Theory of Stochastic Processes
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ON ONE STOCHASTIC OPTIMAL CONTROL PROBLEM WITH VARIABLE DELAY

A stochastic optimal control problem with variable delays in control is considered. The maximum principle for nonlinear stochastic control system with constrains in the right end of trajectory is proved.

1. Introduction

The stochastic differential equations with delay find much exhibits in description of the real systems, more or less are subjected to the influence of the random noises. Many problems in theories of automatic regulation, mechanical engineering, economy, automatics are described by stochastic differential equations with delay. Therefore problems of optimal control for systems, described by such equations, are actual at present [1,2]. Earlier the problems of stochastic optimal control with variable delay in phase [3] and with constant delay in control [4] were considered. The present work is devoted to the problem of stochastic optimal control with variable delay in control with constrains on right endpoint of trajectory. Our objective is to obtain a necessary condition for optimal control, when diffusion coefficient contains the control variable with delay.

2. Statement of the main problem

Let (Ω, F, P) be a complete probability space with filtration $\{F^t : t_0 \leq t \leq t_1\}$ generated by Wiener process w_t, $F^t = \sigma(w_s, t_0 \leq s \leq t)$. Let $L_F^2(t_0, t_1, R^n)$ be a space of predictable processes $x_t(\omega)$ such that $E\int_{t_0}^{t_1} |x_t|^2 \, dt < +\infty$.

Consider the following stochastic system with variable delay in control:

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$$dx_t = g(x_t, u_t, u_{t-h(t)}, t)dt + f(x_t, u_t, u_{t-h(t)}, t)dw_t; \;\; t \in (t_0, t_1], \qquad (1)$$

$$x_{t_0} = x_0 \qquad (2)$$

$$u_t = Q(t), \;\; t \in [t_0 - h(t_0), t_0) \qquad (3)$$

$$u_t(\omega) \in U_\partial \equiv \{u(\cdot) \in L_F^2(t_0, t_1; R^m)|\; u(\omega) \in U \subset R^m, \;\; a.s.\} \qquad (4)$$

where U is a nonempty bounded set, $Q(t)$ is a piecewise continuous non-random function, $h(t) \geq 0$ is a continuously differentiable, non-random function such that $\frac{dh(t)}{dt} < 1$.

Let it is necessary to minimize the following functional inside the set of admissible controls:

$$J(u) = E\left\{p(x_{t_1}) + \int_{t_0}^{t_1} l(x_t, u_t, t)dt\right\} \qquad (5)$$

with a condition

$$Eq(x_{t_1}) \in G \subset R^k, \qquad (6)$$

where G is a closed convex set in R^k.

Let assume that the following requirements are satisfied:

I. The functions l, g, f and their derivatives are continuous in (x, u, t) :

$$l(x, u, t) : R^n \times R^m \times [t_0, t_1] \to R^1;$$

$$g(x, u, \nu, t) : R^n \times R^m \times R^m \times [t_0, t_1] \to R^n;$$

$$f(x, u, \nu, t) : R^n \times R^m \times R^m \times [t_0, t_1] \to R^{n\times n}.$$

II. The functions l, g, f are twice continuously differentiable with respect to x, l_{xx}, g_{xx}, f_{xx}, bounded and of linear growth:

$$(1+|x|)^{-1}(|g(x,u,\nu,t)| + |f(x,u,\nu,t)| + |g_x(x,u,\nu,t)| + |f_x(x,u,\nu,t)|) \leq N;$$

$$(1+|x|)^{-1}(|l(x,u,t)|) + (l_x(x,u,t)) \leq N.$$

III. Function $p(x) : R^n \to R^1$ is twice continuously differentiable and

$$|p(x)| + |p_x(x)| \le N(1 + |x|);\ |p_{xx}(x)| \le N.$$

IV. Function $q(x) : R^m \to R^k$ is twice continuously differentiable and

$$|q(x)| + |q_x(x)| \le N(1 + |x|); |q_{xx}(x)| \le N.$$

At first the stochastic optimal control problem (1)-(5) is being considered.

Theorem 1. *Let conditions I-III hold and (x_t^0, u_t^0) is a solution of problem (1)-(5). Then there exist random processes $(\psi_t, \beta_t) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$ and $(\Phi_t, K_t) \in L_F^2(t_0, t_1; R^n) \times L_F^2(t_0, t_1; R^{n\times n})$, which are the solutions of the following adjoint equations:*

$$\begin{cases} d\psi_t = -H_x(\psi_t, x_t^0, u_t^0, \nu_t^0, t)dt + \beta_t dw_t, \quad t_0 \le t < t_1, \\ \psi_{t_1} = -p_x(x_{t_1}^0), \end{cases} \tag{7}$$

$$\begin{cases} d\Phi_t = -[g_x^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t + \Phi_t g_x(x_t^0, u_t^0, \nu_t^0, t)+ \\ +f_x^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t f_x(x_t^0, u_t^0, \nu_t^0, t)dt + f_x^*(x_t^0, u_t^0, \nu_t^0, t)K_t+ \\ +K_t f_x(x_t^0, u_t^0, \nu_t^0, t) + H_{xx}(\psi_t, x_t^0, u_t^0, \nu_t^0, t)]dt + K_t dw_t,\ t_0 \le t < t_1, \\ \Phi_{t_1} = -p_{xx}(x_{t_1}^0), \end{cases} \tag{8}$$

and $\forall\, u \in U$ a.s. the following equations hold:

$$\begin{cases} \Delta_u H(\psi_\theta, x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) + [\Delta_\nu H(\psi_z, x_z^0, u_z^0, \nu_z^0, z)+ \\ +0.5\Delta_\nu f^*(x_z^0, u_z^0, \nu_z^0, z)\Phi_z \Delta_\nu f(x_z^0, u_z^0, \nu_z^0, z)]\Big|_{z=s(\theta)} s'(\theta)+ \\ +0.5\Delta_u f^*(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta)\Phi_\theta \Delta_u f(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) \le 0, \\ a.e.\, \theta \in [t_0, t_1 - h(t_1)) \\ H(\psi_\theta, x_\theta^0, u, \nu_\theta^0, \theta) - H(\psi_\theta, x_\theta^0, u_\theta^0, \nu_\theta^0, \theta)+ \\ +0.5\Delta_u f^*(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta)\Phi_\theta \Delta_u f(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) \le 0, \\ for\ a.e.\ \theta \in [t_1 - h(t_1), t_1], \end{cases} \tag{9}$$

where $t = s(\tau)$ is a solution of the equation $\tau = t - h(t)$, $\nu_t = u_{t-h(t)}$,

$$\Delta_u y(x_t, u_t, \nu_t, t) = y(x_t, u, \nu_t, t) - y(x_t, u_t, \nu_t, t),$$

$$\Delta_\nu y(x_t, u_t, \nu_t, t) = y(x_t, u_t, u, t) - y(x_t, u_t, \nu_t, t),$$

$$H(\psi_t, x_t, u_t, \nu_t, t) = \psi_t^* \cdot g(x_t, u_t, \nu_t, t) + \beta_t^* \cdot f(x_t, u_t, \nu_t, t) - l(x_t, u_t, t).$$

Proof. Let $u_t = u_t^0 + \Delta u_t$ be some admissible control and $x_t = x_t^0 + \Delta x_t$ be corresponding to this control trajectory of system (1)-(4). Let's use the following identity:

$$\begin{cases} d(\Delta x_t) = [g(x_t, u_t, \nu_t, t) - g(x_t^0, u_t^0, \nu_t^0, t)]dt + [f(x_t, u_t, \nu_t, t) - \\ -f(x_t^0, u_t^0, \nu_t^0, t)]dw_t = [\Delta_u g(x_t^0, u_t^0, \nu_t, t) + \Delta_\nu g(x_t^0, u_t^0, \nu_t^0, t) + \\ +g_x(x_t^0, u_t, \nu_t, t)\Delta x_t + 0.5\Delta x_t^* g_{xx}(x_t^0, u_t, \nu_t, t)\Delta x_t]dt + \\ +[\Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + f_x(x_t^0, u, \nu_t, t)\Delta x_t + \\ +0.5\Delta x_t^* f_{xx}(x_t^0, u_t, \nu_t, t)\Delta x_t]dw_t + \eta_t^1, \\ t \in (t_0, t_1], \quad \Delta x_t = 0, \quad t \in [t_0 - h(t_0), t_0] \end{cases} \tag{10}$$

where

$$\eta_t^1 = \left\{ \int_0^1 [g_x^*(x_t^0 + \mu\Delta x_t, u_t, \nu_t, t) - g_x^*(x_t^0, u_t^0, \nu_t^0, t)]\Delta x_t d\mu + \right.$$

$$+0.5 \cdot \int_0^1 \Delta x_t^* [g_{xx}^*(x_t^0 + \mu\Delta x_t, u_t, \nu_t, t) - g_{xx}^*(x_t^0, u_t^0, \nu_t^0, t)]\Delta x_t d\mu\} dt +$$

$$+ \left\{ \int_0^1 [f_x^*(x_t^0 + \mu\Delta x_t, u_t, \nu_t, t) - f_x^*(x_t^0, u_t^0, \nu_t^0, t)]\Delta x_t d\mu + \right.$$

$$+0.5 \cdot \int_0^1 \Delta x_t^* [f_{xx}^*(x_t^0 + \mu\Delta x_t, u_t, \nu_t, t) - f_{xx}^*(x_t^0, u_t^0, \nu_t^0, t)]\Delta x_t d\mu\} dw_t.$$

According to Ito's formula [5] we have:

$$\begin{aligned} & d(\psi_t^* \cdot \Delta x_t) = d\psi_t^* \Delta x_t + \psi_t^* d\Delta x_t + \{\beta_t^* [\Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + \\ + \ & f_x(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t + 0.5\Delta x_t^* f_{xx}(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t] + \\ + \ & \beta_t^* \int_0^1 \left[f_x(x_t^0 + \mu\Delta x_t, u_t^0, \nu_t^0, t) - f_x(x_t^0, u_t^0, \nu_t^0, t) \right] \Delta x_t d\mu + 0.5 \times \\ \times \ & \beta_t^* \int_0^1 \Delta x_t^* \left[f_{xx}(x_t^0 + \mu\Delta x_t, u_t^0, \nu_t^0, t) - f_{xx}(x_t^0, u_t^0, \nu_t^0, t) \right] \Delta x_t d\mu\} dt \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
&d(\Delta x_t^* \cdot \Phi_t \cdot \Delta x_t) = \Delta x_t^* \cdot d\Phi_t \cdot \Delta x_t + \Delta x_t^* \cdot \Phi_t d\Delta x_t + d\Delta x_t^* \cdot \Phi_t \cdot \Delta x_t + \\
+ &\ \{ K_t^* [\Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + f_x(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t + \\
+ &\ 0.5 \Delta x_t^* f_{xx}(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t] + [\Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + \\
+ &\ f_x(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t + 0.5\Delta x_t^* f_{xx}(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t] \times \\
\times &\ \Phi_t \cdot [\Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + \\
+ &\ f_x(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t + 0.5\Delta x_t^* f_{xx}(x_t^0, u_t^0, \nu_t^0, t)\Delta x_t]\} dt \qquad (12)
\end{aligned}
$$

The almost certainly uniqueness of the solutions of adjoint stochastic equations (7), (8) follow from [6].

Taking into consideration (10)-(12), the expression of increment of a functional (5) along the admissible control takes a form:

$$
\begin{aligned}
&\Delta J\left(u^0\right) = E\left\{ p(x_{t_1}) - p(x_{t_1}^0) + \int_{t_0}^{t_1} [l(x_t, u_t, t) - l(x_t^0, u_t^0, t)]dt \right\} = \\
= &\ -E\int_{t_0}^{t_1} \left[\psi_t^* \Delta_u g(x_t^0, u_t^0, \nu_t^0, t) + \psi_t^* \Delta_\nu g(x_t^0, u_t^0, \nu_t^0, t) + \beta_t^* \Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \right. \\
+ &\ \beta_t^* \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) - \Delta_u l(x_t^0, u_t^0, t) + 0.5 \cdot \Delta_u f^*(x_t^0, u_t^0, \nu_t^0, t) \times \\
\times &\ \Phi_t \Delta_u f(x_t^0, u_t^0, \nu_t^0, t)dt + \Delta_u f^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) + \\
+ &\ \Delta_u f^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t \Delta_u f(x_t^0, u_t^0, \nu_t^0, t) + \Delta_\nu f^*(x_t^0, u_t^0, \nu_t^0, t) \times \\
\times &\ \left. \Phi_t \Delta_\nu f(x_t^0, u_t^0, \nu_t^0, t) \right] dt - \eta_{t_0}^{t_1}, \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
&\eta_{t_0}^{t_1} = E\int_0^1 \left[p_x^*(x_{t_1}^0 + \mu\Delta x_{t_1}) - p_x^*(x_{t_1}^0)\right] \Delta x_{t_1} d\mu + 0.5 \times \\
\times &\ E\int_0^1 \Delta x_{t_1}^* \left[p_{xx}^*(x_{t_1}^0 + \mu\Delta x_{t_1}) - p_{xx}^*(x_{t_1}^0)\right] \Delta x_{t_1} d\mu + \\
+ &\ E\int_{t_0}^{t_1} \left\{ \int_0^1 [l_x^*(x_t^0 + \mu\Delta x_t, u_t, t) - l_x^*(x_t^0, u_t, t)]\Delta x_t d\mu \right\} dt + 0.5 \times \\
\times &\ E\int_{t_0}^{t_1} \left\{ \int_0^1 \Delta x_t^* [l_{xx}^*(x_t^0 + \mu\Delta x_t, u_t, t) - l_{xx}^*(x_t^0, u_t, t)]\Delta x_t d\mu \right\} dt +
\end{aligned}
$$

$$+ \quad E\int_{t_0}^{t_1}\left\{\int_0^1[\psi_t^*(g_x(x_t^0+\mu\Delta x_t,u_t,\nu_t,t)-g_x(x_t^0,u_t^0,\nu_t^0,t))]\Delta x_t d\mu\right\}dt+0.5\times$$

$$\times \quad E\int_{t_0}^{t_1}\left\{\int_0^1\Delta x_t^*[\psi_t^*(g_{xx}(x_t^0+\mu\Delta x_t,u_t,\nu_t,t)-g_{xx}(x_t^0,u_t^0,\nu_t^0,t))]\Delta x_t d\mu\right\}dt+$$

$$+ \quad E\int_{t_0}^{t_1}\int_0^1\beta_t^*[f_x(x_t^\circ+\mu\Delta x_t,u_t,\nu_t,t)-f_x(x_t^0,u_t^0,\nu_t^0,t)]\Delta x_t d\mu dt+0.5\times$$

$$\times \quad E\int_{t_0}^{t_1}\int_0^1\Delta x_t^*\cdot\beta_t^*[f_{xx}(x_t^\circ+\mu\Delta x_t,u_t,\nu_t,t)-f_{xx}(x_t^0,u_t^0,\nu_t^0,t)]\Delta x_t d\mu dt. \quad (14)$$

Let's consider the following spike variation:

$$\Delta u_t=\Delta u_{t,\varepsilon}^\theta=\begin{cases}0, & t\notin[\theta,\theta+\varepsilon),\ \varepsilon>0,\\ u^*-u_t^0, & t\in[\theta,\theta+\varepsilon),\ \ u^*\in L_2\left(\Omega,F^\theta,P;R^m\right),\end{cases}$$

where $x_{t,\varepsilon}^\theta$ is a trajectory corresponding to control $u_{t,\varepsilon}^\theta=u_t^0+\Delta u_{t,\varepsilon}^\theta\cdot u_{t,\varepsilon}^\theta=u_t^0+\Delta u_{t,\varepsilon}^\theta$.

Then (13) takes a form:

$$\Delta_\theta J\left(u^0\right)=-E\int_\theta^{\theta+\varepsilon}\left[\psi_t^*\Delta_{u^*}g\left(x_t^0,u_t^0,\nu_t^0,t\right)+\psi_t^*\Delta_{\nu^*}g\left(x_t^0,u_t^0,\nu_t^0,t\right)+\right.$$

$$+\beta_t^*\Delta_{u^*}f\left(x_t^0,u_t^0,\nu_t^0,t\right)+\beta_t^*\Delta_{\nu^*}f\left(x_t^0,u_t^0,\nu_t^0,t\right)+0.5\Delta_{u^*}f^*(x_t^0,u_t^0,\nu_t^0,t)\Phi_t\times$$
$$\times\Delta_{u^*}f(x_t^0,u_t^0,\nu_t^0,t)dt+\Delta_{u^*}f^*\left(x_t^0,u_t^0,\nu_t^0,t\right)\Phi_t\Delta_{\nu^*}f\left(x_t^0,u_t^0,\nu_t^0,t\right)+$$
$$+\Delta_{\nu^*}f^*\left(x_t^0,u_t^0,\nu_t^0,t\right)\Phi_t\Delta_{u^*}f\left(x_t^0,u_t^0,\nu_t^0,t\right)+$$
$$\left.+\Delta_{\nu^*}f^*\left(x_t^0,u_t^0,\nu_t^0,t\right)\Phi_t\Delta_{\nu^*}f\left(x_t^0,u_t^0,\nu_t^0,t\right)-\Delta_{u^*}l\left(x_t^0,u_t^0,t\right)\right]dt+\eta_\theta^{\theta+\varepsilon}.$$

Lemma 1. *Let conditions I-III hold.*

Then $E\left|x_{t,\varepsilon}^\theta-x_t^0\right|^2\le N\varepsilon,$ *if* $\varepsilon\to0$, *where trajectory of systems (1)-(4) corresponds to control* $u_{t,\varepsilon}^\theta=u_t^0+\Delta u_{t,\varepsilon}^\theta$.

Proof. Let's designate the following:

$\tilde{x}_{t,\varepsilon}=x_{t,\varepsilon}^\theta-x_t^0$.

It is clear that $\forall t\in[t_0,\theta)\quad\tilde{x}_{t,\varepsilon}=0$

Then for $\forall t\in[\theta,\,\theta+\varepsilon)$

$$d\tilde{x}_{t,\,\varepsilon}=\left[g(x_t^0+\varepsilon\tilde{x}_{t,\varepsilon},u^*,\nu_t^0,t)-g(x_t^0,u_t^0,\nu_t^0,t)\right]dt+$$

$$+\left[f(x_t^0+\varepsilon\tilde{x}_{t,\varepsilon},u^*,\nu_t^0,t)-f(x_t^0,u_t^0,\nu_t^0,t)\right]dw_t,\quad t\in(\theta,\theta+\varepsilon)$$

$$\tilde{x}_{\theta,\,\varepsilon}=-(g(x_\theta^0,u^*,\nu_\theta^0,\theta)-g(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta))$$

or

$$\tilde{x}_{\theta+\varepsilon,\,\varepsilon}=\int\limits_\theta^{\theta+\varepsilon}\left[g(x_s^0+\varepsilon\tilde{x}_{s,\varepsilon},u^*,\nu_t^0,s)-g(x_s^0,u_s^0,\nu_s^0,s)\right]ds+$$

$$+\int\limits_\theta^{\theta+\varepsilon}\left[g(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)-g(x_s^0,u_s^0,\nu_s^0,s)\right]ds+\int\limits_\theta^{\theta+\varepsilon}\left[f(x_s^0+\varepsilon\tilde{x}_{s,\varepsilon},u_s^0,\nu_s^0,s)-\right.$$

$$\left.-f(x_s^0,u_s^0,\nu_s^0,s)\right]dw_s+\int\limits_\theta^{\theta+\varepsilon}\left[g(x_s^0,u^*,\nu_t^0,s)-g(x_\theta^0,u^*,\nu_\theta^0,\theta)\right]ds.$$

Therefore from the conditions I-II and the Gronwall's inequality we have

$$E\left|\tilde{x}_{\theta+\varepsilon,\varepsilon}\right|^2\le N\left[\varepsilon^2E\sup_{\theta\le t\le\theta+\varepsilon}\left|x_{t,\varepsilon}^\theta-x_t^0\right|^2+\varepsilon^2E\sup_{\theta\le t\le\theta+\varepsilon}\left|x_t^0-x_\theta^0\right|^2+\right.$$

$$+\sup_{\theta\le t\le\theta+\varepsilon}\varepsilon^2E\left|g(x_t^0,u^*,\nu_t^0,t)-g(x_\theta^0,u^*,\nu_\theta^0,\theta)\right|^2+$$

$$+\varepsilon E\int\limits_\theta^{\theta+\varepsilon}\mid f(x_t^0,u_t^0,\nu_t^0,t)-f(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)\mid^2dt+$$

$$\left.+\varepsilon^2E\int\limits_\theta^{\theta+\varepsilon}\left|g(x_t^0,u_t^0,\nu_t^0,t)-g(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)\right|^2dt\right]$$

Hence:

$$E\left|\tilde{x}_{t+\varepsilon,\,\varepsilon}\right|^2\le\varepsilon N,\quad\varepsilon\to0,\quad\forall t\in[\theta,\theta+\varepsilon)$$

For $\forall t\in[\theta+\varepsilon,t_1]$:

$$d\tilde{x}_{t,\varepsilon}=\left[g(x_t^0+\varepsilon\tilde{x}_{t,\varepsilon},u_t^0,u^*,t)-g(x_t^0,u_t^0,\nu_t^0,t)\right]dt+$$

$$+\left[f(x_t^0+\varepsilon\tilde{x}_{t,\varepsilon},u_t^0,u^*,t)-f(x_t^0,u_t^0,\nu_t^0,t)\right]dw_t$$

Consequently we have:

$$d\tilde{x}_{t,\varepsilon}=\int\limits_0^1 g_x(x_t^0+\mu\varepsilon\tilde{x}_{t,\varepsilon},u_t^0,\nu_t^0,t)\tilde{x}_{t,\varepsilon}d\mu dt+$$

$$+\int\limits_0^1 f_x(x_t^0+\mu\varepsilon\tilde{x}_{t,\varepsilon},u_t^0,\nu_t^0,t)\tilde{x}_{t,\varepsilon}d\mu dt$$

$$\tilde{x}_{\theta+\varepsilon,\varepsilon} = -(g(x^0_{\theta+\varepsilon}, u^0_{\theta+\varepsilon}, u^*, \theta) - g(x^0_{\theta+\varepsilon}, u^0_{\theta+\varepsilon}, \nu^0_{\theta+\varepsilon}, \theta))$$

Hence:

$$E\left|\tilde{x}_{t,\varepsilon}\right|^2 \leq \varepsilon N, \quad \text{for } \forall t \in [\theta+\varepsilon, t_1], \quad \text{if } \varepsilon \to 0$$

Thus

$\sup\limits_{t_0 \leq t \leq t_1} E\left|\tilde{x}_{t,\varepsilon}\right|^2 \leq \varepsilon N$. Lemma 1 is proved.

From Lemma 1 and expression (14) for η_{t_0,t_1} we obtain: $\eta_\theta^{\theta+\varepsilon} = o(\varepsilon)$.

Since ε can be sufficiently small, we obtain that:

$$E\left[\Delta_{u^*} H(\psi_\theta, x^0_\theta, u^0_\theta, \nu^0_\theta, \theta) + \Delta_{\nu^*} H(\psi_z, x^0_z, u^0_z, \nu^0_z, z)\Big|_{z=s(\theta)}\, s'(\theta) - \right.$$

$$-\Delta_{u^*} l(x^0_\theta, u^0_\theta, \theta) + 0.5\Delta_{u^*} f^*(x^0_\theta, u^0_\theta, \nu^0_\theta, \theta)\Phi_\theta \Delta_{u^*} f(x^0_\theta, u^0_\theta, \nu^0_\theta, \theta) +$$

$$\left. +0.5\Delta_{\nu^*} f^*(x^0_z, u^0_z, \nu^0_z, z)\Phi_z \Delta_{\nu^*} f(x^0_z, u^0_z, \nu^0_z, z)\Big|_{z=s(\theta)} s'(\theta)\right] \leq 0,$$

$$\text{a.e. } \theta \in [t_0, t_1 - h(t_1))$$

and

$$E\left[\Delta_{u^*}\, H(\psi_\theta, x^0_\theta, u^0_\theta, \nu^0_\theta, \theta) - \Delta_{u^*} l(x^0_\theta, u^0_\theta, \theta) + \right.$$

$$\left. +0.5\Delta_{u^*} f^*(x^0_\theta, u^0_\theta, \nu^0_\theta, \theta)\Phi_\theta \Delta_{u^*} f(x^0_\theta, u^0_\theta, \nu^0_\theta, \theta)\right] \leq 0,$$

$$\text{a.e. } \theta \in [t_1 - h(t_1), t_1),$$

in other words (9) is fulfilled. Theorem 1 is proved.

Then using the obtained result and variation principle of Ekeland [7] we prove the following theorem for stochastic optimal control problem with endpoint constraint (6).

2. Problem with constraint

Then using the obtained result and variation principle of Ekeland [7] we prove the following theorem for stochastic optimal control problem with endpoint constraint (6).

Theorem 2. *Let the conditions I-IV hold and (x^0_t, u^0_t) be a solution of problem (1)-(6).*

Then there exist nonzero $(\lambda_0, \lambda_1) \in R^{k+1}$ such that $\lambda_0 \geq 0, \lambda_1$ is a normal to the set G at point $Eq(x^0_{t_1})$, $\lambda_0^2 + \left|\lambda_1\right|^2 = 1$ and random processes $(\psi_t, \beta_t) \in L^2_F(t_0, t_1; R^n) \times L^2_F(t_0, t_1; R^{n\times n})$, $(\Phi_t, K_t) \in L^2_F(t_0, t_1; R^n) \times L^2_F(t_0, t_1; R^{n\times n})$ which are solutions of the following adjoint system:

$$\begin{cases} d\psi_t = -H_x(\psi_t, x^0_t, u^0_t, \nu^0_t, t)dt + \beta_t dw_t, \quad t_0 \leq t < t_1, \\ \psi_{t_1} = -\lambda_0 p_x(x^0_{t_1}) - \lambda_1 q_x(x^0_{t_1}) \end{cases} \tag{15}$$

$$\begin{cases} d\Phi_t = -[g_x^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t + \Phi_t g_x(x_t^0, u_t^0, \nu_t^0, t)+ \\ +f_x^*(x_t^0, u_t^0, \nu_t^0, t)\Phi_t f_x(x_t^0, u_t^0, \nu_t^0, t)dt + f_x^*(x_t^0, u_t^0, \nu_t^0, t)K_t+ \\ +K_t f_x(x_t^0, u_t^0, \nu_t^0, t) + H_{xx}(\psi_t, x_t^0, u_t^0, \nu_t^0, t)]dt + K_t dw_t, \quad t_0 \le t < t_1, \\ \Phi_{t_1} = -\lambda_0 p_{xx}(x_{t_1}^0) - \lambda_1 q_{xx}(x_{t_1}^0) \end{cases} \tag{16}$$

such that $\forall u \in U$ *a.s. it is fulfilled:*

$$\begin{cases} \Delta_u H(\psi_\theta, x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) + [\ \Delta_\nu H(\psi_z, x_z^0, u_z^0, \nu_z^0, z)+ \\ +0.5\Delta_\nu f^*(x_z^0, u_z^0, \nu_z^0, z)\Phi_z \Delta_\nu f(x_z^0, u_z^0, \nu_z^0, z)\ ]\,\big|_{z=s(\theta)} s'(\theta)+ \\ +0.5\Delta_u f^*(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta)\Phi_\theta \Delta_u f(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) \le 0, \\ \ a.e.\ \theta \in [t_0, t_1 - h(t_1)) \\ H(\psi_\theta, x_\theta^0, u, \nu_\theta^0, \theta) - H(\psi_\theta, x_\theta^0, u_\theta^0, \nu_\theta, \theta)+ \\ +0.5\Delta_u f^*(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta)\Phi_\theta \Delta_u f(x_\theta^0, u_\theta^0, \nu_\theta^0, \theta) \le 0, \\ \ for\ a.e.\ \ \theta \in [t_1 - h(t_1), t_1] \end{cases} \tag{17}$$

Proof. For any natural number j let's introduce the following approximation functional: $I_j(u) = S_j(Ep(x_{t_1}) + E\int\limits_{t_0}^{t_1} l(x_t, u_t, t)dt, Eq(x_{t_1})) =$

$$= \min_{(c,y)\in \mathcal{E}} \sqrt{|c - 1/j - Ep(x_{t_1}) - E\int_{t_0}^{t_1} l(x_t, u_t, t\ )dt|^2 + \|y - Eq(x_{t_1})\|^2}$$

where $\mathcal{E} = \{(c, y) : c \le J^0,\ y \in G\}$, J^0 is a minimal value of the functional in (1)-(6).

Let $V \equiv (U_\partial, d)$ be the space of controls obtained by means of introducing of the following metric:

$$d(u, \nu) = (l \otimes P)\{(t, \omega) \in [t_0, t_1] \times \Omega : \nu_t \ne u_t\}$$

V is a complete metric space. Now we prove some auxiliary results (Lemmas 2, 3, 4).

Lemma 2. *Let's assume that conditions I-IV hold,* u_t^n *be a sequence of admissible controls from* V*,* x_t^n *be a sequence of corresponding trajectories of the system (1)-(3).*

If $d(u_t^n, u_t) \to 0, \quad n \to \infty,$ *then* $\lim\limits_{n\to\infty} \left\{ \sup\limits_{t_0 \le t \le t_1} E\,|x_t^n - x_t|^2 \right\} = 0,$ *where* x_t *is a trajectory corresponding to an admissible control* u_t*.*

Proof. Let u_t^n be a sequence of admissible controls from V and x_t^n be a sequence of corresponding trajectories. Then for any $t \in (t_0; t_1]$ we have:

$$|x_t^n - x_t| = \left| \int_{t_0}^{t} \left[g(x_s^n, u_s^n, \nu_s^n, s) - g(x_s, u_s, \nu_s, s) \right] ds + \right.$$

$$\left. + \int_{t_0}^{t} \left[f(x_s^n, u_s^n, \nu_s^n, s) - f(x_s, u_s, \nu_s, s) \right] dw_s \right|$$

Let's put both sides of the equation to the second power and take mathematical expectations. Due to assumption II we have:

$$E\left|x_t^n - x_t\right|^2 \le NE\int_{t_0}^{t} \left|\Delta_{u^n} g(x_s, u_s, \nu_s, s)\right|^2 ds +$$

$$+NE\int_{t_0}^{t} \left|\Delta_{\nu^n} g(x_s, u_s, \nu_s, s)\right|^2 ds + NE\int_{t_0}^{t} \left|\Delta_{u^n} f(x_s, u_s, \nu_s, s)\right|^2 ds +$$

$$+NE\int_{t_0}^{t} \left|\Delta_{\nu^n} f(x_s, u_s, \nu_s, s)\right|^2 ds + NE\int_{t_0}^{t} \left|x_s^n - x_s\right|^2 ds$$

Hence from the condition I,II and using the Gronwall's inequality we have $E\left|x_t^n - x_t\right|^2 \le C\exp(C(t - t_0))$, where

$$C = NE\int_{t_0}^{t} \left|\Delta_{u^n} g(x_s, u_s, \nu_s, s)\right|^2 ds + NE\int_{t_0}^{t} \left|\Delta_{\nu^n} g(x_s, u_s, \nu_s, s)\right|^2 ds +$$

$$+NE\int_{t_0}^{t} \left|\Delta_{u^n} f(x_s, u_s, \nu_s, s)\right|^2 ds + NE\int_{t_0}^{t} \left|\Delta_{\nu^n} f(x_s, u_s, \nu_s, s)\right|^2 ds$$

Lemma 2 is proved.

Due to continuity of the functional $J_j : V \to R^n$, according to variation principle of Ekeland we have that there exists a control $u_t^j : d(u_t^j, u_t^0) \le \sqrt{\varepsilon_j}$ and $\forall u \in V$ it is fulfilled: $J_j(u^j) \le J_j(u) + \sqrt{\varepsilon_j} d(u^j, u)$, $\varepsilon_j = \frac{1}{j}$.

This inequality means that (x_t^j, u_t^j) is a solution of the following problem:

$$\begin{cases} I_j(u) = J_j(u) + \sqrt{\varepsilon_j} E \int_{t_0}^{t_1} \delta(u_t, u_t^j) dt \to \min \\ dx_t = g(x_t, u_t, \nu_t, t)dt + f(x_t, u_t, \nu_t, t)dw_t, \quad t \in (t_0, t_1] \\ u_t = Q(t), \quad t \in [t_0 - h(t_0), t_0] \\ u_t \in U_\partial \end{cases} \qquad (18)$$

Function $\delta(u,\nu)$ is determined in the following way:

$$\delta(u,\nu)=\begin{cases}0, u=\nu\\ 1, u\neq\nu\end{cases}$$

Let (x_t^j,u_t^j) be a solution of problem (18). Then according to Theorem 1 we have: there exist random processes $(\psi_t^j,\beta_t^j)\in L_F^2(t_0,t_1;R^n)\times L_F^2(t_0,t_1;R^{n\times n})$, $(\Phi_t^j,K_t^j)\in L_F^2(t_0,t_1;R^n)\times L_F^2(t_0,t_1;R^{n\times n})$ which are solutions of the following systems:

$$\begin{cases} d\psi_t^j=-H_x\left(\psi_t^j,x_t^j,\nu_t^j,u_t^j,t\right)dt+\beta_t^j dw_t,\ t_0\le t\le t_1\\ \psi_{t_1}^j=-\lambda_0^j p_x(x_{t_1}^j)-\lambda_1^j q_x(x_{t_1}^j)\end{cases} \tag{19}$$

$$\begin{cases} d\Phi_t^j=-[g_x^*(x_t^j,u_t^j,\nu_t^j,t)\Phi_t^j+\Phi_t^j g_x(x_t^j,u_t^j,\nu_t^j,t)+f_x^*(x_t^j,u_t^j,\nu_t^j,t)\times\\ \times\Phi_t^j f_x(x_t^j,u_t^j,\nu_t^j,t)dt+f_x^*(x_t^j,u_t^j,\nu_t^j,t)K_t^j+K_t^j f_x(x_t^j,u_t^j,\nu_t^j,t)+\\ +H_{xx}(\psi_t^j,x_t^j,u_t^j,\nu_t^j,t)]dt+K_t^j dw_t,\quad t_0\le t<t_1,\\ \Phi_{t_1}^j=-\lambda_0^j p_{xx}(x_{t_1}^j)-\lambda_1^j q_{xx}(x_{t_1}^j)\end{cases} \tag{20}$$

and non-zero $(\lambda_0^j,\lambda_1^j)\in R^{k+1}$ meet the following requirement:

$$(\lambda_0^j,\lambda_1^j)=\left(-c_j+1/j+Ep\left(x_{t_1}^j\right)+E\int_{t_0}^{t_1} l(x_t^j,u_t^j,t)dt-y_j+Eq\left(x_{t_1}^j\right)\right)/J_j^0 \tag{21}$$

then $\forall u\in U$ a.s. it is fulfilled:

$$\begin{cases} \Delta_u H(\psi_\theta^j,x_\theta^j,u_\theta^j,\nu_\theta^j,\theta)+[\ \Delta_\nu H(\psi_z^j,x_z^j,u_z^j,\nu_z^j,z)+\\ +0.5\Delta_\nu f^*(x_z^j,u_z^j,\nu_z^j,z)\Phi_z^j\Delta_\nu f(x_z^j,u_z^j,\nu_z^j,z)\ ]\big|_{z=s(\theta)}s'(\theta)+\\ +0.5\Delta_u f^*(x_\theta^j,u_\theta^j,\nu_\theta^j,\theta)\Phi_\theta^j\Delta_u f(x_\theta^j,u_\theta^j,\nu_\theta^j,\theta)\le 0,\\ \text{a.e. }\theta\in[t_0,t_1-h(t_1))\\ H(\psi_\theta,x_\theta^0,u,\nu_\theta^0,\theta)-H(\psi_\theta,x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)+\\ +0.5\Delta_u f^*(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)\Phi_\theta\Delta_u f(x_\theta^0,u_\theta^0,\nu_\theta^0,\theta)\le 0,\\ a.e.\ \theta\in[t_1-h(t_1),t_1]\end{cases} \tag{22}$$

Here

$$J_j^0=\sqrt{|c_j-1\,/j-Ep\left(x_{t_1}^j\right)-E\int_{t_0}^{t_1} l(x_t^j,u_t^j,t\,)dt|^2+\left|y_j-Eq\left(x_{t_1}^j\right)\right|^2}.$$

Since $\left\|(\lambda_0^j,\lambda_1^j)\right\|=1$, then we can think that $(\lambda_0^j,\lambda_1^j)\to(\lambda_0,\lambda_1)$.
It is known that S_j is a convex function which is differentiable by Gateaux at the point $(Ep(x_{t_1}^j)+E\int_{t_0}^{t_1} l(x_t^j,u_t^j,t)dt,Eq(x_{t_1}^j))$. Then for all $(c,y)\in\mathcal{E}$:

$(\lambda_0^j,c-\frac{1}{j}-Ep(x_{t_1}^j)-E\int_{t_0}^{t_1} l(x_t^j,u_t^j,t)dt)+(\lambda_1^j,y-Eq(x_{t_1}^j))\le\frac{1}{j}$

Proceeding to the limit in the last inequality we receive that $\lambda_0 \geq 0$ and λ_1 is a normal to the set G at the point $Eq(x^0_{t_1})$.
Since $\psi^j_{t_1} = -\lambda^j_0 p_x(x^j_{t_1}) - \lambda^j_1 q_x(x^j_{t_1})$, then $\psi^j_{t_1} \to \psi_{t_1}$ in $L^2_F(t_0, t_1; R^n)$.
Similarly, from $\Phi^j_{t_1} = -\lambda^j_0 p_{xx}(x^j_{t_1}) - \lambda^j_1 q_{xx}(x^j_{t_1})$, for $j \to \infty$ implies $\Phi^j_{t_1} \to \Phi_{t_1}$ in $L^2_F(t_0, t_1; R^n)$.

Lemma 3. *Let ψ^j_t be a solution of system (19), and ψ_t be a solution of system (15).*

Then $E\int\limits_{t_0}^{t_1} |\psi^j_t - \psi_t|^2 dt + E\int\limits_{t_0}^{t_1} |\beta^j_t - \beta_t|^2 dt \to 0,$ *if* $d(u^j_t, u_t) \to 0,\ j \to \infty.$

Lemma 4. *Let Φ^j_t be a solution of system (20), and Φ_t be a solution of system (16).*

Then $E\int\limits_{t_0}^{t_1} |\Phi^j_t - \Phi_t|^2 dt + E\int\limits_{t_0}^{t_1} |K^j_t - K_t|^2 dt \to 0,$ *if* $j \to \infty.$

According to Lemma 3 and Lemma 4 taking limit in (19), (20) as $j \to \infty$ we obtain equalities (15) and (16).

Consequently, taking limit in (22) as $j \to \infty$ we obtain inequality (17).

Theorem 2 is proved.

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