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SEMI-MARKOV REWARD MODELS FOR DISABILITY INSURANCE

A semi-Markov model for disability insurance is described. Statistical evidences of relevance semi-Markov setting are given. High order semi-Markov backward reward models are invented. Applications of these models to profit-risk analysis of disability insurance contracts are considered.

1. Introduction

Several authors have discussed the use of Markov processes techniques in insurance. We cite only some of them, i.e. Hoem (1969, 1988), Consael and Sonnenscheim (1978), Moller (1992), Norberg (1993), Waters (1984), Wolthuis (1994) and the books Habermann and Pitacco (1999) and Wolthuis (2003). For a wider bibliography the reader can refer to Hoem (1988) and to the two cited books.

Markov models, in discrete and continuous time setting, clearly work well in insurance environment. However, there exists a problem that can not be solved in the frame of Markov models. As a matter of fact, transition times between states should have geometrical or exponential distributions, respectively, in discrete and continuous time Markov models. These distributions possess so-called memoryless property that may make them not relevant in some of applications.

An alternative is the use of more general semi-Markov models for which transition times can be arbitrary distributed on the positive half-line.

The first application of semi-Markov models in insurance field was proposed by Janssen (1966). Hoem (1972) proposed a non-homogeneous semi-Markov model. Iosifescu (1972) defined non-homogeneous semi-Markov processes in a different way and only from theoretic side. Janssen and De Dominicis (1984) and De Dominicis and Manca (1984) gave applications of non-homogeneous semi-Markov processes following the Iosifescu-Manu approach. Dominicis and Manca (1986) gave the definition of non-homogeneous semi-Markov rewards and applied them to insurance disability problems. Semi-Markov insurance applications were given in pension

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schemes by, Sahin and Balcer (1979), Balcer and Shain (1986), De Dominicis, Manca, and Granata (1991), and Janssen and Manca (1997). Other applications of semi-Markov models in health insurance were given in CMIR12 (1991) and more recently in Janssen and Manca (2002, 2004). In particular the last two papers develop the application of semi-Markov processes in insurance giving great relevance to the reward processes. In Blasi, Janssen and Manca, (2004), the multiple states insurance model were seen as an example of the concept of generalized homogeneous stochastic annuities. The definition of this financial tool was given by means of homogeneous semi-Markov reward processes. The latest results in this area can be found in Stenberg, Manca, and Silvestrov (2005) and Janssen and Manca (2006).

An insurance contract ensures the holder benefits in the future from some random events occurring at some random moments in time. Denote the discounted cash flow that occurs between the counter parties as the discounted accumulated reward where both the benefits and premiums are considered to be rewards. When developing an insurance contract between the writer and the receiver the following question must be asked. How shall the reward structure of the contract be determined? Different reward structures will lead to significant changes in such characteristics as expectation, variance, skewness, and kurtosis for the implied discounted accumulated reward. For both parties it is of great importance to be able to handle not only the accumulated reward but also the risk for the insurance contracts.

In this paper, we start from previous results given in Janssen and Manca (2004, 2006), where expectations of accumulated rewards for semi-Markov disability insurance models were studied. We improve these results in several directions.

First, we develop a method for calculating not only expectations but also higher order moments for accumulated rewards for disability insurance contracts. This approach let one to compare contracts not only on the base of expected accumulated rewards, as in the works mentioned above, but also to perform a full-scale profit-risk analysis and comparison of insurance contracts based, for example, on various criteria combining expected rewards and variances of rewards for different contracts. We illustrate this by presenting examples based on the real data.

It should be noted that such results related to higher order semi-Markov reward models were not pointed out in the literature. The most general results of such type relates to more simple semi-Markov rewards accumulated up to the first hitting time into some domain and can be found in Silvestrov (1980a, 1980b, 1996). In this model for accumulated rewards depend only of initial states, and systems of linear equations for moments of different order, recursive by orders of moments, are involved. In the model considered in the present paper, moments of different order for rewards accumulated up to fixed time depend not only of initial states but also of time variable. This make the corresponding systems of linear equations more complicated. Here, they are "doubly recursive", by order of moments and in time. We show, however, that the corresponding effective numerical matrix algorithms can be developed. These results, as we think, have their own value beyond the framework of the present work.

Secondly, we pay more attention to analysis of statistical evidences, which confirm the relevance of semi-Markov setting for disability insurance applications.

2. Semi-Markov reward models

Let us consider a discrete time Markov renewal process $(\eta_n, \kappa_n), n = 0, 1, ...$ that is a homogeneous discrete time Markov chain with the phase space $E \times \{0, 1, \ldots\}$, where $E = \{1, \ldots, m\}$, and transition probabilities,

$$Q_{ij}(t) = P\{\eta_{n+1} = j, \kappa_{n+1} \leq t | \eta_n = i, \kappa_n = s\} \\ = P\{\eta_{n+1} = j, \kappa_{n+1} \leq t | \eta_n = i\}. \tag{1}$$

A semi-Markov process $\eta(t), t \geq 0$ can be associated with the Markov renewal process (η_n, κ_n). In this case random variables η_n are interpreted as positions of this process at moments of jumps, κ_n as a inter-jump times, $\tau_n = \kappa_1 + \cdots + \kappa_n$ as moments of jumps, $\nu(t) = \max(n : \tau_n \leq t)$ as a number of jumps in the time interval $[0, t]$, and the semi-Markov process is defined in the following way,

$$\eta(t) = \eta_{\nu(t)}, \ t \geq 0. \tag{2}$$

It is also useful to introduce the counting process $\kappa(t), t \geq 0$, which counts time between the moment of the last jump occurred before moment t and the moment t,

$$\kappa(t) = t - \tau_{\nu(t)}, \ t \geq 0. \tag{3}$$

We shall use also the following transition characteristics which have an obvious probabilistic interpretation,

$$p_{ij} = Q_{ij}(\infty), \ b_{ij}(t) = Q_{ij}(t) - Q_{ij}(t-1), \ Q_i(t) = \sum_{j \in E} Q_{ij}(t). \tag{4}$$

We exclude instant transitions, i.e. assume that probabilities $Q_i(0) = 0, i \in E$.

We accept the case where the semi-Markov process $\eta(t)$ has absorption states. If a state i is absorption then transition probabilities $p_{ij} = 0, j \neq i$ and probabilities $Q_{ii}(t) = 0, t = 0, 1, \ldots$.

Let us also define conditional transition characteristics,

$$b_{ij,u}(t) = \begin{cases} \frac{b_{ij}(u+t)}{1-Q_i(u)}, & \text{if } 1 - Q_i(u) \neq 0, \\ 0, & \text{if } 1 - Q_i(u) = 0 \text{ and } t = 1, 2, \ldots \text{ or } j \neq i, \\ 1, & \text{if } 1 - Q_i(u) = 0 \text{ and } t = 0, j = i, \end{cases} \tag{5}$$

and

$$1 - Q_{i,u}(t) = \begin{cases} \frac{1-Q_i(u+t)}{1-Q_i(u)}, & \text{if } 1 - Q_i(u) \neq 0, \\ 0, & \text{if } 1 - Q_i(u) = 0. \end{cases} \tag{6}$$

We will consider two major types of rewards, permanence and instant rewards. The permanence reward $\psi_i(t)$, is associated with continuous maintaining in state i time which is not less than t, while instant rewards, $\gamma_{ij}(t)$, are collected when we change state from state i to state j occurs after continuous maintaining in state i exactly time t.

The simplest case is where the permanence reward in state i at time t, $\psi_i(t) = \psi_i$ only depends on the current state i and the instant reward $\gamma_{ij}(t) = \gamma_{ij}$ only depends on the state i before the jump and the state j after the jump.

Let $e^{-t\delta}$ denote the discount factor for t periods with common fixed continuous compounded interest rate δ. Let $\xi(s,t)$, $0 \leq s \leq t < \infty$ denote the accumulated discounted reward during the time interval $(s,t]$ defined by the following relation,

$$\xi(s,t) = \sum_{s<u\leq t} e^{-u\delta}\psi_{\eta(u)}(\kappa(u)) + \sum_{s<\tau_n\leq t} e^{-\tau_n\delta}\gamma_{\eta_n,\eta_{n+1}}(\kappa_n). \tag{7}$$

Let us also denote by $\xi_{i,u}(s,t)$ the random variable, which has the distribution the same with the conditional distribution for the random variable $\xi(s,t)$ given that at moment s the process is at state i and the last (before s) jump occurred at the moment $s-u$ if $1-Q_i(u) > 0$ or 0 if $1-Q_i(u) = 0$.

By the definition, the random variables $\xi(t,t) = 0$ for all t and, therefore, we apply the convention that $\xi_{i,u}(t,t) = 0$ for any i, u, t.

Let us use the symbol $\alpha \stackrel{d}{=} \beta$ to denote that two stochastic variables α and β have the same distribution.

Lemma. *The accumulated discounted reward process $\xi(s,t)$ satisfies the following stochastic relation, for $i, j \in E, u = 0, 1, \ldots, 0 \leq s \leq t \leq T < \infty$,*

$$\xi_{i,u}(s,t) \stackrel{d}{=} e^{-\delta s}\xi_{i,u}(0, t-s). \tag{8}$$

Lemma 1 shows that we can reduce consideration to staidies of random variables $\xi_{i,u}(0,t)$.

The object of our interest are power moments,

$$V_{i,u}^{(k)}(t) = E[\xi_{i,u}(0,t)]^k, \ 0 \leq t \leq T, \ k = 1, 2 \ldots. \tag{9}$$

To simplify formulas we introduce notations,

$$a_{i,u}(t) = \sum_{s=1}^{t} \psi_i(s+u)e^{-\delta s}, \quad \tilde{a}_{ij,u}(t) = a_{i,u}(t) + e^{-\delta t}\gamma_{ij}(t+u). \tag{10}$$

We shall also use more compact matrix notations. Let $\mathbf{V}_u^{(k)}(t)$ denote the $m \times 1$ vector with $E[\xi_{i,u}(t)]^k$ at position i, $\mathbf{D}_u(t)$ denote the $m \times m$ diagonal matrix with entry $1 - Q_{i,u}(t)$ in the center diagonal and zero elsewhere,

$\mathbf{B}_u(t)$ be the $m \times m$ matrix with elements $b_{ij,u}(t)$ in position $< i, j >$, and $\mathbf{A}_u^{(k)}(t)$ be the diagonal matrix with $(a_{i,u}(t))^k$ in the center diagonal and zero elsewhere, $\tilde{\mathbf{A}}_u^{(k)}(t)$ be the $m \times m$ matrix with elements with elements $(\tilde{a}_{ij,u}(t))^k$ position $< i, j >$, and $\mathbf{1}_m$ is the $m \times 1$ vector with all components equal 1.

Theorem 1. *The vectors of expected discounted accumulated rewards*

$$\mathbf{V}_u^{(1)}(t), \qquad u, t = 0, \ldots, T,$$

are uniquely determined by the following recursion relation:

$$\begin{aligned}\mathbf{V}_u^{(1)}(t) &= \mathbf{D}_u(t)\mathbf{A}_u^{(1)}(t)\mathbf{1}_m \\ &+ \sum_{s=1}^{t} \mathbf{B}_u(s)\tilde{\mathbf{A}}_u^{(1)}(s)\mathbf{1}_m + \sum_{s=1}^{t} e^{-\delta s}\mathbf{B}_u(s)\mathbf{V}_0^{(1)}(t-s).\end{aligned} \tag{11}$$

This relation should be used in the following recursion order: (a) *for* $u = 0$ *sequentially for* $t = 0, 1, \ldots, T$, (b) *for every* $u = 1, \ldots, T$ *sequentially for* $t = 0, 1, \ldots, T$.

Proof. Let us introduce the random variable $\vartheta_{i,u}$ which has the same distribution as the time to the next jump given that the process $\eta(s)$ already have spent in the state i for time u and let $\zeta_{i,u}$ denote the corresponding state the process $\eta(s)$ end up in after the jump. According to the definition these random variables have the following distributions,

$$P\{\vartheta_{i,u} > t\} = 1 - Q_{i,u}(t), \tag{12}$$

and

$$P\{\vartheta_{i,u} = t, \zeta_{i,u} = j\} = b_{ij,u}(u+t). \tag{13}$$

Let us construct a stochastic relation for the random variables $\xi_{i,u}(t)$. We will have to consider two cases, if no jump occurs before moment t, or if at least one jump occurs between moment 0 up to moment t. Using indicator variables $\chi(\cdot)$ for random events we can write down the following stochastic relation,

$$\begin{aligned}\xi_{i,u}(0,t) &\stackrel{d}{=} \chi(\vartheta_{i,u} > t)a_{i,u}(t) + \sum_{j\in E}\sum_{s=1}^{t} \chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)\tilde{a}_{ij,u}(s) \\ &+ \sum_{j\in E}\sum_{s=1}^{t} e^{-\delta s}\chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)\xi_{j,0}(0, t-s), \\ &\qquad i \in E, \qquad t = 0, \ldots, T,\end{aligned} \tag{14}$$

where the random variables $\chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)$ and $\xi_{j,0}(0, t-s)$ are independent.

The first term in (14) represents the discounted reward received for maintaining in state i for t moments. The second term is the sum over all possible times when the first jump can occur, and the corresponding reward for maintaining in state i for this amount of time including the possible instant reward at the jump. The third term is due to the fact that the process restarts and possess Markov property at moments of jumps. All terms are conditioned with the help of the indicator random variable denoting when and where the first jump will occur.

The corresponding relation for expectations can now be calculated using this stochastic relation and independence relationships mentioned earlier,

$$\begin{aligned} E[\xi_{i,u}(0,t)] &= (1 - Q_{i,u}(t))a_{i,0}(t) + \sum_{j\in E}\sum_{s=1}^{t} b_{ij,u}(t)\tilde{a}_{ij,0}(s) \\ &\quad + \sum_{j\in E}\sum_{s=1}^{t} e^{-\delta s} b_{ij,u}(t) E[\xi_{j,0}(0,t-s)], \\ &\quad i \in E, \qquad u,t = 0,\dots,T. \end{aligned} \tag{15}$$

This relation, rewritten in matrix form is equivalent to relation (11). □

The following theorem gives an analogous result for higher order moments.

Theorem 2. *The vectors of k-order moments for discounted accumulated rewards $\mathbf{V}_u^{(k)}(t), u, t = 0,\dots,T$ are uniquely determined for $k = 1, 2, \dots$ by the following recursion relation*:

$$\begin{aligned} \mathbf{V}_u^{(k)}(t) &= \mathbf{D}_u(t)\mathbf{A}_u^{(k)}(t)\mathbf{1}_m + \sum_{s=1}^{t}(\mathbf{B}_u(s)\cdot\tilde{\mathbf{A}}_u^{(k)}(s))\mathbf{1}_m \\ &\quad + \sum_{s=1}^{t}\sum_{l=1}^{k-1}\binom{k}{l} e^{-\delta s(k-l)}(\mathbf{B}_u(s)\cdot\tilde{\mathbf{A}}_u^{(k)}(s))\mathbf{V}_0^{(k-l)}(t-s) \\ &\quad + \sum_{s=1}^{t} e^{-k\delta s}\mathbf{B}_u(s)\mathbf{V}_0^{(k)}(t-s). \end{aligned} \tag{16}$$

This relation should be used in the following recursion order:

(a) *for $k = 1$ and $u = 0$ sequentially for $t = 0, 1, \dots, T$,*
(b) *for $k = 1$ and every $u = 1, \dots, T$ sequentially for $t = 0, 1, \dots, T$,*
(c) *for $k = 2$ and $u = 0$ sequentially for $t = 0, 1, \dots, T$,*
(d) *for $k = 2$ and every $u = 1, \dots, T$ sequentially for $t = 0, 1, \dots, T$,*
(e) *sequentially for any higher moment order $k > 2$ and $u = 0$ sequentially for $t = 0, 1, \dots, T$,*
(f) *for $k > 2$ and every $u = 1, \dots, T$ sequentially for $t = 0, 1, \dots, T$.*

Proof. The following stochastic relation can be written down for random variables $\xi^k_{i,u}(t)$,

(17)

$$\xi^k_{i,u}(0,t) \stackrel{d}{=} \chi(\vartheta_{i,u} > t)(a_{i,u}(t))^k + \sum_{j\in E}\sum_{s=1}^{t} \chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)(\tilde{a}_{ij,u}(s))^k$$

$$+ \sum_{j\in E}\sum_{s=1}^{t}\sum_{l=1}^{k-1} \binom{k}{l} e^{-\delta s(k-l)} (\tilde{a}_{ij,u}(s))^{k-l} \chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)\xi^{k-l}_{j,0}(0, t-s)$$

$$+ \sum_{j\in E}\sum_{s=1}^{t} \chi(\vartheta_{i,u} = s, \zeta_{i,u} = j) e^{-\delta sk} \xi^k_{j,0}(0, t-s),$$

$$i \in E, \qquad u, t = 0, \ldots, T,$$

where the random variables

$$\chi(\vartheta_{i,u} = s, \zeta_{i,u} = j) \quad \text{and} \quad \xi_{j,0}(0, t-s)$$

are independent.

Here we have used the fact that the product of indicators

$$\chi(\vartheta_{i,u} > t)\chi(\vartheta_{i,u} = s, \zeta_{i,u} = j) = 0$$

for $s \leq t$ and

$$\chi(\vartheta_{i,u} = s, \zeta_{i,u} = j)\chi(\vartheta_{i,u} = s', \zeta_{i,u} = j') = 0$$

if $j \neq j'$ or $s \neq s'$, while powers of these indicators coincides with these indicators themselves.

The corresponding relation for expectations can now be calculated using this stochastic relation and independence relationships mentioned earlier,

$$E[\xi_{i,u}(0,t)]^k = b_{i,u}(t)(a_{i,u}(t))^k + \sum_{j\in E}\sum_{s=1}^{t} b_{ij,u}(s)(\tilde{a}_{ij,u}(s))^k$$

(18)
$$+ \sum_{j\in E}\sum_{s=1}^{t}\sum_{l=1}^{k-1} b_{ij,u}(s) \binom{k}{l} e^{-\delta s(k-l)} (\tilde{a}_{ij,0}(s))^{k-l} E[\xi_{j,0}(0, t-s)]^{k-l}$$

$$+ \sum_{j\in E}\sum_{s=1}^{t} \chi(\vartheta_{i,u} = s, \zeta_{i,u} = j) e^{-\delta sk} E[\xi_{j,0}(0, t-s)]^k,$$

$$i \in E, \qquad u, t = 0, \ldots, T.$$

This relation, rewritten in matrix form is equivalent to relation (16). □

3. Disability insurance

In the papers by Janssen and Manca (2002, 2004) it is shown how to apply continuous time semi-Markov reward processes in multiple life insurance. In the paper Blasi, Janssen and Manca (2004) a real case study based on historical disability data is described. We extend these studies in the directions listed above, in the introduction.

The historical data gives the disability history of 840 persons that had silicosis problems lived in Campania, a region in Italy. Each individual with silicosis were examined by a doctor. The doctor determined approximately the degree of disability in percentage for each patient ranging from 0% to 100%. Depending on the degree of disability the policy maker have determined 5 possible states, which differs by reward payments. Also a "death" state with no rewards payed to in this state should be added in the model. These states are categorized in Table 1.

Given 6 states defined above we attach a reward policy to the disability degree. The rewards that is given to construct the example represents the money amount that is paid per one time period to the disabled person as a function of his degree of illness. In fact, a year is the duration of one time period. Table 1 also defines two variants of the rewards (per year in Euro) used for two different insurance contracts,

Table 1. Disability states and rewards.

states	disability degree	contract - I	contract - II
1	$[0\%, 10\%)$	1000	1200
2	$[10\%, 30\%)$	1500	1600
3	$[30\%, 50\%)$	2000	2000
4	$[50\%, 70\%)$	2500	2400
5	$[70\%, 100\%]$	3000	2800
6	death	0	0

This subdivision is similar of the ones used in Yntema (1962) and Janssen (1966) that were the first to apply respectively Markov and semi-Markov environment in disability problems. Transition between states occurs after a visit to the doctor that can be seen as the check to decide in which state the disable person is in. This gives naturally an example where virtual transitions are possible, i.e., the individual have neither become sufficiently better or worse to change state.

The data mentioned above give for every individual durations of time intervals between visits to the doctor. The states of disability determined by the doctor during these visits. It should be taken into account that an actual change in rewards payed to the individual, resulted by one or several visit to doctor during the same period of time, can occur only in the end of this period of time.

This means that, reward process can be considered in discrete time and times between transitions should be counted in numbers of years, when the individual was classified and payed according to a given disability class.

The specific feature of the original data is that every observed trajectory ends with a visit to a doctor, where either a new disability state was determined, or a new degree of disability in percentage was determined which did not caused a change disability state (virtual transition) or an actual new state or degree of disability were not determined. A part of these cases should be interpreted as transition to the "death" state.

The original data mentioned above can be transformed to the data representing stepwise realisations of the discrete time process described above. We use a semi-Markov model to describe this process. Therefore, we a priory accept Markov property at moments of transitions. In order to keep a reasonable relation between the original "sample size" and a number of parameters of the model that must be estimated from these data, we also a priory accept a simple semi-Markov model with distribution of transition (sojourn) times that do depend only on an actual state of the process but do not depend on a "destination" state into which the process occurs after transition.

The difference between the semi-Markov model described above and a Markov model is in the assumption about the distribution of sojourn times. We do not accept geometrical form for these distributions but do prefer to use the distribution estimated in non-parametric way from sample data.

In this case, the semi-Markov model is determined by a 6×6 matrix of transition probabilities $\mathbf{P} = ||p_{ij}||$, for so-called embedded Markov chain controlling transitions in the phase space, and 5 discrete distributions $\mathbf{b}_i = < b_i(1), \ldots, b_i(T), \bar{b}_i(T+1) >$. Here p_{ij} is a probability of transition from disability state i to disability state j; $b_i(t)$ is a probability that such transition occurs after t years, $\bar{b}_i(T+1) = 1 - b_i(1) - \cdots - b_i(T)$ is the corresponding tail probability; T is the time horizon which is the subject of actual reward studies. In our example, we take $T = 10$.

Note that the state 6 is an absorption state and, therefore, probabilities $p_{6i} = 0, i = 1, \ldots, 5$ and probabilities $b_i(t) = 0, t = 1, 2, \ldots$.

The transition matrix $\mathbf{P}$ is estimated from the original data in the following way. First we construct a matrix counting numbers n_{ij} of jumps between the states in observed trajectories, this is done by adding 1 in the position $< i, j >$ for each time that a person that was in state i makes a visit to the doctor and the doctor gives the disability degree that corresponds to the state j. By normalizing these numbers by the corresponding numbers $n_i = \sum_j n_{ij}$ of visits in state i, we get sample estimates $\hat{p}'_{ij}$ for conditional transition probabilities p'_{ij} for states $i, j = 1, \ldots, 5$. Here p'_{ij} is a probability of transition from state i to state j conditioned by the event that a disable person did not died during the corresponding sojourn time in state i. To get estimates $\hat{p}_{ij}$ for probabilities p_{ij} one should multiply sample estimates

$\hat{p}'_{ij}$ by quantities $1 - \hat{p}_{i6}$ where $\hat{p}_{i6}$ is a sample estimate for probability to death during sojourn time in the state i.

Unfortunately, the original data do not contain exact information about death cases. These cases are hidden among the cases where the actual new disability state was not determined during the final visit to a doctor. We estimate the transition probabilities $p_{i6}, i = 1, \ldots, 5$ in the following way. First, the mean values m_i of the sojourn times for states $i = 1, \ldots, 5$ are estimated by the standard sample means $\hat{m}_i$ for every disability state $i = 1, \ldots, 5$. Here actual observed values for sojourn times create the corresponding samples (remind that a priory assumption about independence of these times on the destination states was accepted). Then transition probabilities p_{i6} are estimated by products $\hat{p}_{i6} = p\hat{m}_i$ where p is an average one year death probability for a reasonable age range for a given type of insurance contracts. For simplicity, the age range in chosen as 50 years (from 20 to 70 years) and a natural value for $p = 0.02$ is used in this case. In fact, some demographical data related to the actual historical period of observation and given geographical region could be used but this would require a special analysis which is beyond of the goal of this paper.

Table 2 represent the value of the estimate $\hat{\mathbf{P}}$ of the transition matrix $\mathbf{P}$ obtained from sample data in the way described above.

TABLE 2. Estimated transition matrix $\hat{\mathbf{P}} = ||\hat{p}_{ij}||$.

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$i=1$	0.0000	0.9489	0.0000	0.0000	0.0000	0.0511
$i=2$	0.0000	0.5532	0.3483	0.0154	0.0051	0.0779
$i=3$	0.0000	0.0156	0.6376	0.2628	0.0104	0.0736
$i=4$	0.0000	0.0211	0.0352	0.5354	0.3311	0.0772
$i=5$	0.0000	0.0000	0.0000	0.0183	0.9132	0.0685
$i=6$	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000

The transition matrix for the embedded Markov-chain can be visualized, see Figure 1, which show by arrows transitions with positive transition probabilities.

The probabilities $b_i(t), t = 1, \ldots, T$ and $\bar{b}_i(T+1)$ are estimated with the use of natural frequency estimates $\hat{b}_i(t)$ and $\hat{\bar{b}}_i(t)$. In the first, quotients of numbers $n_i(t)$ of cases with sojourn time in a disability state i taken exactly value t, in the second greater than t, respectively, normalized by a total number n_i of visits into state i observed in the sample data.

Table 3 represents the estimates of the distributions $\mathbf{b}_i, i = 1, \ldots, 5$ obtained from the sample data in the way described above.

In order to confirm the relevance of semi-Markov setting, we use p-value technique for checking the "semi-Markov" hypothesis, which means that the sojourn times are not geometrically distributed. As was mentioned above, a natural estimator for the probability $b_i(t)$ is $\hat{b}_i(t) = n_i(t)/n_i$, where $n_i(t)$ is the number of cases in our sample data when the sojourn time in this

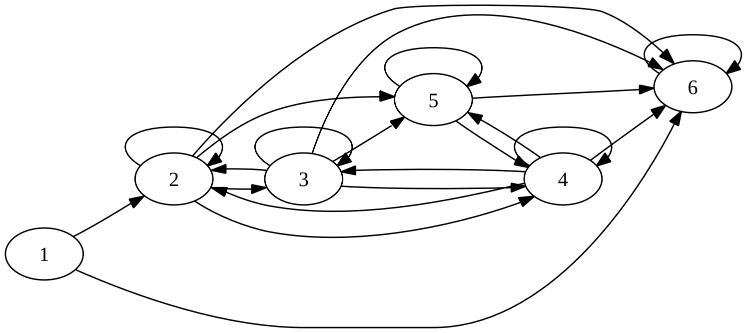

FIGURE 1. Visualization of the transition matrix for the embedded Markov chain.

TABLE 3. Estimated distributions $\hat{\mathbf{b}}_i$.

	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$	$t=6$	$t=7$	$t=8$	$t=9$	$t=10$	$t>10$
$\hat{b}_1(t)$	0.0000	0.4444	0.5556	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\hat{b}_2(t)$	0.0855	0.2124	0.2080	0.1799	0.1091	0.1224	0.0265	0.0162	0.0177	0.0029	0.0192
$\hat{b}_3(t)$	0.0464	0.2781	0.2362	0.1965	0.0905	0.0817	0.0265	0.0155	0.0044	0.0066	0.0177
$\hat{b}_4(t)$	0.0323	0.2452	0.2258	0.2129	0.1097	0.1032	0.0258	0.0065	0.0129	0.0065	0.0194
$\hat{b}_5(t)$	0.0164	0.4098	0.2295	0.2131	0.0328	0.0328	0.0164	0.0164	0.0000	0.0164	0.0164

state took the value t. Note that quantities $n_i(t)$ are random numbers. Under "geometrical hypothesis" the equality $b_i(1)(1-b_i(1)) = b_i(2)$ must hold. Thus, the statistic $\hat{b}_i(1)(1-\hat{b}_i(1)) - \hat{b}_i(2)$ should take small values for sample sizes large enough. Moreover, by evaluating asymptotical variances for this statistic and by applying some results for statistics with random sample size, it can be shown that statistic $\sqrt{n_i}(\hat{b}_i(1)(1-\hat{b}_i(1)) - \hat{b}_i(2))/\sqrt{b_i(1)(1-b_i(1))^2(2-b_i(1))}$ should have asymptotically the standard normal distribution. Since, probability $b_i(1)$ is unknown it can be replaced by its estimate $\hat{b}_i(1)$. So, the statistic $\hat{S}_i = \sqrt{n_i}(\hat{b}_i(1)(1-\hat{b}_i(1)) - \hat{b}_i(2))/\sqrt{\hat{b}_i(1)(1-\hat{b}_i(1))^2(2-\hat{b}_i(1))}$ should have, under the "geometrical" hypothesis, approximately the standard normal distribution. The corresponding proof can be accomplished, for example, with the use of results given in Silvestrov (2004).

The proposition about asymptotic normality of statistic $\hat{S}_i$ can be utilized in the following way. In the case, when $\hat{s}_i$ is the value of this statistics obtained from sample date, one can calculate the quantity $2(1 - F(|\hat{s}_i|))$, where $F(s)$ is the standard normal distribution. If this value is small enough, this would be an evidence to reject the hypothesis that the distribution $\mathbf{b}_i$ is geometrical, i.e., it would be an evidence to the benefit of "semi-Markov" hypothesis.

This method applied, for example, to the disability state 2 gives the values 58, 144 and 678, for statistics $n_2(1), n_2(2)$, and n_2, respectively. Sequentially, $\hat{s}_2 = -9.440$ and $2(1 - F(|\hat{s_2}|)) \approx 3.7 \cdot 10^{-21}$. According to the remarks above, this value can be interpreted as a strong evidence to the benefit of semi-Markov hypothesis.

Let us $e^{-t\delta}$ denotes the discount factor for t periods with common fixed continuous compounded interest rate δ. Recall that we denote $\xi_{iu}(s,t)$, $s \leq t$ as the accumulated discounted reward during the time interval $(s,t]$ given that at time s the process is at state $i \in E$ and the previous jump accursed u moments ago. In section 2, we developed the method for finding higher order moments $E[\xi_{i,u}(s,t)]^k$, $k = 0, 1, \ldots, 0 \leq s \leq t \leq T$ as functionals of transition characteristics for the corresponding semi-Markov process.

The expectations $E[\xi_{i,u}(s,t)]$ and variances $Var[\xi_{i,u}(s,t)]$ are the objects of special interest. These characteristics can be used in a profit-risk analysis and comparison of insurance contracts that have obvious advantages against a profit analysis and comparison of insurance contracts based only on expectations of accumulated rewards. For, example one can use combined risk-profit characteristics $C^{(a)}_{i,u}(s,t) = E[\xi_{i,u}(s,t)] - a\sqrt{Var[\xi_{i,u}(s,t)]}$ with reasonably chosen weighting parameter a for comparison of contracts for time interval $(s,t]$. For example, the values $1, 2$ and 3 for parameter a are inspired by the so-called one-, two- and three sigma rules.

Table 4 presents characteristic $E[\xi_{1,0}(0,t)]$, $Var[\xi_{1,0}(0,t)]$, and $C^{(3)}_{1,0}(0,t)$ for two different contracts with rewards given if Table 1.

The simple interest rate per year 3% is chosen and transformed into continuous compounded interest rate $\delta = \log(1 + 0.03)$.

It is readily seen from the data given in Table 4 that, for the case $i = 1, u = 0$, contract I has a slightly better expectations of accumulated rewards but significantly worse variances of accumulated rewards than contract II for the time interval $(0, 10]$, while the combined risk-profit characteristic $C^{(3)}_{1,0}(0, 10)$ takes a better value for contract II.

The comparison of these contracts made on the simpler base of expectations of accumulated rewards $E[\xi_{1,0}(0, 10)]$, would recommend to choose contract I, while more accurate comparison of these contracts taking into account risk factors and based on the the combined risk-profit characteristics $C^{(3)}_{1,0}(0, 10)$ would recommend to choose contact II.

As explained in Habermann and Pitacco (1999) the Markov environment cannot consider all the previous evolution that system had before nor the time u spent inside a state before a transition. The semi-Markov environment lets one evaluate characteristics of accumulated rewards as function of the state i and the time u spent in a given disability state.

Table 5 show that expected values of accumulated rewards can significantly depend on time u spent in a given disability state 2 for contract I:

Table 4. Profit-risk characteristics of discounted accumulated rewards for contacts I and II.

	$E[\xi_{1,0}(t)]$		$Var[\xi_{1,0}(t)]$		$C^{(3)}_{1,0}[\xi_{1,0}(t)]$	
t	I	II	I	II	I	II
1	970	1067	0	0	970	1067
2	1912	2103	0	0	1912	2103
3	2998	3163	77470	34106	2163	2609
4	4263	4257	251952	162127	2757	3049
5	5500	5316	636019	434561	3108	3339
6	6714	6343	1286450	879215	3312	3530
7	7907	7337	2270228	1531392	3387	3624
8	9076	8300	3645316	2425160	3348	3628
9	10220	9235	5462352	3592225	3208	3549
10	**11339**	**10142**	**7760581**	**5062041**	**2982**	**3392**

Table 5. Expectations and variances of accumulated discounted rewards.

	$E[\xi_{2,u}(t)]$			$Var[\xi_{2,u}(t)]$		
t	$u=0$	$u=1$	$u=2$	$u=0$	$u=1$	$u=2$
1	1456	1456	1456	0	0	0
2	2875	2886	2891	21910	59292	75512
3	4268	4291	4303	137129	287425	357793
4	5636	5671	5688	441487	783425	944535
5	6978	7023	7048	1025020	1631242	1925198
6	8292	8348	8375	1964034	2906036	3373795
7	9580	9640	9669	3326448	4670956	5335672
8	10836	10900	10932	5168873	6964287	7850892

In conclusion, we would like to note that some alternative semi-Markov models also may be considered. In particular, we also examined the model with more complicated mechanism of transitions, where transitions into "death" state 6 and other disability states $1, \ldots, 5$ occur on the base of "competition" between death transitions and transitions to other disability states. More precisely, the sojourn time in a state i is modelled as a minimum of two independent variables. The first one is a random variable with some unknown distribution $\tilde{\mathbf{b}}_{\mathbf{i}}$ given in non-parametric form, and another is a geometrically distributed random variable with parameter p, where again p is the average one year death probability for a reasonable age range for a given type of insurance contracts. In this case, distributions of transition times to the "death" state differs of distribution of transition time to other disability states.

We can report that the actual estimates for transition characteristics and expected accumulated rewards are very close to those obtained for a basic semi-Markov model. In particular, the differences in expected accumulated rewards for two models are in the limits of -2% – +1 %.

It should be noted that the distributions of sojourn times are formed by very complicated mechanisms determined by stochastic disability dynamics for individuals, rules of insurance medical service in a country, and many other factors. That is why, the non-parametric semi-Markov models may be involved. It should, however, be noted that these models have a disadvantage since they may involve too many parameters required to be estimated from sample data.

Our conjecture is that some semi-parametric models for distributions of sojourn times could also be applied as an alternative. In particular, the so-called "burned" geometrical distribution, which have a non-geometrical form for probabilities $b_i(t)$ for a few small values of t and a geometrical form for probabilities $b_i(t)$ otherwise. Looking at sample data available in the example described above, we conjecture that such a model could possibly be used for distributions of sojourn times for disability states $2, 3$, and 4.

4. Conclusions

In this paper a first step for the application of the higher order backward semi-Markov rewards in insurance has been done. Reward processes represent the first moment of the total revenues that are given in a stochastic financial operations. Not only the first moment was considered, we also showed how to develop algorithms to calculate higher order moments for discounted accumulated rewards. A real-world example was given in Section 3. Calculating higher moments is important, this gives the tools to compare different insurance contracts and analyze not only profit but also risk properties of individual contracts. Our setup also makes it possible to see how small changes in the underlying parameters such as duration of contract, fees and benefits influence the cash flow between the counter parties. When developing new insurance policies such properties are of great importance.

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