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ASYMPTOTIC EQUIVALENCE OF THE SOLUTIONS OF THE LINEAR STOCHASTIC ITO EQUATIONS IN THE HILBERT SPACE

We obtain the sufficient conditions of asymptotic equivalence in mean square and with probability one of linear ordinary and stochastic Ito equations in the Hilbert space.

1. Introduction

Qualitative theory of linear stochastic differential equations takes significant place in the general questions of stochastic equations. Stability investigation is one of the most important parts of this theory. A lot of papers are devoted to this subject.

A new approach to the investigation of asymptotical behavior of the solutions of linear stochastic equations is used. More precisely, finding an ordinary differential equation, whose solutions have asymptotical behavior similar to the one of the solutions of the stochastic equation, is proposed. Hence, the question of the stochastic equation stability is reduced to the one of the ordinary differential equation.

Stochastic systems, whose solutions have similar asymptotical behavior, analogically to the ordinary differential equations, are called *asymptotically equivalent*.

The paper [1] is devoted to this approach for the systems of stochastic equations in the space $\mathbf{R}^n$. The present work is a generalization of the former result to the Hilbert phase space. The sufficient conditions of the stochastic differential Ito equations asymptotical equivalence in mean square and with probability 1 are obtained.

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2. Definitions.

Let $(H, (\cdot,\cdot), \|\cdot\|)$ be a real separable Hilbert space (with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$). Let also $\{e_n | n \geq 1\}$ be some orthonormal basis in H.

We consider a stochastic differential equation in Hilbert space H [3,p.247]

$$dy = (A + B(t))ydt + D(t)ydW_t \tag{1}$$

on the probability space $(\Omega, \mathbf{F}, P)$, where $t \geq 0$, $x \in H$, and $A : H \to H$ is a linear bounded operator; linear operators $B(t), D(t) : H \to H$ are bounded for each $t \geq 0$; W_t is a scalar Wiener process, defined for $t \geq 0$ on probability space $(\Omega, \mathbf{F}, P)$; $\{\mathcal{F}_t, t \geq 0\}$ is a filtration corresponding to W_t.

Under conditions of the theorem below, the stochastic equation (1) has the (strong) unique solution $y(t) \equiv y(t, \omega) \in H$ with initial condition $y(0) = y_0$.

Consider a linear differential equation in H

$$\frac{dx}{dt} = Ax. \tag{2}$$

We compare solutions of the stochastic differential equation (1) and solutions of the ordinary differential equation (2) with some properly chosen random initial conditions.

Definition. *If for each solution $y(t)$ of the equation (1) there is solution $x(t)$ of the equation (2), such that*

$$\lim_{t\to\infty} E|x(t) - y(t)|^2 = 0,$$

then the equation (1) is called asymptotically equivalent *to the equation (2) in mean square.*

Definition. *If for each solution $y(t)$ of the equation (1) there is solution $x(t)$ of the equation (2), such that*

$$\mathrm{P}\{\lim_{t\to\infty} |x(t) - y(t)| = 0\} = 1,$$

then the equation (1) is called asymptotically equivalent *to the equation (2) with probability* 1.

3. Decomposition of Hilbert space.

Consider a complex covering of space H (see [2,p.26]). Let $\tilde{H}$ be a complex covering of space H, i.e. $\tilde{H} = \{x_1 + ix_2 | x_1, x_2 \in H\}$.

The spectrum of A we denote by $\sigma(A)$. Since A is a real operator, we obtain that $\sigma(A)$ is symmetrical set relative to real axis.

Assume that $\sigma(A)$ is not connected as a set. The closed part of the spectrum, whose complement is closed in $\sigma(A)$ is said to be a *spectral set* (see [2,p.32]).

Let

$$\sigma(A) = \sigma_1(A) \bigcup \sigma_2(A),$$

where $\sigma_1(A), \sigma_2(A)$ are disjoint spectral sets.

It follows from [2,p.33] that the space $\tilde{H}$ can be decomposed in a direct sum of subspaces invariant with respect to operator A

$$\tilde{H} = \tilde{H}_1 \oplus \tilde{H}_2,$$

such that the spectrum $\sigma(A|\tilde{H}_i)$ of the restriction of A on the subspace $\tilde{H}_i$ coincides with $\sigma_i(A), i = 1, 2$.

Let $H_i = Re\tilde{H}_i, i = 1, 2$. If $\sigma_{1,2}(A)$ are symmetric sets w.r.t. real axis (further, we will use this construction for symmetric spectral sets only), then H_i are subspaces invariant with respect to operator A. Thus,

$$H = H_1 \oplus H_2.$$

Let P_i be a projector on the subspace $H_i, i = 1, 2$. It is easy to see that $I = P_1 + P_2$, where I is identity operator on H. It is proved in [2,p.25] that one can modify the scalar product $(\cdot, \cdot)$ in H to the topologically equivalent one $(\cdot, \cdot)'$, where projectors P_1 and P_2 become orthoprojectors and the scalar product in H is given by the formula

$$(x, y)' = (P_1 x, P_1 y) + (P_2 x, P_2 y).$$

Obviously, $(P_i x, P_i y)$ is a scalar product in $H_i, i = 1, 2$. Without loss of generality we can assume that $(\cdot, \cdot)' = (\cdot, \cdot)$.

It is clear that H_1, H_2 are Hilbert subspaces in H. From here we will call the projector P_i and the subspace $H_i = P_i H$ the *spectral projector* and *invariant subspace*, corresponding each other and spectral set $\sigma_i(A), i = 1, 2$.

4. Asymptotical equivalence of the stochastic equations in Hilbert space

The following theorem is a generalization of Levinson theorem (see, for instance, [4,p.159]) on the asymptotical equivalence of the ordinary differential equation to the case of stochastic differential equations in a Hilbert space.

Theorem. *Assume the solutions of equation (2) are bounded on* $[0, \infty)$ *and the spectrum of operator* A *consists of two spectral sets*

$$\sigma(A) = \sigma_-(A) \bigcup \sigma_0(A),$$

such that $\sigma_-(A) \subset \{z \in \boldsymbol{C} | Rez < 0\}$ *and* $\sigma_0(A) \subset \{z \in \boldsymbol{C} | Rez = 0\}$.
Let A_0 *be a restriction of operator* A *on invariant subspace* H_0, *corresponding to the spectral set* $\sigma_0(A)$. *If* A_0 *is similar to some skew-hermitian operator (i.e.* $A_0 = S^{-1}(iQ)S, Q = Q^*$*) and*

$$\int_0^\infty \|B(t)\| dt \le K_1 < \infty, \int_0^\infty \|D(t)\|^2 dt \le K_1 < \infty, \tag{3}$$

for some $k > 0$, *then equation (1) is asymptotically equivalent to equation (2) in mean square and with probability* 1.

Proof. It follows from the conditions of the theorem and [3,p.272] that there exists the unique solution of equation (1) with initial condition $y(0) = y_0$, which can be represented in the form

$$y(t) = X(t)y(0) + \int_0^t X(t-\tau)B(\tau)y(\tau)d\tau + \int_0^t X(t-\tau)D(\tau)y(\tau)dW_\tau, \tag{4}$$

where $t \ge 0$ and $X(t) = e^{At}$ is an operator exponent (see [2,p.41]). Taking into account the boundedness of the solutions on the positive semiaxis, we get that there is a constant $K_2 > 0$ such that for all $t \ge 0$ the following estimate holds

$$\|e^{At}\| \le K_2.$$

Let's prove that all the solutions of equation (1) are bounded in mean square under condition (3).

For this we use the presentation of the solution of equation (1) in the form (4). Let's estimate expectation of $|y(t)|^2$.

$$E|y(t)|^2 \le 3\|X(t)\|^2 E|y(0)|^2 + 3E\left|\int_0^t X(t-\tau)B(\tau)y(\tau)d\tau\right|^2 +$$

$$+3E\left|\int_0^t X(t-\tau)D(\tau)y(\tau)dW_\tau\right|^2 \le 3K_2^2 E|y(0)|^2 +$$

$$+3E\left(\int_0^t \sqrt{\|X(t-\tau)\|}\sqrt{\|X(t-\tau)\|}\sqrt{\|B(\tau)\|}\sqrt{\|B(\tau)\|}|y(\tau)|d\tau\right)^2 +$$

$$+3\int_0^t E\left|X(t-\tau)D(\tau)y(\tau)\right|^2 d\tau \le 3K_2^2 E|y(0)|^2 +$$

$$+3\int_0^t \|X(t-\tau)\|\|B(\tau)\|E|y(\tau)|^2 d\tau \int_0^t \|X(t-\tau)\|\|B(\tau)\|d\tau +$$

$$+3\int_0^t \|X(t-\tau)\|^2\|D(\tau)\|^2 E|y(\tau)|^2 d\tau \le$$

$$\le 3K_2^2\left(E|y(0)|^2+\int_0^\infty \|B(\tau)\|d\tau\int_0^t \|B(\tau)\|E|y(\tau)|^2d\tau+\right.$$

$$\left.+\int_0^t \|D(\tau)\|^2 E|y(\tau)|^2 d\tau\right).$$

The Gronuoll-Bellman inequality implies

$$E|y(t)|^2 \le 3K_2^2 E|y(0)|^2 e^{3K_2^2\int_0^t (K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau} \le$$

$$\le 3K_2^2 E|y(t_0)|^2 e^{3K_2^2\int_0^\infty (K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau} \le \widetilde{K}E|y(0)|^2, \tag{5}$$

where $\widetilde{K}=3K_2^2 e^{3K_2^2\int_0^\infty (K_1\|B(\tau)\|+\|D(\tau)\|^2)d\tau}$.

Furthermore, the conditions of the theorem and the computations above imply that space H is decomposable in the direct sum of Hilbert subspaces

$$H = H_- \oplus H_0 = P_-H \oplus P_0H,$$

where H_- is an invariant subspace corresponding to the spectral set $\sigma_-(A)$, P_- is an orthoprojector on H_-, H_0 is an invariant subspace, corresponding to the spectral set $\sigma_0(A)$, P_0 is an orthoprojector on H_0.

It can be obtained from [2,p.122] that equation (2) is equivalent to the system of two independent equations

$$\frac{dx_-}{dt} = A_-x_-, \frac{dx_0}{dt} = A_0x_0, \tag{6}$$

where $x_- = P_-x, x_0 = P_0x, A_- = P_-A, A_0 = P_0A$.

Then

$$e^{At} = \begin{pmatrix} e^{A_-t} & 0 \\ 0 & e^{A_0t} \end{pmatrix} = X_-(t) + X_0(t),$$

where

$$X_-(t) = \begin{pmatrix} e^{A_-t} & 0 \\ 0 & 0 \end{pmatrix}, X_0(t) = \begin{pmatrix} 0 & 0 \\ 0 & e^{A_0t} \end{pmatrix}.$$

The evolutionary property of matrix exponent allows to transform the equation (4) in the following form

$$y(t) = X(t)\left[y(t_0) + \int_0^\infty X_0(0-\tau)B(\tau)y(\tau)d\tau+\right.$$

$$+\int_0^\infty X_0(0-\tau)D(\tau)y(\tau)dW_\tau\Bigg]+$$

$$+\int_t^t X_-(t-\tau)B(\tau)y(\tau)d\tau+\int_0^t X_-(t-\tau)D(\tau)y(\tau)dW_\tau- \tag{7}$$

$$-\int_t^\infty X_0(t-\tau)B(\tau)y(\tau)d\tau-\int_t^\infty X_0(t-\tau)D(\tau)y(\tau)dW_\tau.$$

For each solution $y(t) \equiv y(t,\omega)$ of the equation (1) with the initial condition $y(0) = y_0$ we correspond the solution $x(t)$ of the equation (2) with the initial condition

$$x(0) = y(0) + \int_0^\infty X_0(-\tau)B(\tau)y(\tau)d\tau + \int_0^\infty X_0(-\tau)D(\tau)y(\tau)dW_\tau. \tag{8}$$

Since A_0 is similar to skew-hermitian operator, it follows from [2,p.113] that there is a constant $K_3 > 0, t \in (-\infty,\infty)$ such that $\|X_0\| \le K_3$.

From the stated above and inequality (5) we get that all improper integrals involved in (7) are convergent in mean square.

Since the solution $x(t)$ of linear equation (2) and the solution $y(t) \equiv y(t,\omega)$ of stochastic equation (1) are defined by initial conditions, equation (8) defines correspondence modulo stochastic equivalence between the solution set $\{y(t) \equiv y(t,\omega)\}$ of equation (1) and solution set $\{x(t)\}$ of equation (2).

The rest of the proof is analogical to the one in [1].

5. Example.

Consider equation (2) in the space $H = (L_2[0,1])^3$, such that $x = x(t,u) = (x_1,x_2,x_3), x_i \in L_2[0,1], t \ge 0, u \in [0,1]$ and operator A is defined as follows

$$A = \begin{pmatrix} 0 & I & 0 \\ T^2 & 0 & 0 \\ 0 & 0 & V \end{pmatrix}$$

where I is the identity operator in $L_2[0,1]$, operators T and V are defined as

$$(Tz)(u) = z(u) - \int_0^1 \varphi(u)\varphi(s)z(s)ds, \tag{9}$$

$$(Vz)(u) = (u-2)z(u),$$

here $\varphi(u)$ is a continuous not identically zero function on $[0,1]$ such that $c_0 = \int_0^1 \varphi^2(s)ds \ne 1$

It is easy to see, that $\sigma(V) = [-2, -1]$.

Consider operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ T^2 & 0 \end{pmatrix}$$

which can be represented in the form $\mathcal{A} = S(iQ)S^{-1}$, where

$$Q = \begin{pmatrix} T & 0 \\ 0 & -T \end{pmatrix}, S = \begin{pmatrix} I & I \\ -iT & iT \end{pmatrix}, S^{-1} = \frac{1}{2}\begin{pmatrix} I & iT^{-1} \\ I & -iT^{-1} \end{pmatrix}.$$

Under stated above conditions the inverse to T operator exists and is defined by relation

$$(T^{-1}y)(u) = y(u) + \frac{1}{1-c_0}\int_0^1 \varphi(u)\varphi(s)y(s)ds.$$

Thus, operator $\mathcal{A}$ is similar to skew-hermitian operator, which implies that its spectrum belongs to the imaginary axis.

Hence, the space H can be decomposed into a direct sum of subspaces

$$H = H_1 \oplus H_2,$$

where $H_1 = \{(x_1, x_2, 0) | x_1, x_2 \in L_2[0, 1]\}, H_2 = \{(0, 0, x_3) | x_3 \in L_2[0, 1]\}$.

The restriction of operator A on the subspace H_1 is the operator $\mathcal{A}$, and the restriction on H_2 is the operator V. Therefore, $\sigma(A) = \sigma_-(A) \bigcup \sigma_0(A)$, where $\sigma_-(A) = [-2, -1]$ and $\sigma_0(A)$ is some closed subset of imaginery axis. Hence, equation (2) satisfies the conditions of the theorem.

Thus, for arbitrary operators $B(t), D(t) : H \to H$, which satisfy estimates (3) stochastic equation (1) will be asymptotically equivalent to equation (2).

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