambda) \in L_1(\mathbb{R}^d)$, $\psi(\lambda_1, \lambda_2) f_3(\lambda_1, \lambda_2) \in L_1(\mathbb{R}^{2d})$) and based on the observations $X(t)$, $t \in \mathcal{D}_T = [-T, T]^d$.

The estimation of spectral functionals is relevant to many statistical problems. These functionals can be used to represent some characteristics of random fields in nonparametric setting. On the other hand, these functionals appear in parametric estimation in spectral domain, e.g., when so-called minimum contrast estimators are studied.

Our main concern in the present paper is the bias evaluation for considered estimates and conditions which guarantee an appropriate rate of

convergence of the bias to zero, especially for the spatial data ($d \geq 2$), when the bias of estimates can be subject to so-called edge effects (see, e.g., [9], [10], [11]).

2. Bias control in the estimation of integrals of a spectrum

We will study here the estimator for the functional $J_2(\varphi)$ based on the tapered periodogram.

Consider the tapered data $h_T(t)X(t), t \in \mathcal{D}_T$, where $h_T(t) = h\left(\frac{t_1}{T}, ..., \frac{t_d}{T}\right)$ $= \prod_{i=1}^{d} h_1\left(\frac{t_i}{T}\right)$, $t = (t_1, ..., t_d) \in \mathbb{R}^d$, is a taper (i.e., we suppose that the taper factorizes), and $h_1(t)$, $t \in \mathbb{R}$, is a positive measurable symmetric ($h_1(t) = h_1(-t)$) function of bounded variation with bounded support: $h_1(t) = 0$ for $|t| > 1$.

Denote

$$H_k(\lambda) = \int h(t)^k e^{-i(\lambda, t)} dt, \ \ H_{k,T}(\lambda) = \int h_T(t)^k e^{-i(\lambda, t)} dt = T^d H_k(T\lambda).$$

Let

$$d_T^h(\lambda) = \int h_T(t) X(t) e^{-i(\lambda, t)} dt, \quad I_{2,T}^h(\lambda) = \frac{1}{(2\pi T)^d H_2(0)} d_T^h(\lambda) d_T^h(-\lambda)$$

be the finite Fourier transform and the periodogram of the second order based on tapered data (it is supposed that $H_2(0) \neq 0$).

Consider the following estimator for the functional $J_2(\varphi)$:

$$\hat{J}_{2,T}(\varphi) = \int_{\mathbb{R}^d} \varphi(\lambda) I_{2,T}^h(\lambda) d\lambda.$$

The bias of $\hat{J}_{2,T}(\varphi)$ can be represented as follows:

$$E\hat{J}_{2,T}(\varphi) - J_2(\varphi) =$$

$$= \int \int \varphi(\lambda) \left(f_2(\lambda + u) - f_2(\lambda)\right) K_{2,T}^h(u) \, du d\lambda \qquad (1)$$

$$= \int \int f_2(\lambda) \left(\varphi(\lambda + u) - \varphi(\lambda)\right) K_{2,T}^h(u) \, du d\lambda \qquad (2)$$

$$= \int \left(g_2(u) - g_2(0)\right) K_{2,T}^h(u) \, du, \qquad (3)$$

where $K_{2,T}^h(u) = ((2\pi T)^d H_2(0))^{-1} \left|H_{1,T}(u)\right|^2$, and we have denoted $g_2(u) = \int f_2(\lambda)\varphi(\lambda + u) \, d\lambda$.

Therefore, to evaluate the bias we need to analyze the asymptotic behavior of the expressions (1)-(3). Here we can apply standard arguments if we impose conditions of regularity on the functions f_2, φ, or, more generally,

on their convolution g_2. We will also need some restriction on the kernels $K_{2,T}^h(u)$, more precisely, on the kernels $K_2^h(u) = ((2\pi)^d H_2(0))^{-1} |H_1(u)|^2$ $= \prod_{i=1}^d k_2^h(u_i)$, where $k_2^h(u) = \left(2\pi \int h_1(t)^2 dt\right)^{-1} \times \left|\int h_1(t) e^{-itu} dt\right|^2$.

We state some conditions which assure the desirable rate of convergence of the bias of $J_{2,T}(\varphi)$ in the following theorem.

Theorem 1. *Let the kernel $k_2^h(u)$ satisfies the condition*

(i) $\int |u|^l k_2^h(u) \, du < \infty, l = 1, 2,$

and one of the following conditions holds:

(ii) *f_2 is twice boundedly differentiable and $\varphi \in L_1(\mathbb{R}^d)$;*

(iii) *φ is twice boundedly differentiable;*

(iv) *the convolution $g_2(u)$ is twice boundedly differentiable at zero.*

Then, as $T \to \infty$,

$$E\hat{J}_{2,T}(\varphi) - J_2(\varphi) = O\left(T^{-2}\right). \tag{4}$$

Proof. Consider the expression (1). We have

$$\int (f_2(\lambda + u) - f_2(\lambda)) K_{2,T}^h(u) \, du$$

$$= \frac{T^d}{(2\pi)^d H_2(0)} \int (f_2(\lambda + u) - f_2(\lambda)) |H_1(Tu)|^2 \, du$$

$$= \frac{1}{(2\pi)^d H_2(0)} \int \left(f_2\left(\lambda + \frac{u}{T}\right) - f_2(\lambda)\right) K_2^h(u) \, du. \tag{5}$$

If the condition (ii) holds, we can write in view of Taylor's theorem:

$$f_2\left(\lambda + \frac{u}{T}\right) - f_2(\lambda) = T^{-1} \sum_{j=1}^d u_j f_j'(\lambda) + O\left(T^{-2}\right) |u|^2$$

(uniformly in λ in O-term). Since the function $k_2^h(u)$ is even and condition (i) holds, the integral (5) can be evaluated as $O\left(T^{-2}\right)$ and since $\varphi \in L_1(\mathbb{R}^d)$ we obtain that the expression (1) is also $O\left(T^{-2}\right)$, as $T \to \infty$. Analogously we can deduce (4) from the expressions (2) or (3) applying the conditions (iii) or (iv) respectively.

Remark. We can see that if the standard normalizing factor $T^{d/2}$ is applied (under the conditions of Theorem 1), then the bias will be of order $T^{d/2-2}$, that is we can handle dimensions $d = 1, 2, 3$ using the tapered periodogram in the estimator for $J_2(\varphi)$.

We note that the representation for the bias in the form (1), where the kernels $K_{2,T}^h(u)$ are involved, and its further asymptotic analysis with the appeal to the properties of these kernels (and smoothness properties of the spectral density) is the approach the most commonly used in the literature

for bias evaluation of estimators for the spectral density and spectral integrals (see, [10], [11], among others, and also [4]–[7] for the case of untapered periodogram). However, when estimating the integrals of spectral densities, one has a possibility for a trade-off between the smoothness properties of a spectral density f and that of weight function φ: as Theorem 1 shows, one can relax conditions on f imposing at the same time stronger conditions on φ. This allows to treat the case of processes with long-range dependence.

We can write down another representation which appears to be quite helpful for the solution of the problem of bias control, namely, the following one

$$E\hat{J}_{2,T}(\varphi) - J_2(\varphi) \;=\; \frac{1}{(2\pi T)^d\, H_2(0)} \int \varphi(\lambda)\, e^{-i(\lambda,u)} \int c_2(u) \qquad (6)$$

$$\times \int \left(h_T(t+u) - h_T(t)\right) h_T(t)\, dt\, du\, d\lambda.$$

One can see that it is possible to achieve the desirable rate of convergence to zero of the above expression by imposing appropriate restrictions on a taper. We present here such possibilities for the case $d = 1$.

Theorem 2. *Let $d = 1$ and the following conditions hold:*

(i) $\int |u|\, |c_2(u)|\, du < \infty$;

(ii) *there exists a finite K such that $\int |h(t+u) - h(t)|\, dt < K|u|\ \forall u$;*

(iii) $\varphi \in L_1(\mathbb{R})$.

   Then, as $T \to \infty$, $E\hat{J}_{2,T}(\varphi) - J_2(\varphi) = O(T^{-1})$.

Proof follows immediately from the representation (6).

Note that the condition (ii) is a form of integrated Lipschitz condition. It is satisfied, for example, by functions $h(t)$ with uniformly bounded first derivatives and by $h(t)$ of bounded variation.

Theorem 2, which is a straightforward generalization of the result in [8] (obtained in connection with the estimation of the spectral density itself), relies on the very restrictive condition (i), which means that the process is weakly dependent. However, dealing with the integrals of the periodogram, we have the possibility to avoid the condition (i), replacing it with the condition on the function φ, as the next theorem states.

Theorem 3. *Let $d = 1$, the condition (ii) of Theorem 2 holds; $\varphi \in L_2(\mathbb{R})$ and its Fourier transform $\tilde{\varphi}$ satisfies the condition: $\int |u|\, |\tilde{\varphi}(u)|\, du < \infty$.*

   Then, as $T \to \infty$, $E\hat{J}_{2,T}(\varphi) - J_2(\varphi) = O(T^{-1})$.

Proof. The left hand side of (6) can be written in the following form:

$$\frac{1}{2\pi T H_2(0)} \int\int f_2(\gamma)\, e^{i\gamma u} d\gamma \int \varphi(\lambda)\, e^{-i\lambda u}$$

$$\times \int \left(h_T(t+u) - h_T(t)\right) h_T(t)\, dt\, du\, d\lambda = I,$$

and taking into account (ii), we have $|I| \leq \frac{1}{2\pi T H_2(0)} \int f_2(\gamma) \, d\gamma \int |u| \, |\tilde{\varphi}(u)| \, du$, which implies the statement of the theorem.

The next theorem shows that the better rate of convergence for the bias can be achieved with another set of conditions.

Theorem 4. *Let $d = 1$, the taper h is twice boundedly differentiable and suppose that $\varphi \in L_2(\mathbb{R})$ is an even function and its Fourier transform $\tilde{\varphi}$ satisfies the condition: $\int |u|^l |\tilde{\varphi}(u) \, c_2(u)| \, du < \infty$, $l = 1, 2$.*

Then, as $T \to \infty$, $E \hat{J}_{2,T}(\varphi) - J_2(\varphi) = O(T^{-2})$.

Proof is analogous to the proof of Theorem 1 and based again on Taylor's theorem which is applied now to $h_T(t + u) - h_T(t)$ in the left hand side of (6).

Note that with Theorem 4, as well as with Theorem 3 one can treat the processes with long range dependence.

We will not present here the asymptotics for the variance of $\hat{J}_{2,T}(\varphi)$; this standard expression, where the second-order and fourth-order spectral densities are involved, can be found, for example, in [2], [10] (under different sets of conditions)

3. Bias control in the estimation of integrals of a bispectrum

We study two estimates for the functional $J_3(\psi)$. Firstly, we consider the estimate

$$\hat{J}_{3,T}^h(\psi) = \int_{R^{2d}} \psi(\lambda_1, \lambda_2) \, I_{3,T}^h(\lambda_1, \lambda_2) \, d\lambda_1 d\lambda_2, \qquad (7)$$

based on the tapered periodogram of the third order

$$I_{3,T}^h(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^{2d} T^d H_3(0)} \, d_T^h(\lambda_1) d_T^h(\lambda_2) d_T^h(-\lambda_1 - \lambda_2),$$

and we suppose $H_3(0) \neq 0$.

Then we can write analogously to the second-order case the following representation for the bias of $\hat{J}_{3,T}^h(\psi)$:

$$
\begin{aligned}
&E \hat{J}_{3,T}(\psi) - J_3(\psi) \\
&= \int_{R^{4d}} \psi(\lambda_1, \lambda_2) \left(f_3(\lambda_1 + u_1, \lambda_2 + u_2) - f_3(\lambda_1, \lambda_2) \right) \qquad (8) \\
&\quad \times \Phi_{3,T}^h(u_1, u_2) \, du_1 du_2 d\lambda_1 d\lambda_2 \\
&= \int_{R^{4d}} f_3(\lambda_1, \lambda_2) \left(\psi(\lambda_1 + u_1, \lambda_2 + u_2) - \psi(\lambda_1, \lambda_2) \right) \qquad (9) \\
&\quad \times \Phi_{3,T}^h(u_1, u_2) \, du_1 du_2 d\lambda_1 d\lambda_2 \\
&= \int_{R^{2d}} \left(g_3(u_1, u_2) - g_3(0, 0) \right) \Phi_{3,T}^h(u_1, u_2) \, du_1 du_2, \qquad (10)
\end{aligned}
$$

where $\Phi^h_{3,T}(u_1, u_2) = \left((2\pi)^{2d} T^d H_3(0)\right)^{-1} H_{1,T}(u_1) H_{1,T}(u_2) H_{1,T}(-u_1 - u_2)$ and $g_3(u_1, u_2) = \int_{R^{2d}} \psi(\lambda_1, \lambda_2) f_3(\lambda_1 + u_1, \lambda_2 + u_2) d\lambda_1 d\lambda_2$. Analogously to the second-order case the asymptotic analysis of the above expression leads to the next theorem. We will denote $k^h_3(u) = \left((2\pi)^2 \int h_1(t)^3 dt\right)^{-1} \times \int h_1(t) e^{-itu} dt \int h^2_1(t) e^{-itu} dt$.

Theorem 5. *Let $k^h_3(u)$ satisfies the condition*

(i) $\int |u|^l k^h_3(u) du < \infty, l = 1, 2,$

and one of the following conditions holds:

(ii) *f_3 is twice boundedly differentiable and $\psi \in L_1\left(\mathbb{R}^{2d}\right)$;*

(iii) *φ is twice boundedly differentiable;*

(iv) *the convolution $g_3(u_1, u_2)$ is twice boundedly differentiable at zero.*

Then, as $T \to \infty$,

$$E\hat{J}^h_{3,T}(\varphi) - J_2(\varphi) = O\left(T^{-2}\right). \tag{11}$$

Proof. Analogously to Theorem 1, the proof is based on Taylor's formula and properties of the kernels

$$
\begin{aligned}
\Phi^h_{3,T}(u_1, u_2) &= \frac{1}{(2\pi)^{2d} T^d H_3(0)} H_{1,T}(u_1) H_{1,T}(u_2) H_{1,T}(-u_1 - u_2) \\
&= \frac{1}{(2\pi)^{2d} H_3(0)} T^{2d} H_1(Tu_1) H_1(Tu_2) H_1(T(-u_1 - u_2)) \\
&= \frac{1}{(2\pi)^{2d} \left(\int h_1(t)^3 dt\right)^d} T^{2d} \prod_{i=1}^{d} k_1(Tu_1^{(i)}) k_1(Tu_2^{(i)}) k_1(T(-u_1^{(i)} - u_2^{(i)})),
\end{aligned}
$$

where $k_1(u) = \int h_1(t) e^{-itu} dt$. Taking into account the above representation for $\Phi^h_{3,T}(u_1, u_2)$, consider the expression (8). Changing the variables: $Tu_1 = v_1, Tu_2 = v_2$ in (8) and developing f_3 in Taylor's series up to the second order term, we have the possibility to integrate over v_1, v_2 the products $\prod_{i=1}^{d} k_1(v_1^{(i)}) k_1(v_2^{(i)}) k_1(-v_1^{(i)} - v_2^{(i)})$, so that we obtain the sum of the terms of the form

$$const \times \int_{R^{2d}} \psi(\lambda_1, \lambda_2) \int v_l^{(i)} k^h_3(v_l^{(i)}) dv_l^{(i)} d\lambda_1 d\lambda_2, l = 1, 2, i = 1, ..., d,$$

which equal to zero (as k^h_3 is even), and the sum of the terms of the form

$$\int_{R^{2d}} \psi(\lambda_1, \lambda_2) \int \left(v_l^{(i)}\right)^2 k^h_3(v_l^{(i)}) dv_l^{(i)} d\lambda_1 d\lambda_2, l = 1, 2, i = 1, ..., d, \tag{12}$$

supplied by the multiplier of order $O\left(T^{-2}\right)$, in view of (i) and (ii) the integrals (12) are bounded, therefore, the result (11) follows. Analogously, the derivation of (11) can be based on (iii) or (iv).

Next, we consider the estimator for $J_3(\psi)$ of the form

$$\hat{J}_{3,T}^w(\psi) = \int_{R^{2d}} \psi(\lambda_1, \lambda_2) \hat{f}_{3,T}^w(\lambda_1, \lambda_2)\, d\lambda_1 d\lambda_2, \qquad (13)$$

where the estimator for the bispectrum $\hat{f}_{3,T}^w(\lambda_1, \lambda_2)$ we propose to construct in the following way:

$$\begin{aligned}
\hat{f}_{3,T}^w(\lambda_1, \lambda_2) &= \frac{1}{|\mathcal{D}_T|} \int_{\mathcal{D}_T} \int_{\mathcal{D}_T} \int_{\mathcal{D}_T} w_T(u)\, w_T(v)\, e^{-i(u,\lambda_1) - i(v,\lambda_2)} \\
&\quad \times X(t)\, X(t+u)\, X(t+v)\, dt\, du\, dv, \qquad (14)
\end{aligned}$$

it is supposed here (for convenience in notations) that we are given the observations $\left\{ X(t), t \in \mathcal{D}_{2T} = [-2T, 2T]^d \right\}$; the averaging weights are defined as follows: $w_T(u) = w\left(\frac{t}{T}\right) = \prod_{i=1}^d w_1\left(\frac{t_i}{T}\right) = \prod_{i=1}^d w_{1,T}(t_i), t \in \mathcal{D}_T$, where the function $w_1(t)$ has bounded support, $w_1(t) = 0$ for $|t| > 1$, and its Fourier transform (supposed to exist) $W_1(u) = \int w_1(t) e^{-it\lambda} dt$ is an even function such that $\int_R W_1(u)\, du = 1$. We define $W_{1,T}(u) = \int w_{1,T}(t) e^{-it\lambda} dt = TW_1(Tu)$, note that $W_{1,T}(u)$ is even and integrates to 1: $\int_R W_{1,T}(u)\, du = 1$. Denote $W_T(u) = \prod_{i=1}^d W_{1,T}(u_i)$.

The bias of the estimator $\hat{J}_{3,T}^w(\psi)$ can be represented in the form:

$$\begin{aligned}
&E\hat{J}_{3,T}^w(\psi) - J_3(\psi) \\
&= \int_{R^{4d}} \psi(\lambda_1, \lambda_2)\, (f_3(\lambda_1 + u_1, \lambda_2 + u_2) - f_3(\lambda_1, \lambda_2)) \\
&\quad \times W_T(u_1)\, W_T(u_2)\, du_1 du_2 d\lambda_1 d\lambda_2 \\
&= \int_{R^{4d}} f_3(\lambda_1, \lambda_2)\, (\psi(\lambda_1 + u_1, \lambda_2 + u_2) - \psi(\lambda_1, \lambda_2)) \\
&\quad \times W_T(u_1)\, W_T(u_2)\, du_1 du_2 d\lambda_1 d\lambda_2 \\
&= \int_{R^{2d}} (g_3(u_1, u_2) - g_3(0,0))\, W_T(u_1)\, W_T(u_2)\, du_1 du_2,
\end{aligned}$$

where $g_3(u_1, u_2) = \int f_3(\lambda_1, \lambda_2) \psi(\lambda_1 + u_1, \lambda_2 + u_2)\, d\lambda_1 d\lambda_2$. Comparing the above formula with (8)-(10), we note that the kernel under the integral sign factorizes, that is instead of $\Phi_{3,T}^h(u_1, u_2)$ we have a product $W_T(u_1) W_T(u_2)$, which makes the analysis simpler and completely the same as in the second-order case. Asymptotic analysis of the above representation leads to our next result.

Theorem 6. *Let the function $W_1(u)$ be such that*

(i) $\int |u|^l W_1(u)\, du < \infty, l = 1, 2,$

and one of the following conditions holds:

(ii) f_3 *is twice boundedly differentiable and* $\psi \in L_1\left(\mathbb{R}^{2d}\right)$;

(iii) ψ *is twice boundedly differentiable;*

(iv)   *the convolution* $g_3\left(u_1, u_2\right)$ *is twice boundedly differentiable at zero.*

   Then, as $T \to \infty,\ E\hat{J}_{3,T}^h(\varphi) - J_2\left(\varphi\right) = O\left(T^{-2}\right).$

Proof is analogous to the proof of Theorem 1.

Remark. Note that our estimator for the bispectrum (14) was tailored in such a specific way with the purpose to obtain the expression for the bias of $\hat{J}_{3,T}^w\left(\psi\right)$, which is analogous to the expressions (1)-(3) for the bias of $\hat{J}_{2,T}\left(\varphi\right)$, and therefore, allows us to handle the bias exactly as in the second-order case. The estimate (14) is of the form of weighted third-order sample moments, however with a truncated region of integration. Due to this truncation and specific weights we obtain the desirable rate of convergence of the bias for $d = 1, 2, 3$. The idea to use a truncated versions of a periodogram of the second order can be found in [12] for the discrete-time case (there the restriction on the region of summation of sample covariances is defined by means of some monotonically increasing function). Here we "simplified" the truncation in comparison with the mentioned authors, in our scheme the observation from $\mathcal{D}_{2T}\backslash\mathcal{D}_T$ are treated as "less significant" as they involved in the expression (14) to less extent (for the case of discrete time stochastic processes the estimate for a spectrum and a bispectrum of the form (14) can be found in [1]). Also, the second-order truncated periodogram is not guaranteed to be non-negative, but dealing with the third-order spectra we do not need to worry about this. We note also that other schemes of truncation (e.g., like that used in [12]) can be of use here and can be combined with the procedure of averaging of estimates built on overlapping d-dimensional intervals covering the region of observation, which can lead to better estimates.

   We will not present here the asymptotic expressions for the variance of estimates (7) and (13). For (7) this expression, which includes the spectral densities of second-, third-, fourth- and sixth-order, can be found in [2] (see also [3] for more detailed expression for the case of untapered periodogram which can be easily rewritten for tapered case as only the factor depending on the taper should be supplied), and for (13) the analogous expression can be written (however, this expression will not depend on weight functions, that is appear in the form, similar to the untapered case). Note that for the case of estimate (13) conditions guaranteeing the corresponding asymptotic representation for the variance become of simpler form than those for the estimate (7) due to the fact that all kernels, which appear under the integral sign in the corresponding derivations, factorize.

4. Conclusions

The biases of the estimators for integrals of a spectrum based on the tapered periodogram and estimators for integrals of a bispectrum based on the tapered third-order periodogram and on weighted sample third-order moments have been considered. For dimensions $d = 1, 2, 3$ conditions have been obtained which guarantee the appropriate rate of convergence of a bias to zero.

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