Theory of Stochastic Processes
Vol. 12 (28), no. 3–4, 2006, pp. 203–238

DMITRII SILVESTROV, JOZEF TEUGELS, VIKTORIYA MASOL,
AND ANATOLIY MALYARENKO

INNOVATION METHODS, ALGORITHMS, AND SOFTWARE FOR ANALYSIS OF REINSURANCE CONTRACTS[1]

A Monte Carlo based approach to evaluate and/or compare the riskiness of reinsurance treaties for both the ceding and the reinsurance companies is introduced. An experimental program system *Reinsurance Analyser* based on the indicated approach is presented. The program allows analyzing applications of a large set of reinsurance contracts under a variety of claim flow models. The effect of applications is compared by risk measures, provided that the parameters of the contracts are balanced by an average reinsurer's load quantity.

1. Introduction

The aim of reinsurance is to protect an insurance company, hereafter called the ceding company or the first line insurer, against large losses caused either by excessively large claims or by a large number of claims. A reinsurance treaty determines the rules according to which claims are split between the ceding and the reinsurance companies. One of the questions arising in connection with reinsurance is which reinsurance treaty, applied to a given claim flow, provides a less risky position to the reinsurer. It is therefore of interest to develop methods that compare the riskiness of different treaties and that select the optimal treaty. By way of procedure we take the point of view of the reinsurer.

The most common measure for judging the riskiness of a reinsurance treaty is the variance of the deductible or reinsured claim amount. In this case, the optimality criterion consists in minimizing the variance while the mean value of the claim amount under consideration is fixed. Minimization of the variance of the deductible with respect to the classical reinsurance treaties such as quota-share, excess-of-loss, stop-loss, surplus, or their combinations has been investigated by many authors (see, e.g., Pesonen (1984), Daykin, Pentikäinen, and Pesonen (1993), Denuit and Vermandele (1998),

[1]Invited lecture.

2000 *Mathematics Subject Classification.* Primary 62P05; Secondary 62G32, 91B70.

Key words and phrases. Comparison of reinsurance treaties, large claims, heavy-tailed distributions, stochastic modelling, program system.

This work is partly supported by the *Perturbed Risk Processes, Extremal Processes and Reinsurance Contracts* Project financed by the Royal Swedish Academy of Sciences.

Gajek and Zagrodny (2000), Kaluszka (2001), Verlaak and Beirlant (2003), Kaluszka (2004), and others). The ceding insurer's and reinsurer's optimal choice of classical reinsurance was derived in Hesselager (1990). However, this author used the minimization of the probability of ultimate ruin as optimality criterion.

Not much is known on the optimal choice of a treaty when the set of feasible treaties contains large claims reinsurance (ECOMOR, largest claims reinsurance, etc.). In this case, the application of the "mean-variance" criterion is complicated by the fact that no handy and/or general analytical expressions exist for the expectation and the variance of the (re-)insured amount. In 1972, Berliner showed a high degree of similarity between excess-of-loss and largest claims reinsurance covers in terms of the correlation coefficient of the reinsured claim amount for the Pareto-Poisson model. By the latter we understand that the claim epochs follow a Poisson stream while the claim sizes follow a Pareto distribution. In Kremer (1982), the asymptotic equivalence of the net risk premium of the largest claims cover and the net risk premium of the excess-of-loss treaty plus additive term is established. This result was subsequently generalized in Kremer (1984) to the case of a generalization of the largest claims cover. Further results on the optimal choice of a treaty when large claims reinsurance is involved can be found in Kremer (1990) and Kremer (1991). In the papers referred to above, the largest claims cover is compared to the excess-of-loss treaty with the help of either the asymptotic efficiency or the limiting efficiency. It is shown that, from the insurer's viewpoint, the excess-of-loss treaty is better than the largest claims cover with respect to an asymptotic efficiency measure; however the treaties are equivalent with regard to the limiting efficiency.

The riskiness of a combination of quota-share and a large claim reinsurance contract was studied in Ladoucette and Teugels (2006a). For a survey of the analytical results concerning large claim reinsurance we refer to Teugels (2003) and Ladoucette and Teugels (2006b).

In this paper we introduce a Monte Carlo based approach to evaluate and/or compare the riskiness of reinsurance treaties for both the ceding and the reinsurance companies. However, when making a decision on the optimality and the expediency of a treaty, we restrict attention to the reinsurer's interest. Not only is this the main novelty in our approach, but often first line insurance can be considered as a form of reinsurance where the client accepts a certain franchise or deductible. Due to the Monte Carlo technique the implementation of the indicated approach extends to a large set of reinsurance treaties. In particular, simulation methods allow to cope with the mathematical complexity of large claims reinsurance.

Furthermore, we present an experimental software *Reinsurance Analyser* and show results of experimental studies made with the help of the program. The Reinsurance Analyser becomes an especially handy tool when one compares reinsurance contracts of high mathematical complexity, typical for large claims reinsurance. Moreover, the use of computer programs allows us to analyze applications of a reinsurance contract under a variety of claim flow models.

Throughout the paper we assume that claim flows can be described by the classical Sparre Andersen risk model where inter-claim times are assumed to form a renewal process. The compound Poisson model is a Sparre Andersen model where the inter-claim intervals are exponentially-distributed. The distribution of claim sizes and the distribution of the inter-claim intervals are modelled as mixtures of distributions chosen from a broad class containing both light- and heavy-tailed distributions. Each distribution playing a role in the mixture has a certain probability but the sum of the probabilities in anyone mixture is equal to 1. Hence, these probabilities may be considered as weights of the distributions in the mixture. The mixture approach for modelling claim flows allows us to take account of claims of different nature like frequent small, medium claims and rare extreme large claims. More concretely, light-tailed distributions describing small and medium claims have higher probabilities in the mixture while heavy-tailed distributions for the large claims have lower probabilities.

The riskiness of a reinsurance treaty is evaluated on a specified time interval called the *evaluation interval* by a set of risk measures (e.g., variance, dispersion, coefficient of variation, value at risk, etc.) The evaluation interval can be determined using different principles. In the present paper we consider two such types, i.e., claim-type, containing a pre-specified number of claims, and time-type. The claim amount retained by the first line insurer (respectively, reinsurer) within an evaluation interval will be referred to as the *interval deductible* (*interval reinsured claim amount*).

In order to make a fair comparison between two reinsurance treaties by any feasible risk measure, we need to calibrate them. We therefore choose the parameters of the treaties in such a way that the *average reinsurer's quota loads* (i.e., the percentage of the interval claim amount covered by the reinsurer in average) on the same evaluation interval are equal. Moreover these balanced treaties are to be applied to the same claim flows. Note that if the first or the second moment of the claim size distribution is infinite, the same might happen to the corresponding moment of the distribution of the reinsured amount. This is the case, for example, for excess-of-loss, largest claims reinsurance treaties. It is therefore sensible to consider ratios of the corresponding sample estimates of risk measures as they can display asymptotic stabilization even in the case of heavy tailed distributions.

The approach to evaluate and compare reinsurance treaties as presented in this paper has been used as a background for the experimental software Reinsurance Analyser (ReAn). The program ReAn is written in the programming language Java with the use of SSJ and JFreeChart class libraries. Due to the facilities of Java the software is platform independent and is well-equipped for distributed calculations through the Internet. The description of the program and the results of experimental studies are given in the paper as well.

The remaining of the paper is structured as follows. The claim flows modelling and the types of reinsurance contracts are described in Section 2. Special attention is paid to the extreme value contracts. Section 3 treats the evaluation of reinsurance treaties. In Sections 4 and 5 we formulate the problems which can be solved with the help of the Reinsurance Analyser. The algorithms for solving these problems are also included. In particular, the approach to compare reinsurance treaties is discussed in Section 5. The description of the program Reinsurance Analyser is given in Section 6. Finally, results of experimental studies are presented and commented in Section 7.

2. Claim Flows and Reinsurance Treaties

Let $T_1, T_2, \ldots,$ be the inter-claim intervals (T_1 is the moment when the first claim arrives). If $X_1, X_2, \ldots,$ is the sequence of corresponding claim sizes and $N(t) = \max\{n : T_n \leq t\}$, $t \geq 0$, is the number of claims arrived to an insurance company up to time t, then the *total* or *aggregate* claim amount over that time period is

$$X(t) := \sum_{n=1}^{N(t)} X_n, \quad t \geq 0,$$

and $X(t) = 0$ if $N(t) = 0$.

The moment of the appearance of the n-th claim is given by

$$Z_n = T_1 + T_2 + \ldots + T_n, \quad n = 1, 2, \ldots. \tag{1}$$

Any reinsurance treaty is a rule according to which each individual claim X_n is split into a "lower" and an "upper" part

$$X_n = X_n^I + X_n^R, \quad n = 1, 2, \ldots,$$

where X_n^I is the *deductible* payed by the first line insurer, and X_n^R is the *reinsured amount.* Consequently, the aggregate claim $X(t)$ is split as

$$X(t) = X^I(t) + X^R(t) = \sum_{n=1}^{N(t)} X_n^I + \sum_{n=1}^{N(t)} X_n^R, \quad t \geq 0.$$

Figure 1 illustrates a reinsurance arrangement on a claim-by-claim basis.

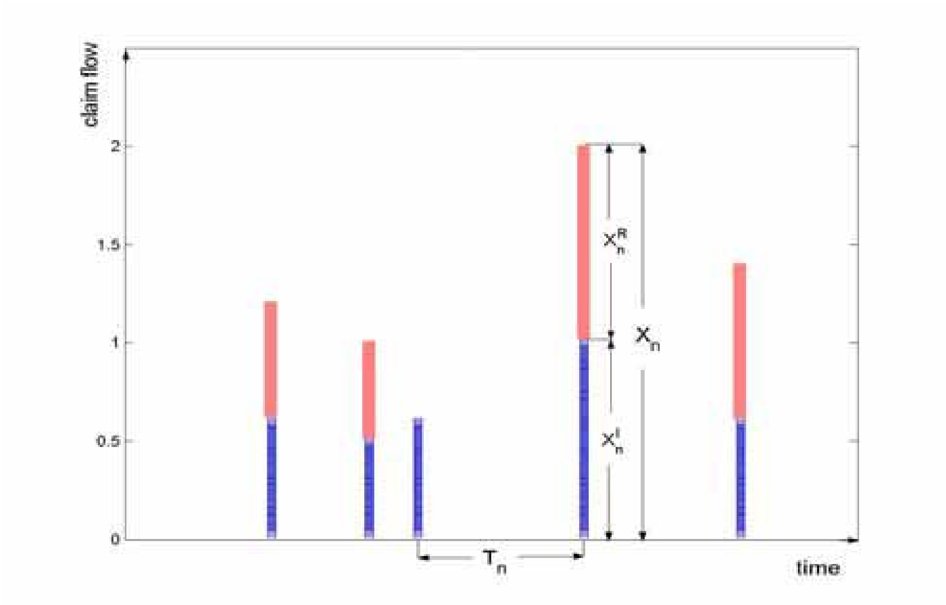


FIGURE 1. The claim flow split between the ceding insurer and the reinsurer.

2.1 Sparre Andersen model. We assume that for $n = 1, 2, \ldots,$

A: (X_n, T_n) are independent identically distributed random vectors with non-negative components;

B: X_n and T_n are independent random variables which have finite expectations (but not necessarily finite variances) and distribution functions F and G respectively.

As mentioned before, $N(t)$ is a renewal process and the claim flow model is called the *Sparre Andersen model* (see, e.g., Rolski, Schmidli, Schmidt, and Teugels (1999)).

The Sparre Andersen model generalizes the classical compound Poisson model where $N(t)$ is the Poisson process, i.e., $G(t) = 1 - e^{-\lambda t}$, an exponential distribution function.

2.2 Claim size and inter-claim interval distributions. The distribution functions F and G are modelled as mixtures of several distributions

$$F(x) = p_1 F_1(x) + \ldots + p_m F_m(x),$$

$$G(t) = q_1 G_1(t) + \ldots + q_k G_k(t),$$

where $p_1 + \ldots + p_m = 1$, $q_1 + \ldots + q_k = 1$; $F_i(x)$, $i = 1, 2, \ldots, m$, and $G_j(t)$, $j = 1, 2, \ldots, k$, are distribution functions from the following classes:

- **Light-tailed distributions** (Uniform(a, b), Exponential(λ), Gamma$(\alpha \leq 1,\ \beta)$, Weibull$(\lambda, \tau \geq 1)$, etc.);

- **Heavy-tailed distributions** (Strict Pareto(α), Alternative Pareto(α, β), Weibull(λ, $\tau \leq 1$), Gamma($\alpha \geq 1, \beta$), Reciprocal Gamma(α, β), etc.);
- **Super-heavy tailed distributions** (Logarithmic (α), etc.).

The quantities $p_1, p_2, \ldots, p_m$, and $q_1, q_2, \ldots, q_k$ will be referred to as the probabilities or weights assigned to the distributions in the claim size and inter-claim interval distribution mixtures, respectively. Intuitively, with probability p_i a particular claim size X_n arrives from the claim flow with claim size distribution F_i, $i = 1, 2, \ldots, m$. Alternatively, the probability p_i may be thought of as the proportion of claim sizes having distribution F_i, $i = 1, 2, \ldots, m$. Analogous reasonings can be made on a particular inter-claim interval T_n with probabilities q_j and inter-claim distributions given by G_j, $j = 1, 2, \ldots, k$.

The exponential and the strict Pareto distributions are used as benchmarks to split distributions into light-, heavy-, and super-heavy tailed:

- if
$$\limsup_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} < \infty \quad \text{for some } \lambda > 0,$$
where $\bar{F}(x) = 1 - F(x)$ denotes the tail of the distribution function F, we call F *light-tailed*;
- if
$$\limsup_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda x}} > 0 \quad \text{for all } \lambda > 0,$$
we call F *heavy-tailed*, and
- if
$$\limsup_{x\to\infty} \frac{\bar{F}(x)}{x^{-\lambda}} > 0 \quad \text{for all } \lambda > 0,$$
we say that F has a *super-heavy tail*.

According to these definitions, the exponential distribution itself is light-tailed, the strict Pareto distribution given by $\bar{F}(x) \sim x^{-\alpha}$, $\alpha > 0$, is heavy-tailed, and the logarithmic distribution given by $\bar{F}(x) \sim (\ln x)^{-\alpha}$, $\alpha > 0$, is super-heavy tailed.

The results of simulations presented in Section 7, are obtained for a compound Poisson model where the distribution F of the claim sizes is a mixture of an exponential F_1 and a reciprocal gamma F_2 with probabilities $p_1 > p_2$.

Recall that a random variable X has a reciprocal (α, β)-gamma distribution if its density is given by

$$f_X(x) = \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{x}\right)^{\alpha+1} \beta^\alpha \exp\{-\beta/x\}, \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0,$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp\{-x\} dx$. Alternatively, $Y = \frac{1}{X}$ has a classical (α, β)-gamma distribution with density

$$f_Y(y) = \frac{1}{\Gamma(\alpha)} \cdot y^{\alpha-1} \beta^\alpha \exp\{-\beta y\}, \quad y \geq 0.$$

The parameter α is known as the shape parameter while β is the scale parameter.

The right tail of the reciprocal gamma distribution follows a power law with exponent $-(\alpha+1)$. Its expected value is

$$\mathbf{E}X = \frac{\beta}{\alpha - 1}, \quad \alpha > 1,$$

while the variance is

$$VarX = \frac{\beta^2}{(\alpha-2)(\alpha-1)^2}, \quad \alpha > 2.$$

Hence, for $1 < \alpha < 2$ the expected value is finite but the variance is not.

2.3 Classical reinsurance treaties. The classical reinsurance arrangements such as quota-share, surplus, excess-of-loss, stop-loss, and their combinations are commonly used and well known. We will consider the following two types in more detail.

(1) An **excess-of-loss treaty** is determined by a positive number $M \geq 0$ called the *retention level* (or the *priority*). With this type of treaty the reinsurer pays that part of each claim that exceeds the retention M; as such

$$X_n^R = \{X_n - M\}_+, \quad n = 1, 2, \ldots,$$

where $\{x\}_+ = \max\{x, 0\}$, $x \in \mathbb{R}$. The ceding insurer covers claims that do not exceed the retention M and pays the amount M when $X_n > M$; more specifically,

$$X_n^I = \min\{X_n, M\}, \quad n = 1, 2, \ldots.$$

In the following we will use the notation excess-of-loss [M] to refer to this treaty.

(2) In a **quota-share treaty**, the reinsurer accepts a certain proportion a of each individual claim, while the cedent retains the remaining $1-a$, $0 < a < 1$; therefore

$$X_n^R = a \cdot X_n,$$

and

$$X_n^I = (1-a) \cdot X_n.$$

The parameter a is called the *proportionality factor*.

2.4 Extreme value reinsurance. If one is interested in coverage against excessive claims, then one should look for extreme value treaties based on the largest claims in the portfolio. Such treaties are largest claims reinsurance, generalized largest claims reinsurance, ECOMOR, drop-down excess-of-loss, etc. The commonly used classical reinsurance treaties do not involve largest claims. However, extreme value reinsurance is far less popular. One of the main reasons is the technical complexity of the dependent order statistics

$$X_1^* \leq X_2^* \leq \cdots \leq X_{N(t)}^*$$

of the individual claims $X_1, X_2, \ldots, X_{N(t)}$, that are used to define extreme value reinsurance treaties.

For example, in a **largest claims treaty** the reinsurer covers the r, $r = 1, 2, \ldots, N(t)$, entire largest claims from the sequence $X_1, X_2, \ldots, X_{N(t)}$.

With the **ECOMOR** (excédent du **co**ût **mo**yen **r**elatif) **treaty** introduced in Thépaut (1950) the reinsurer pays only that part of the r largest claims $X_{N(t)-r+1}^*, \ldots, X_{N(t)}^*$, $r = 1, 2, \ldots, N(t) - 1$, that exceeds the $(r+1)$-th largest claim $X_{N(t)-r}^*$; more specifically,

$$X^R(t) = \sum_{i=1}^{r} \left\{ X_{N(t)-i+1}^* - X_{N(t)-r}^* \right\}, \quad r = 1, 2, \ldots, N(t) - 1,$$

and $X^R(t) = 0$ for $r \geq N(t)$. As such, the ECOMOR can be viewed as an excess-of-loss with a random retention level determined by the $(r+1)$-th largest claim.

Having in mind to tackle the technical complexity of extreme value reinsurance and simplify its application in practice, we introduce in this paper a modified approach for determining extreme value treaties. In general, by an *extreme value reinsurance treaty* we will understand a reinsurance arrangement on a claim by claim basis according to which the reinsurer's share X_n^R of an individual claim X_n is defined as a function of X_n and order statistics of the l previous claims $X_{n-l}, X_{n-l+1}, \ldots, X_{n-1}$, $l = 1, 2, \ldots$, $n = 1, 2, \ldots$. The order statistics will be called the *ordered past sample* and denoted by

$$X_{n-l}^* \leq X_{n-l+1}^* \leq \cdots \leq X_{n-1}^*.$$

The parameter l will be called the *past-sample size*. The notation EV[l] will be used to refer to an extreme value reinsurance treaty satisfying the above general definition.

The crucial point in the last definition is that the rule for splitting claims between the cedent and the reinsurer is given for each individual claim and involves the order statistics from a fixed past claim epoch. At the same time, it does not change the essence of the extreme value treaties. For example, the largest claims treaty can be presented in the form

$$X_n^R = \begin{cases} 0, & \text{if } X_n < X_{n-r}^*, \\ X_n, & \text{if } X_n \geq X_{n-r}^*, \quad r = 1, 2, \ldots, l, \end{cases} \tag{2}$$

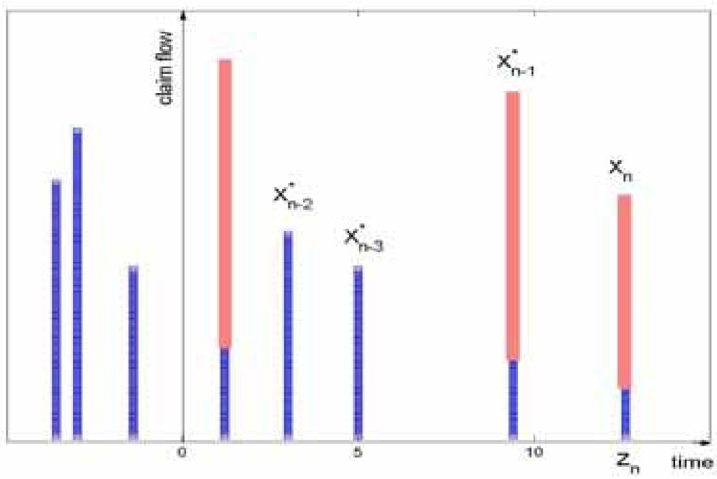


FIGURE 2. LC[l, r, c] reinsurance contract: $l = 3$, $r = 2$, $c = 0.75$.

while the ECOMOR treaty introduced in Thépaut (1950) can be presented as

$$X_n^R = \begin{cases} 0, & \text{if } X_n < X^*_{n-r}, \\ X_n - X^*_{n-r}, & \text{if } X_n \geq X^*_{n-r}, \end{cases} \tag{3}$$

and be still viewed as an excess-of-loss with a random retention, i.e.,

$$X_n^R = \left\{X_n - X^*_{n-r}\right\}_+, \quad r = 1, 2, \ldots, l.$$

In this paper we consider a generalization of the contract (2) arranged in the following way. The reinsurer pays a certain proportion c, $0 \leq c \leq 1$, of each individual claim X_n only if this claim overshoots the $(r+1)$-th largest claim among the l claims prior to X_n; more strictly

$$X_n^R = \begin{cases} 0, & \text{if } X_n < X^*_{n-r}, \\ c \cdot X_n, & \text{if } X_n \geq X^*_{n-r}, \quad r = 1, 2, \ldots, l, \end{cases} \tag{4}$$

where c is called the *proportionality factor*. Hence, the first line insurer covers

$$X_n^I = \begin{cases} X_n, & \text{if } X_n < X^*_{n-r}, \\ (1-c) \cdot X_n, & \text{if } X_n \geq X^*_{n-r}, \quad r = 1, 2, \ldots, l. \end{cases}$$

The role played by the continuous parameter c will be discussed in subsection 4.2.

Hereafter we will use the notation LC[l, r, c] or the prefix LC- to refer to the **l**argest **c**laims reinsurance treaty defined by (4).

Figure 2 illustrates the application of the LC -contract where the past-sample size $l = 3$, the number of the largest claim $r = 2$, and the proportionality factor $c = 0.75$.

Remark 2.1. In analogy to the largest claim reinsurance, the ECOMOR contract given by (3) can be generalized to an ECOMOR[l, r, c] where the reinsured amount is equal to

$$X_n^R = \begin{cases} 0, & \text{if } X_n < X^*_{n-r}, \\ c \cdot \left(X_n - X^*_{n-r}\right), & \text{if } X_n \geq X^*_{n-r}, \quad r = 1, 2, \ldots, l. \end{cases}$$

Remark 2.2. Note that a past sample can be determined not only by a fixed number l of claims prior to the claim X_n, but also by a time interval of a fixed length, say $H > 0$, counted back from the moment Z_n, i.e., the interval $(Z_n - H, Z_n]$. In this case one obtains an extreme value contract with a random past-sample size.

3. Evaluation of Reinsurance Treaties

The effect of a reinsurance treaty is evaluated on certain intervals $I_m = (t'_m, t''_m]$, $m = 1, 2, \ldots$, called the *evaluation intervals*. Evaluation intervals may be determined in a variety of ways. Two of them, namely claim-type and time-type, will be considered in detail while others will only be mentioned as possible alternatives.

3.1 Claim-type evaluation intervals. Claim-type evaluation intervals are formed in such a way that each of them contains a certain number, say $k = 1, 2, \ldots$, of successive claims

$$I_m = \left(Z_{(m-1)\cdot k}, Z_{m\cdot k}\right], \quad m = 1, 2, \ldots,$$

where Z_n is the moment when the n-th claim occurs.

Note that the distribution of the length of a claim-type interval is determined by the inter-claim interval distribution G, and does not therefore depend on k. Its actual length $Z_{m\cdot k} - Z_{(m-1)\cdot k}$, $m = 1, 2, \ldots$, can be different as shown in Figure 3.

3.2 Time-type evaluation intervals. Time-type evaluation intervals are determined by a certain time period, say $h > 0$,

$$I_m = ((m-1) \cdot h, m \cdot h], \quad m = 1, 2, \ldots.$$

The number of claims in one time-type evaluation interval may be different from the number of claims in another one as seen in Figure 4. This time it is the distribution of this number that depends on the inter-claim interval distribution G.

3.3 Money-type evaluation intervals. Money-type evaluation intervals are determined by the amount of money, say $e > 0$, that the ceding (or, reinsurance) company should pay within each interval such that the last claim in the interval is the one that first overshoots the amount e.

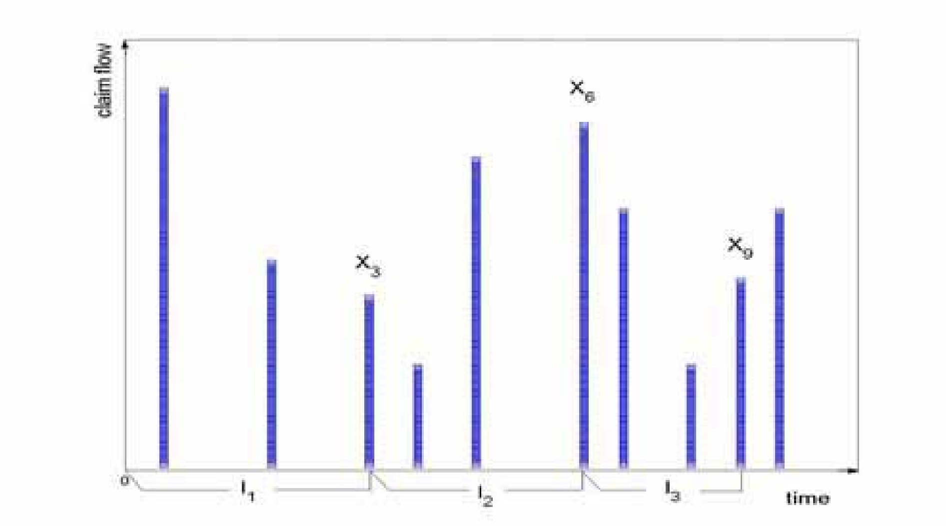


FIGURE 3. Claim-type evaluation intervals: $k = 3$.

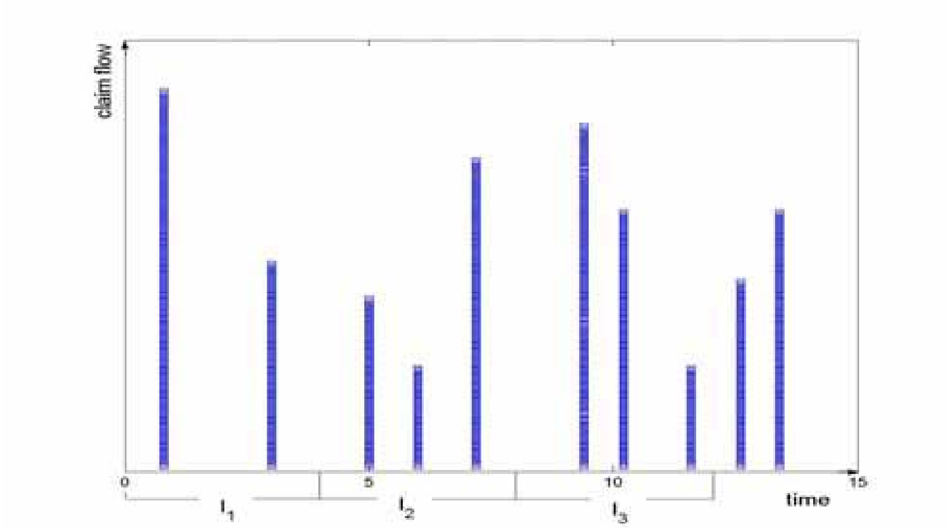


FIGURE 4. Time-type evaluation intervals: $h = 4$ (time units).

3.4 The average reinsurer's quota load and contract characteristics. For any evaluation interval $I_m = (t'_m, t''_m]$, $m = 1, 2, \ldots$, denote by $\widetilde{X}_m$ the *aggregate interval claim amount*

$$\widetilde{X}_m = \sum_{n: t'_m < Z_n \le t''_m} X_n, \quad m = 1, 2, \ldots.$$

Then the *interval reinsured claim amount* can be obtained as

$$\widetilde{X}^R_m = \sum_{n: t'_m < Z_n \le t''_m} X^R_n, \quad m = 1, 2, \ldots,$$

while the *interval deductible* equals

$$\widetilde{X}_m^I = \widetilde{X}_m - \widetilde{X}_m^R, \quad m = 1, 2, \ldots .$$

Denote by $\mathfrak{M}$ the set of *contract characteristics* such as

- expectation,
- median,
- 25% and 75% quantiles,
- value at risk,
- variance,
- dispersion,
- coefficient of variation,
- skewness,
- kurtosis, etc.

of the interval deductible and interval reinsured claim amount.

Having a sample of interval deductibles $\widetilde{X}_1^I, \ldots, \widetilde{X}_N^I$ and a sample of interval reinsured amounts $\widetilde{X}_1^R, \ldots, \widetilde{X}_N^R$ (where N is the sample size), we apply Monte Carlo methods to estimate the characteristics from the set $\mathfrak{M}$ and the *average reinsurer's quota load* Q_R. The estimates are then used to evaluate and analyse reinsurance treaties.

Below, we will consider the reinsurance treaties described in subsections 2.3 and 2.4, i.e., the excess-of-loss, the quota-share, and the largest claims treaty defined by (4), when the underlying claim process is of Sparre Andersen type satisfying conditions (A) and (B) of subsection 2.1.

The *average reinsurer's quota load* Q_R is the percentage of the aggregate interval claim amount that the reinsurer covers on average. It is calculated as

$$Q_R = \mathbf{P}\text{-}\mathbf{lim}_{N\to\infty} \frac{\widetilde{X}_1^R + \cdots + \widetilde{X}_N^R}{\widetilde{X}_1 + \cdots + \widetilde{X}_N} \cdot 100\% =$$

$$= \frac{\mathbf{P}\text{-}\mathbf{lim}_{N\to\infty} \frac{\widetilde{X}_1^R + \cdots + \widetilde{X}_N^R}{N}}{\mathbf{P}\text{-}\mathbf{lim}_{N\to\infty} \frac{\widetilde{X}_1 + \cdots + \widetilde{X}_N}{N}} \cdot 100\% = \frac{\mathbb{E}^* \widetilde{X}_1^R}{\mathbb{E}^* \widetilde{X}_1} \cdot 100\%, \tag{5}$$

where $\mathbb{E}^* \widetilde{X}_1^R$ ($\mathbb{E}^* \widetilde{X}_1$) is the expected value of the reinsured (aggregate) claim amount on the evaluation interval I_1 in the "stationary" regime. If the evaluation intervals are of claim-type then

$$\mathbb{E}^* \widetilde{X}_1^R = k \cdot \mathbb{E}^* X_1^R; \tag{6}$$

and if evaluation intervals are of time-type then

$$\mathbb{E}^* \widetilde{X}_1^R = h \cdot \frac{\mathbb{E}^* X_1^R}{\mathbb{E} T_1}. \tag{7}$$

Thus, in both cases

$$Q_R = \frac{\mathbb{E}^* X_1^R}{\mathbb{E}^* X_1} \cdot 100\%. \tag{8}$$

Note that for the reinsurance contracts with simple structure where the reinsurer's share of a claim is defined by the size of this claim only (e.g., excess-of-loss, quota-share), $\mathbb{E}^* X_1^R = \mathbb{E} X_1^R$ and $\mathbb{E}^* X_1 = \mathbb{E} X_1$ in formulas (5)–(8). In this case formulas (5)–(8) can be proved with the use of classical results of renewal theory, given, for example, in Feller (1966) or in Silvestrov (1980a). Indeed, under these treaties, the sequences $(X_1, X_2, \ldots)$ and $\left(X_1^R, X_2^R, \ldots\right)$ are sequences of independent and identically distributed random variables. However, under an LC-contract, the random sequence $\left(X_1^R, X_2^R, \ldots\right)$ belongs to the class of so-called sequences of $(l+1)$-dependent random variables. The last model can also be considered as a particular case of a Markov renewal process, since the random sequence $\bar{X}_n^l = (X_{n-l}, X_{n-l+1}, \ldots, X_{n-1}, X_n)$ is a homogeneous Markov chain. The generalization of the law of large numbers for such models as given, for instance in Shurenkov (1989) or in Silvestrov (1980b), can be applied to prove the relations (5)–(8) in this case as well.

An additional theoretical problem arises when the claim size distribution mixture contains heavy-tailed distributions with infinite variance. In this case, one can observe a "masking" effect of quasi-stability of estimates

$$\frac{\widetilde{X}_1^R + \cdots + \widetilde{X}_N^R}{\widetilde{X}_1 + \cdots + \widetilde{X}_N}$$

of the quantity Q_R for large values of N. It turns out that the estimates of Q_R are not stable and may have sudden jumps due to the appearance of outliers in the simulated samples. This effect can appear when the claim size distribution is modelled by a mixture

$$F = p_1 F_1 + p_2 F_2,$$

where F_1 is a light-tailed distribution, F_2 is a heavy-tailed distribution, $p_1 \gg p_2$, while

$$p_1 \cdot m_1 \approx p_2 \cdot m_2,$$

where m_i denotes the mean value of the distribution function F_i, $i = 1, 2$.

In view of the above discussion, the estimate $\hat{N}$ of the sample size N that is sufficient to provide a reliable estimate for the average reinsurer's quota load Q_R does require additional theoretical investigation, especially for the EV[l]-type contracts and LC-contracts.

It is also of interest to estimate the distribution functions of the interval deductible and the interval reinsured amount. One can plot histograms and logarithmic histograms of the distributions in question with the help of the Reinsurance Analyser. The (ordinary) histograms of the interval deductible and the reinsured claim amount are based on samples $\widetilde{X}_1^I, \ldots, \widetilde{X}_N^I$ and $\widetilde{X}_1^R, \ldots, \widetilde{X}_N^R$, respectively. The log-histograms are plotted on the basis of the samples obtained from the initial data by the transformation $\widetilde{X}' = \ln\left(\widetilde{X} + 1\right)$. A log-histogram turns out to be more illustrative in cases where the underlying distributions have a large or even infinite variance or/and are very skewed.

4. Direct and Inverse Problems

As mentioned in the Introduction, the prime use of the Reinsurance Analyser is to compare reinsurance treaties. In addition, the program can be used as a tool for solving the following two problems.

4.1 Problem 1 (Direct: estimation of the contract characteristics). Given:

(a) the claim size and inter-claim interval distribution functions F and G respectively,
(b) the reinsurance treaty R and its parameters,
(c) the sample size N and the type of evaluation intervals I_m, $m = 1, 2, \ldots, N$,

estimate the average reinsurer's quota load Q_R and contract characteristics from the set $\mathfrak{M}$.

The solution algorithm is based on Monte Carlo simulations and consists of the following steps

(A) simulate samples $\widetilde{X}_1^R, \ldots, \widetilde{X}_N^R$, and $\widetilde{X}_1^I, \ldots, \widetilde{X}_N^I$ of the chosen sample size N, namely,
  (A1) simulate $T_1, T_2, \ldots$, and $X_1, X_2, \ldots$ from the given claim flow distribution,
  (A2) split claim flow into deductibles $X_1^I, X_2^I, \ldots$, and reinsured amounts $X_1^R, X_2^R, \ldots$ according to the rules of the reinsurance treaty R,
  (A3) form samples $\widetilde{X}_1^R, \ldots, \widetilde{X}_N^R$, and $\widetilde{X}_1^I, \ldots, \widetilde{X}_N^I$ according to the type of the evaluation intervals,
(B) estimate the average reinsurer's quota load Q_R,
(C) estimate contract characteristics from the set $\mathfrak{M}$,
(D) plot (logarithmic) histograms of the interval deductible and the reinsured claim amount.

4.2 Problem 2 (Inverse: estimation of the contract parameters and contract characteristics). Given:

(a) the claim size and inter-claim interval distribution functions F and G respectively,
(b) the type of reinsurance treaty R,
(c) the value Q for the average reinsurer's quota load,
(d) the sample size N and the type of evaluation intervals I_m, $m = 1, 2, \ldots, N$,

find the parameter(s) of the reinsurance treaty R such that the average reinsurer's quota load Q_R is equal to the given value Q, and then estimate characteristics of the reinsurance treaty R composing the set $\mathfrak{M}$.

The algorithm for solving Problem 2 can be divided into two parts

I: find the parameter(s) of the chosen treaty R such that $Q_R = Q$,

II: estimate characteristics from the set $\mathfrak{M}$ with already known contract parameters.

Part II repeats the algorithm for solving Problem 1 described above. Part I is non-trivial and requires closer consideration. The main idea is to use a dichotomy procedure to estimate the parameter(s) of the given treaty R.

In general, if a treaty has two or more parameters it is supposed that only one is free while the others are fixed.

Let p denote the parameter to be estimated.

(i) Suppose, first, that p takes on non-negative real values $p \geq 0$ (for example, the excess-of-loss treaty with retention level $M \geq 0$). In this case Part I of the algorithm consists of the following steps:

(I.1) set p equal to an initial value p_0;

(I.2) simulate samples $\widetilde{X}_1^R, \dots, \widetilde{X}_N^R$, and $\widetilde{X}_1^I, \dots, \widetilde{X}_N^I$ with respect to the treaty $R[p_0]$;

(I.3) estimate the average reinsurer's quota load $Q_{R[p_0]}$;

(I.4) compare $Q_{R[p_0]}$ to Q;
if $|Q_{R[p_0]} - Q| < \epsilon$, where ϵ is the admissible error, then proceed with the treaty evaluation (Part II); otherwise

(I.5) find the next approximation to p using the dichotomy procedure and repeat steps (I.2)–(I.4).

Figure 5 gives a graphical illustration to the estimation procedure for the retention level M of the excess-of-loss treaty.

(ii) Next, suppose that the parameter p is a non-negative integer. In this case, the inverse problem to estimate p given the value Q does not always have a solution. To circumvent this shortcoming, an additional continuous parameter has to be introduced such that the essence of the treaty is still kept.

For example, the largest claims cover defined by (2) has an integer parameter r taking values $r = 1, 2, \dots$. The larger the value of r, the larger the reinsurer's quota load Q_R. When solving the inverse problem, it may happen that the equality $Q_R = Q$ can never be reached: for some integer value $\mathbf{r}$ of r the reinsurer's quota load $Q_{R[\mathbf{r}]}$ is less than Q, $Q_{R[\mathbf{r}]} < Q$, while for the next integer value of r holds $Q_{R[\mathbf{r}+1]} > Q$ (see, also, Figure 6(a)). However, the introduction of a new continuous parameter c and generalization of the treaty as given by (4) make the inverse problem solvable for any given value Q and keep the essence of the initial treaty as well.

On the other hand, the appearance of a new parameter implies that the inverse problem may have more than one solution.

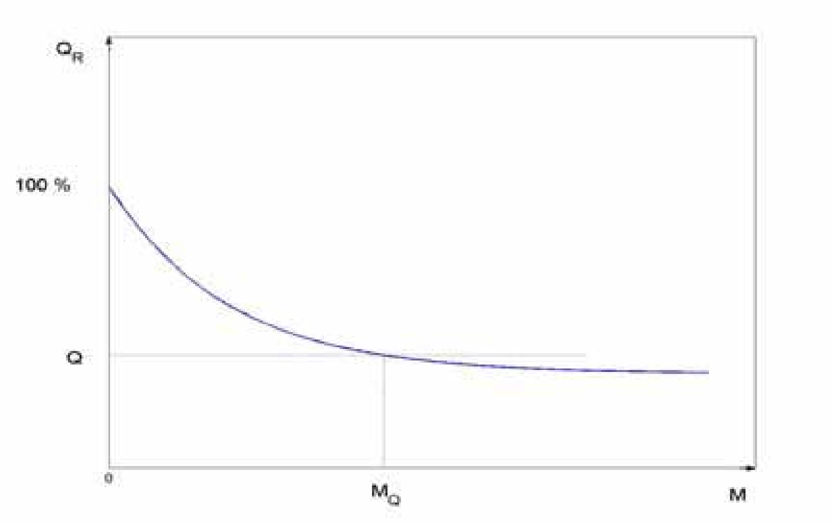


FIGURE 5. Estimation of the retention level M of the Excess-of-loss [M].

For the LC-treaty, we suggest two alternative solving procedures to cope with the disadvantage caused by the non-uniqueness of the solution. Let the past-sample size l be fixed, then either

1) given the value of r, estimate the proportionality factor c following (I.1)–(I.5) for $p = c$ in which case there exists either a unique or no solution,

   or

2) find the value $\mathbf{r}$ of r such that $Q_{LC[l,\mathbf{r}-1,1]} < Q \leq Q_{LC[l,\mathbf{r},1]}$, and estimate the corresponding value $c(\mathbf{r})$ of c such that the equality $Q_{LC[l,\mathbf{r},c(\mathbf{r})]} = Q$ holds.

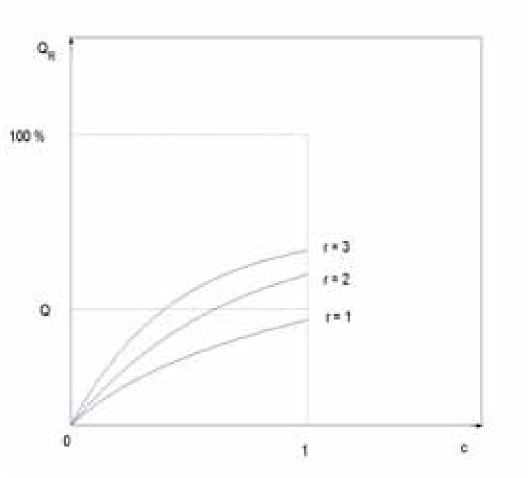
Both alternatives are illustrated in Figure 6(b).

5. Comparison of Reinsurance Treaties

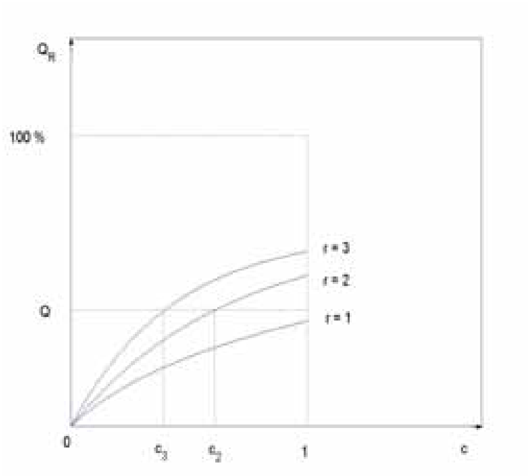
The Reinsurance Analyser allows us to compare the riskiness of reinsurance treaties by a set of *risk measures*, i.e., the value-at-risk and variance-based contract characteristics from the set $\mathfrak{M}$. It is sensible to consider the variance and functions of both the variance and the mean value. The latter takes into account both characteristics while the former completely overlooks the expectation.

The *conditions of fair comparison* of a pair of reinsurance contracts assume

C1: that the contracts are balanced by the average reinsurer's quota load, i.e., their parameters are chosen or estimated such that the average reinsurer's quota loads of both contracts are equal;
C2: the contracts are applied to the same simulated sample;
C3: the same evaluation intervals are used.



(a) Proportionality factor $c = 1$.



(b) Proportionality factor c, $0 \leq c \leq 1$.

FIGURE 6. LC[l, r, c] with the fixed past sample size l.

On the basis of the above, let us compare R_1 and R_2, two different reinsurance treaties.

Given:

(a) the claim size and inter-claim interval distribution functions F and G,
(b) the value Q for the average reinsurer's quota load,
(c) the sample size N and the type of evaluation intervals I_m, $m = 1, 2, \ldots, N$,

then the treaties R_1 and R_2 are compared in a solution algorithm that consists of the following steps:

(A) estimate the parameters of treaties R_1 and R_2 such that $Q_{R_1} = Q_{R_2} = Q$,

(B) simulate samples $\widetilde{X}_1^{R_i}, \dots, \widetilde{X}_N^{R_i}$, and $\widetilde{X}_1^{I_i}, \dots, \widetilde{X}_N^{I_i}$, $i = 1, 2$, of the chosen sample size N,

(C) estimate characteristics from the set $\mathfrak{M}$ for both the R_1 and R_2 contracts,

(D) plot (logarithmic) histograms of the interval deductibles and the reinsured claim amounts,

(E) compare the contracts R_1 and R_2 by (C) and (D).

Since we allow claim flows to contain excessively large claims, we model claim sizes using heavy-tailed distributions with an infinite second or, even an infinite first moment. This might in turn imply the infiniteness of the corresponding moment of the distribution of the interval reinsured claim amount. This is the case for reinsurance arrangements like excess-of-loss, largest claims, etc. In order to compare a pair of treaties in this case, we consider the ratios of the estimates of the corresponding risk measure. For example, we might take the ratio of the sample variances

$$\frac{S_N^{R_1}}{S_N^{R_2}}, \tag{9}$$

where $S_N^{R_i}$ is the sample variance of the interval reinsured claim amount with respect to the contract R_i, $i = 1, 2$, based on a sample size N. As pointed out in subsection 3.4, the sequences $\left(X_1^{R_1}, X_2^{R_1}, \dots\right)$ and $\left(X_1^{R_2}, X_2^{R_2}, \dots\right)$ used in the reinsurance contracts with a simple structure (such as excess-of-loss, quota-share, etc.) are sequences of independent and identically distributed random variables. If the underlying claim size distribution F has a finite variance then the quantity (9) converges weakly to a constant that equals to the ratio of real variances

$$\textbf{P-lim}_{N\to\infty} \frac{S_N^{R_1}}{S_N^{R_2}} = \frac{Var^* \widetilde{X}^{R_1}}{Var^* \widetilde{X}^{R_2}}, \tag{10}$$

where $Var^* \widetilde{X}^{R_i}$, $i = 1, 2$, is the (finite) variance of the interval reinsured claim amount with respect to the contract R_i, $i = 1, 2$.

However, the problem becomes non-trivial when the underlying claim size distribution F is heavy-tailed with infinite variance. As was shown in Albrecher and Teugels (2006), even in this case one can observe a stabilization of the ratios of type (10) based on the same simulated sample. The fact that the ratios of type (10) are stable can be used for the comparison of reinsurance contracts even when the claim size distribution is heavy-tailed. In the general case one requires a deeper theoretical approach, especially for the case of the EV[l]-type reinsurance contracts.

6. Description of the Program Reinsurance Analyser

In this section we describe the experimental program system **Re**insurance **An**alyser (ReAn) and give some advice on how to work with it.

The prime use of the ReAn is to compare the riskiness of reinsurance contracts. Also, it can be used as a tool to estimate

1) the average reinsurer's quota load Q_R for the given treaty (Problem 1, Direct);
2) one or several parameters of the reinsurance treaties, given the average reinsurer's quota load Q_R (Problem 2, Inverse).

The approach used in the program is based on global stochastic modelling of various flows of claims described in Section 2 and involves the method for evaluating reinsurance treaties presented in Section 3.

The input information required is

- sample size;
- claim size and inter-claim interval distributions;
- type of evaluation intervals;
- type(s) of reinsurance treaty(-ies);
- parameters of the treaty or the average reinsurer's quota load;
- premium schemes (optional).

The output information is then

- an estimate of the average reinsurer's quota load or estimates of the parameters of the reinsurance treaties;
- estimates of characteristics from the set $\mathfrak{M}$;
- (logarithmic) histograms of the distributions of the interval deductible and the reinsured claim amount, etc.

6.1 Program characteristics and user's interface. As mentioned in the Introduction, the program is written in the programming language Java with the use of SSJ and JFreeChart class libraries. Thanks to the flexibility of Java the software is platform independent and well-equipped for distributed calculations through the Internet.

The ReAn has a standard Swing graphical user's interface (see, e.g., Figure 7). The main interactive element of the program is **Contract window**. The Contract window consists of two panels, the upper tabbed panel and the lower common panel. There are four tabs on the upper panel called, subsequently, **Claim Flow**, **Contract 1**, **Contract 2**, and **Functional Characteristics**. The lower panel contains information and facilities such as money and time units, sample size, *Start* button and execution progress bar, which are common to all four tabs in the Contract window. The panel is called the **Common panel** and remains accessible independently of the currently activated tab.

6.2 Claim flow tab. Claim size and inter-claim interval distributions are described on the Claim flow tab. Recall that each of these two distributions is modelled as a mixture of several distributions, where each entry in the mixture has a certain probability. The mixtures are formed by the user in, respectively, *Claim size distribution* and *Inter-claim interval distribution* tables on the Claim flow tab.

Initially both tables are empty. To add a new distribution, the user invokes *Distribution club* dialog by clicking *Add...* button and picks out a distribution from the list, as shown in Figure 7. For convenience, comprehensive information about each distribution is available in the program's *Help System*. The distributions listed in the dialog are divided into three groups according to their tails: light, heavy and super-heavy. Option *Database* should be chosen if one has a database of real-life past claims and would like to model a claim flow with the same distribution as the flow in the database.

Remark 6.1. In the Distribution Club dialog, the items marked by black colour in the list are already realised in the program while the items marked by grey are reserved for future development.

Add... and *Delete* buttons, and *Save/Change* combo box serve for editing already formed mixtures. Using them, one can add or delete distributions, change their parameters or weights (provided that the sum of the weights in one mixture remains equal 1). The *load* and *total mean* are automatically calculated by the program, the former as the product of the mean value and the corresponding assigned probability (weight), and the latter as the sum of loads in one mixture (see Figure 7).

6.3 Contract 1 tab. The Contract 1 tab contains information about the first reinsurance treaty, the type(s) of evaluation intervals, the average reinsurer's quota load and the cedent's and reinsurer's premium calculation schemes. It further displays some results of simulations. The user decides here which of the three problems formulated in Sections 4–5 he would like to solve with the help of the program.

Contract 1 tab contains information that is classified as input or output depending on the problem chosen by the user. In the subsequent subsections, we will consider the peculiarities of each of the three cases. However, in any case the following data should be specified on the Contract 1 tab by the user:

(**A**) type of reinsurance treaty;
(**B**) type of evaluation intervals;
(**C**) premium calculation principles with safety coefficients (optional).

A. The type of reinsurance treaty is picked out from the list of available treaties in the *Contract Club Dialog* (see Figure 8). Treaties in the list are grouped according to their structure, namely:

1) proportional reinsurance;

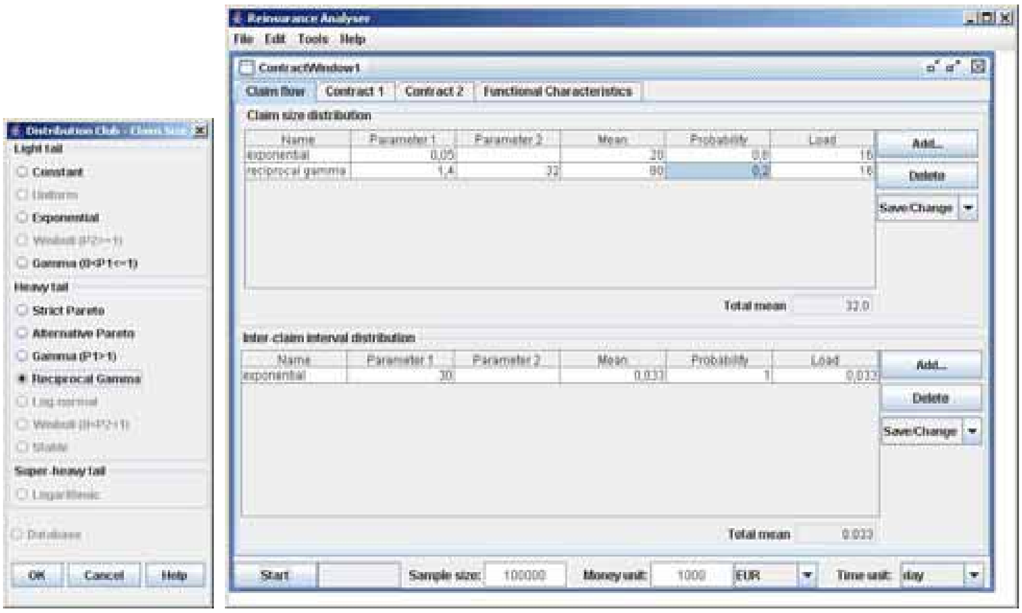


FIGURE 7. Distribution Club Dialog. Claim flow tab.

2) non-proportional reinsurance;
3) combination of treaties.

For convenience, the rule for calculating the reinsurer's share is given as a tool-tip near the name of the corresponding reinsurance contract.

Remark 6.2. As for the Distribution Club, the contracts marked by grey colour in the Contract Club dialog are reserved for future development.

B. In order to specify the type of evaluation intervals, the user activates the proper check box and defines the corresponding parameter (for example, the number of claims k for claim-type evaluation intervals) on the *Insurer's/Reinsurer's quota by* panel. Note that the time parameter h, defined in subsection 3.2, and the money parameter e, defined in subsection 3.3, are measured in time and money units, respectively, displayed on the Common panel.

C. The user has the additional options to involve premiums in the analysis and comparison of the contracts. In order to exclude them from the consideration, the *Ignore premiums* check box should be activated. Otherwise, one defines premium calculation principles along with the safety coefficients for both ceding and reinsurance companies. The list of available premium calculation principles contains

1) *expected value* principle

$$P = (1 + a)\mathbb{E}X;$$

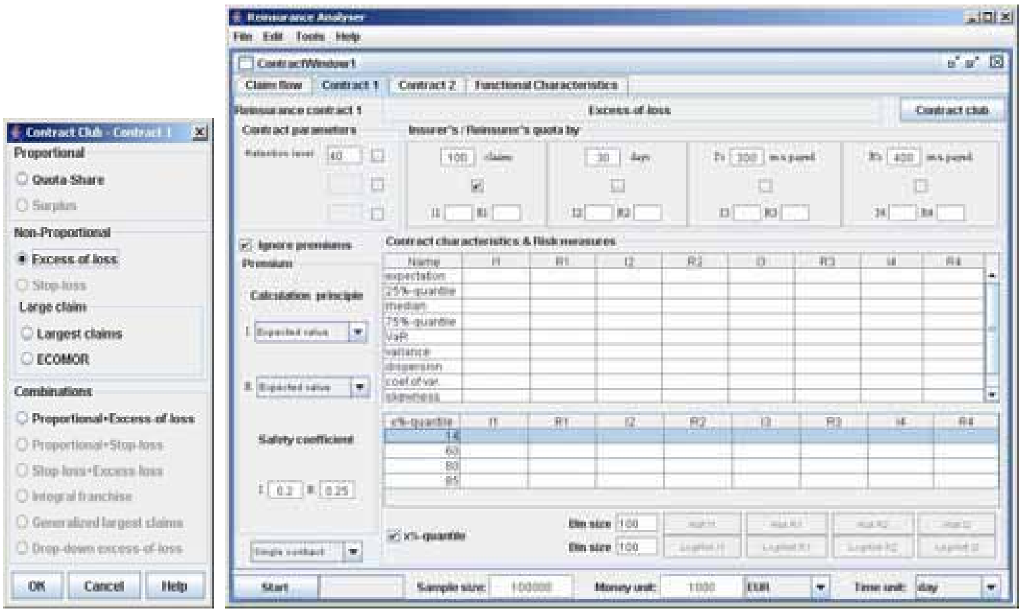


FIGURE 8. Contract Club Dialog. Contract 1 tab.

2) *mean value* principle

$$P = (\mathbb{E}X^2 + VarX)^{1/2};$$

3) *standard deviation* principle

$$P = \mathbb{E}X + a\mathbb{D}X;$$

4) *variance* principle

$$P = \mathbb{E}X + aVarX;$$

5) *modified variance* principle

$$P = \mathbb{E}X + a\frac{VarX}{\mathbb{E}X},$$

where P is the premium, a, $0 \leq a \leq 1$, is the safety coefficient. Further, X denotes either the total claim amount, in the case of the ceding company's premium, or the reinsured amount, in the case of the reinsurer's premium.

Also, the user can extend the list of available contract characteristics by four additional quantiles (see Figure 8).

6.4 Problem 1. Assume one would like to solve Problem 1 (Direct) formulated in subsection 4.1 using the ReAn program. Then, in addition to (**A**)–(**C**), one defines the parameter(s) of the chosen reinsurance treaty and switches the indicator *Single contract/ Compare contracts* to the first position *Single contract*.

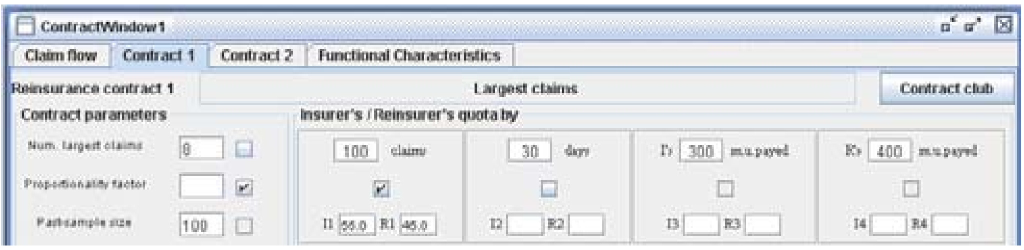
Note that the ReAn program allows to solve Problem 1 for several types of evaluation intervals at the same time. To realise this option, the user just has to activate all the proper check boxes on the *Insurer's/Reinsurer's quota by* panel and define the parameters of the chosen types of evaluation intervals.

6.5 Problem 2. Suppose now that Problem 2 (Inverse) formulated in subsection 4.2 has to be solved. Then, in addition to (**A**)–(**C**), the user defines the average reinsurer's quota load Q_R in the text field under the check box corresponding to the type of evaluation intervals specified earlier (e.g., in the text field *R1* if claim-type evaluation intervals are chosen), and activates the check box next to the treaty's parameter to be estimated, as shown in Figure 9(a). The indicator *Single contract/ Compare contracts* should be switched to position *Single contract*. In contrast to the direct problem, only one type of evaluation intervals can be chosen when solving the inverse problem.

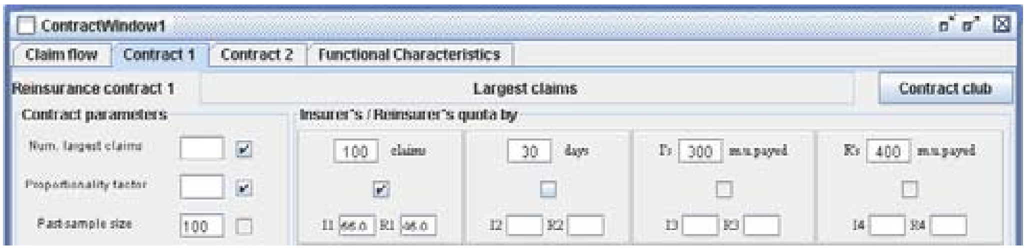
In general, only one contract parameter can be estimated at a time. However, for the LC-treaty, the user can choose to estimate either one (proportionality factor c) or two (the number of largest claims r and proportionality factor c) parameters. Both alternatives were described in detail in subsection 4.2. Here we only recall that the program finds the smallest possible value of r in the latter case. In order to realise the second alternative the user activates two proper check boxes as shown in Figure 9(b).

6.6 Contract 2 tab. When two reinsurance treaties have to be compared by means of the ReAn, one of them is described on the Contract 1 tab and the other is described on the Contract 2 tab. The Contract 2 tab has the same interface as the Contract 1 tab, and it contains the same kind of information as the Contract 1 tab but for the other treaty. The only difference between these two tabs is that the data common for both treaties (e.g., the average reinsurer's quota load) can be defined and edited only on the Contract 1 tab.

6.8 Problem 3. Comparison of treaties. In order to compare the riskiness of two reinsurance treaties, the user defines their types on the Contract 1 and Contract 2 tab respectively. Further, he specifies the average reinsurer's quota load and type of evaluation intervals, and switches the indicator *Single contract/ Compare contracts* to position *Compare contracts* as shown in Figure 10. Recall, that to provide a fair comparison, the type of evaluation intervals and the average reinsurer's quota load should be the same for both treaties. Thus, this data, defined on the Contract 1 tab, is transmitted automatically to the Contract 2 tab without a possibility to be edited there (compare Figure 10 and Figure 11). As an alternative, one can define parameters of the first treaty instead of the reinsurer's quota load. Then the latter will be estimated by the program.



(a) Estimation of the proportionality factor c, given the past-sample size $l = 100$ and the number of largest claims $r = 8$.



(b) Estimation of the number of largest claims r and the proportionality factor c, given the past-sample size $l = 100$.

FIGURE 9. Inverse problem. Estimation of the parameter(s) of the LC[l, r, c], given the average reinsurer's quota load $Q_R = 45\%$.

6.8 Program output. In order to start calculations, the user clicks *Start* button. The estimates of the contract(s) characteristics are displayed in the *Contract characteristics and Risk measures* table(s). The (log)histograms of the distribution of the interval deductible and reinsured claim amount can be obtained by clicking on (*LogHist...*) *Hist...* buttons.

In case the comparison problem has been solved, the user can invoke *Compare Dialog* by choosing command *Compare* in the *Tools Menu* as shown in Figure 11. The *Compare Dialog* displays the estimates of the risk measures for both contracts along with the ratios of the corresponding estimates.

6.9 Functional Characteristics tab (optional). The results of the functioning of both ceding and reinsurance companies are represented on the Functional Characteristics tab. As input data the tab contains the initial capital and ruin level for both ceding and reinsurance company, and the time horizon T. Also, the premium calculation principles for either of the companies can be changed here, and it will in turn imply the change of the corresponding premium scheme defined on the Contract 1 tab. The output information displayed on the Functional Characteristics tab is the ruin probability of ceding and reinsurance company before time horizon T. This tab is reserved for future development.

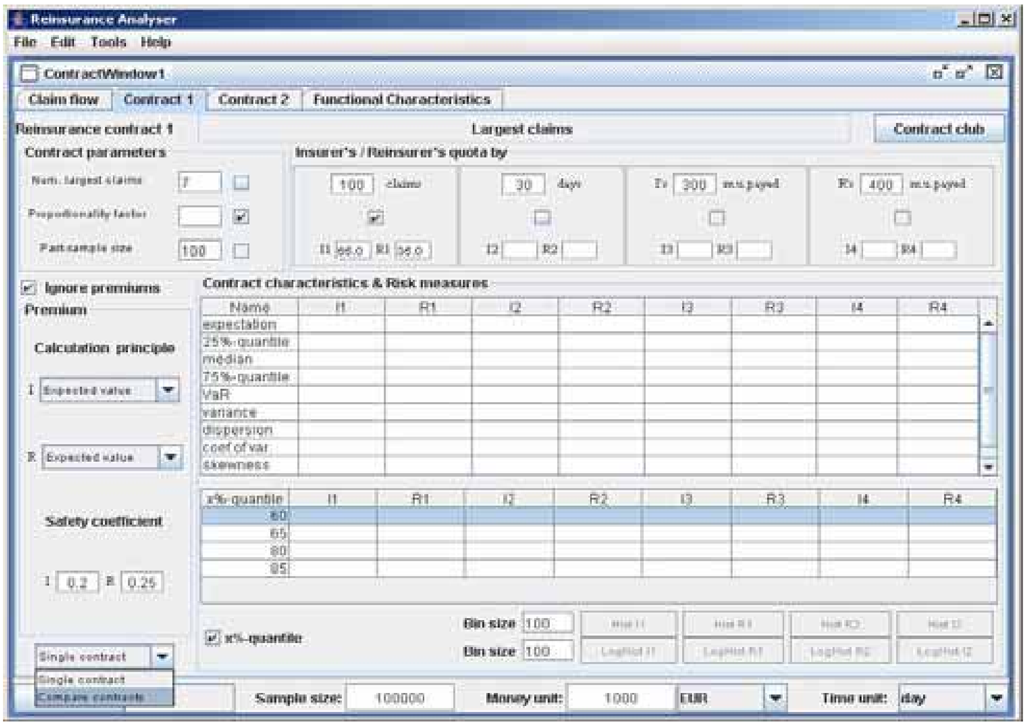


FIGURE 10. Comparison of LC[l, r, c] and Excess-of-loss [M]. Reinsurer's quota load $Q_R = 35\%$. Contract 1: LC[100, 7, c], c to be estimated.

7. Experimental Studies

In this section we give some illustrations of the estimation of the average reinsurer's quota load, parameters and characteristics of reinsurance contracts. We also present the results of the experimental studies made with the help of the ReAn program on the comparison of the riskiness of a pair of reinsurance contracts. In particular, we compare the classical excess-of-loss cover to the largest claims reinsurance arrangement defined in (4), both contracts being applied to claim flows modelled by means of heavy-tailed distributions.

In the simulation experiments presented below, the sample size is $N = 10^5$ when the inverse problem for the largest claims reinsurance contract is not involved and $N = 10^6$ otherwise, a monetary unit equal EUR 1000, and a time unit equal 1 day. Evaluation intervals are formed by $k = 100$ claims, $I_m = (Z_{100(m-1)}, Z_{100m}]$, $m = 1, 2, \ldots, N$; neither the cedent's nor the reinsurer's premiums are involved.

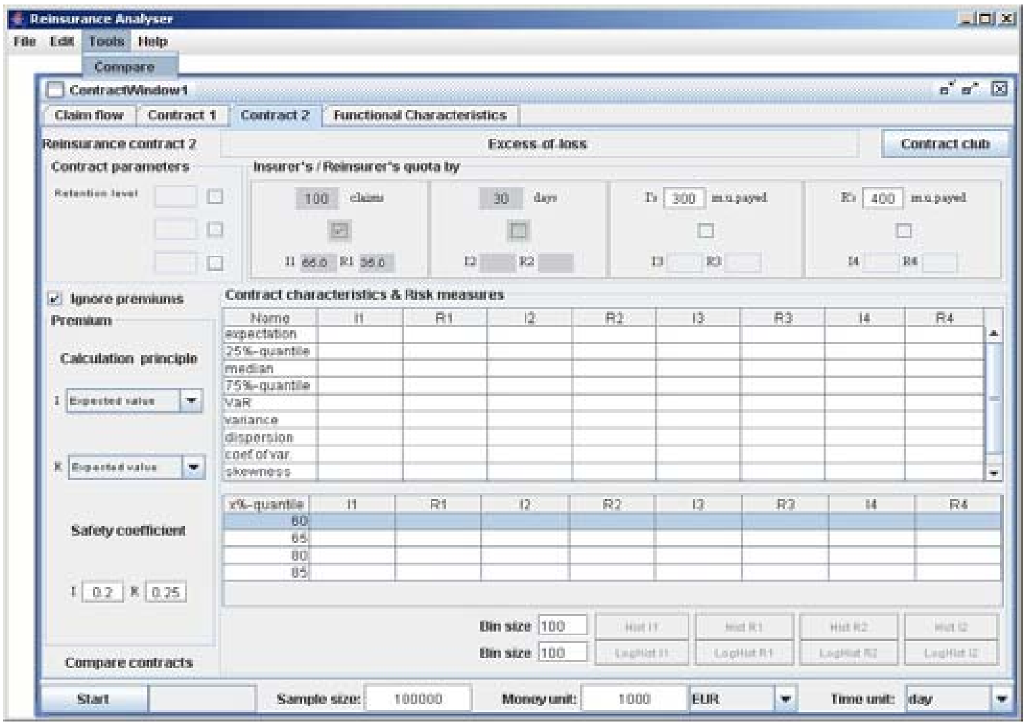


FIGURE 11. Comparison of LC[l, r, c] and Excess-of-loss [M]. Reinsurer's quota load $Q_R = 35\%$. Contract 2: Excess-of-loss [M], M to be estimated.

Under such conditions, it takes on an ordinary computer (e.g., Intel Pentium 4, CPU 2.80 GHz, RAM 512 MB) about 10–20 seconds to solve Problem 1 and 1–2 minutes to solve Problem 2 for any reinsurance contract. Solution of the comparison problem takes from 1 to 4 minutes.

The claim flows are modelled as follows (see, also, Figure 7):

(a) Inter-claim intervals $T_1, T_2, \ldots$ are independent and exponentially distributed with parameter $\lambda_1 = 30$, i.e., the expected length of an inter-claim interval equals $\mathbb{E}T_1 = \mathbb{E}T_2 = \ldots = \frac{1}{30} \approx 0.033$ of a day.

(b) Claim sizes $X_1, X_2, \ldots$ are independent random variables, and their distribution is given as a mixture of an exponential with parameter $\lambda_2 = 0.05$ and a reciprocal gamma with parameters $\alpha = 1.4$ and $\beta = 32$. The light-tailed exponential distribution used for modelling small and middle claims has the higher weight 0.8, while the heavy-tailed reciprocal gamma used for modelling excessive claims has the lower weight 0.2. The values of the parameters were selected such that both distributions give the same load.

(c) (T_n, X_n), $n = 1, 2, \ldots$, are independent random vectors.

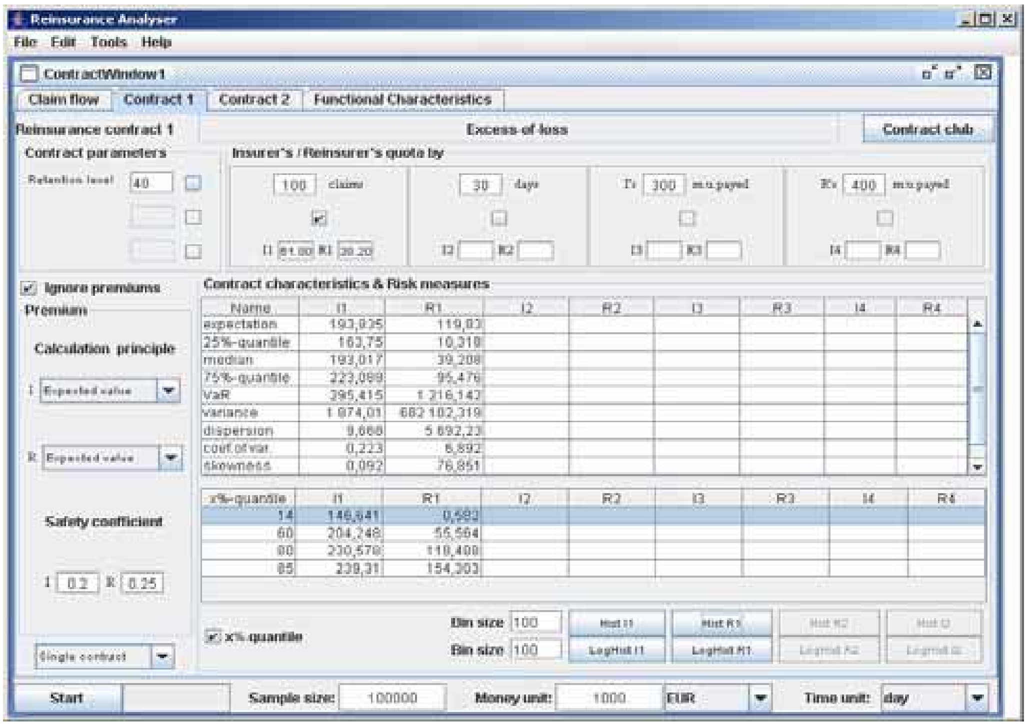


FIGURE 12. Estimation of Q_R for Excess-of-loss [40].

Remark 7.1. The inter-claim interval distribution is not involved in the modelling of the claim flows if the evaluation intervals are of claim-type.

Remark 7.2. The claim size model used in the simulations reflects a realistic situation when the majority, 80%, of claims arriving to the insurance company are often claims of small or middle size, while the remaining 20% are large.

7.1 Estimation of the average reinsurer's quota load. Figure 12 shows the results of the evaluation of the excess-of-loss contract with retention level $M = 40$. In particular, the estimated value of the reinsurer's quota load is 38.2%. The logarithmic histogram of the interval reinsured claim amount is shown in Figure 13.

In Figures 14 and 15, the reader can see the effect of applying the LC[100; 7; 1] under the same conditions as the excess-of-loss cover in the previous example. In particular, the estimated value of the reinsurer's quota load is 42.8%.

Note that with both excess-of-loss and LC-treaty many claims are entirely retained by the ceding insurer (in the examples given here, it is about 13% and 48% respectively). As a consequence, both histograms of the interval reinsured claim amount have an outlier at zero. In Figure 15(b), it is shown

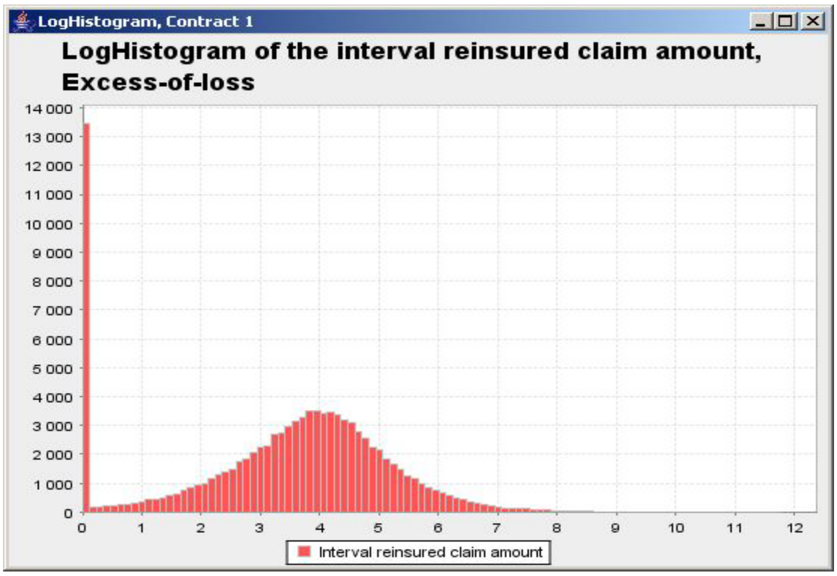


FIGURE 13. Log-histogram of the interval reinsured claim amount under the Excess-of-loss [40].

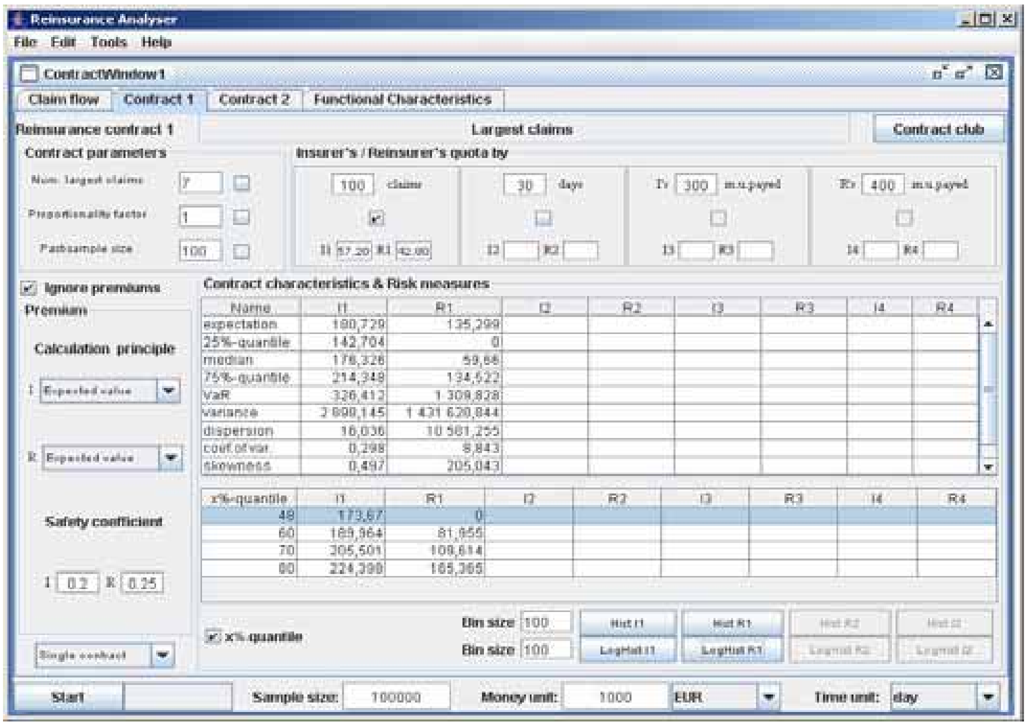
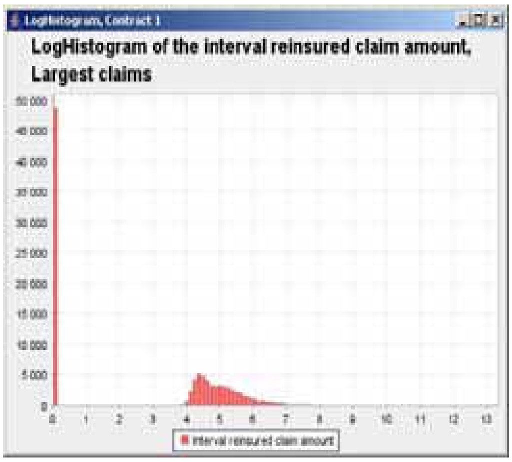
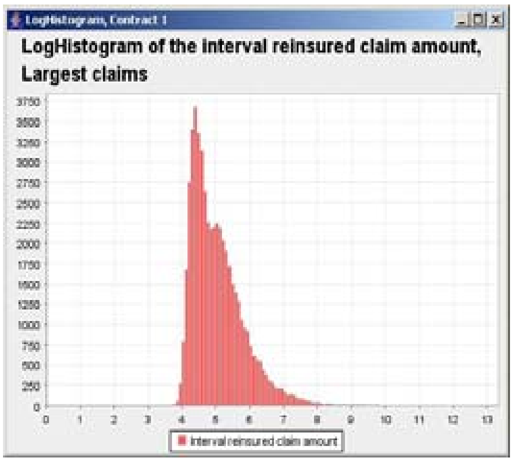


FIGURE 14. Estimation of Q_R load for LC[100; 7; 1].



(a) Including reinsurer's zero-shares.



(b) Excluding reinsurer's zero-shares.

FIGURE 15. Log-histograms of the interval reinsured claim amount under the LC[100, 7, 1].

how the same histogram for the LC-treaty would look after the exclusion of the reinsurer's zero shares from the sample, used for plotting the histogram.

7.2 Estimation of the parameters of reinsurance treaties. In this subsection we present results of the estimation of the retention level M of an excess-of-loss treaty, and the proportionality factor c of an LC[l, r, c], given the average reinsurer's quota load $Q_R = 45\%$.

As seen in Figure 16, the estimated retention level of the excess-of-loss equals $M = 31.3$ which is less than the retention level $M = 40$ providing $Q_R = 38.2\%$ in the example given in Figure 12.

Figure 17 shows that the average reinsurer's quota load $Q_R = 45\%$ can also be reached by applying an LC[100, 8, 0.986] where the number of largest claims $r = 8$ has been defined by the user, and the proportionality factor $c = 0.986$ has been estimated by the program.

Note that the average reinsurer's quota load $Q_R = 45\%$ cannot be attained by applying an LC[l, r, c] with $c = 1$ and an integer value of r. This confirms the sensibility of having an additional continuous parameter for the largest claims contract (see also discussion in subsection 4.2).

In spite of the fact that both the excess-of-loss and largest claims treaties provide an equal average reinsurer's quota load in the examples above, they are still not comparable. As was pointed out in Section 5, both treaties should be applied to exactly the same claim flow. The corresponding examples are given in the next subsection.

7.3 Comparison of the riskiness of reinsurance treaties. In Figures 18–21 the reader can see the results of the comparison of the LC-contract and excess-of-loss. The estimates of the proportionality factor c

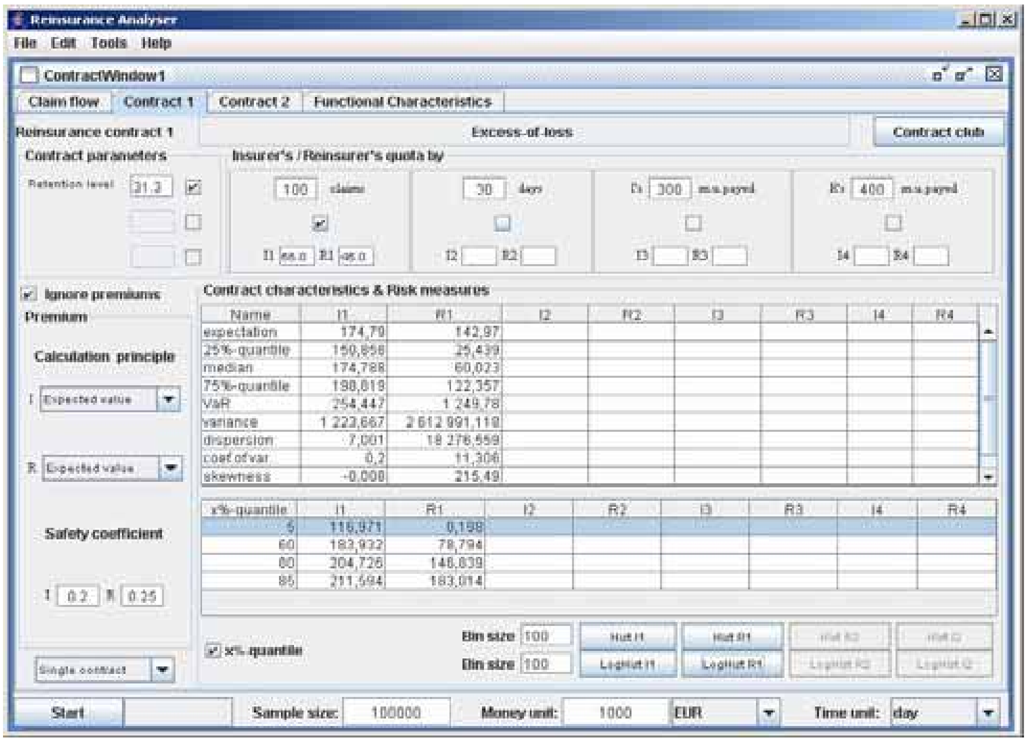


FIGURE 16. Estimation of the retention level M of the Excess-of-loss [M], given the average reinsurer's quota load $Q_R = 45\%$.

and contract characteristics of the LC[l, r, c] are given in Figure 18, and the estimates of the retention level and contract characteristics of the excess-of-loss are shown in Figure 19.

The estimates of the risk measures of both treaties and the ratios of the corresponding estimates are displayed in the Compare Dialog shown in Figure 20. One can observe the difference in 0.065 units between the estimates of the expected value of the interval reinsured claim amount obtained under the LC-contract and the excess-of-loss. The difference is caused by the fact that the admissible precision of the estimate was chosen to be five hundredth percent of the expectation of the total interval claim.

Logarithmic histograms of the interval reinsured claim amount under the LC-contract and the excess-of-loss are given in Figures 21(a) and 21(b) respectively. (Note that both figures display the histograms after the exclusion of the outliers at zero.)

If we compare the histograms, we see that in spite of some common features, they are essentially different in shape and structure. The similarities are that each distribution is very skewed; the (highest) peak is situated between 60% and 65% quantile values while the expectation is lying on the right-hand side from the (highest) peak between 75% and 85% quantiles.

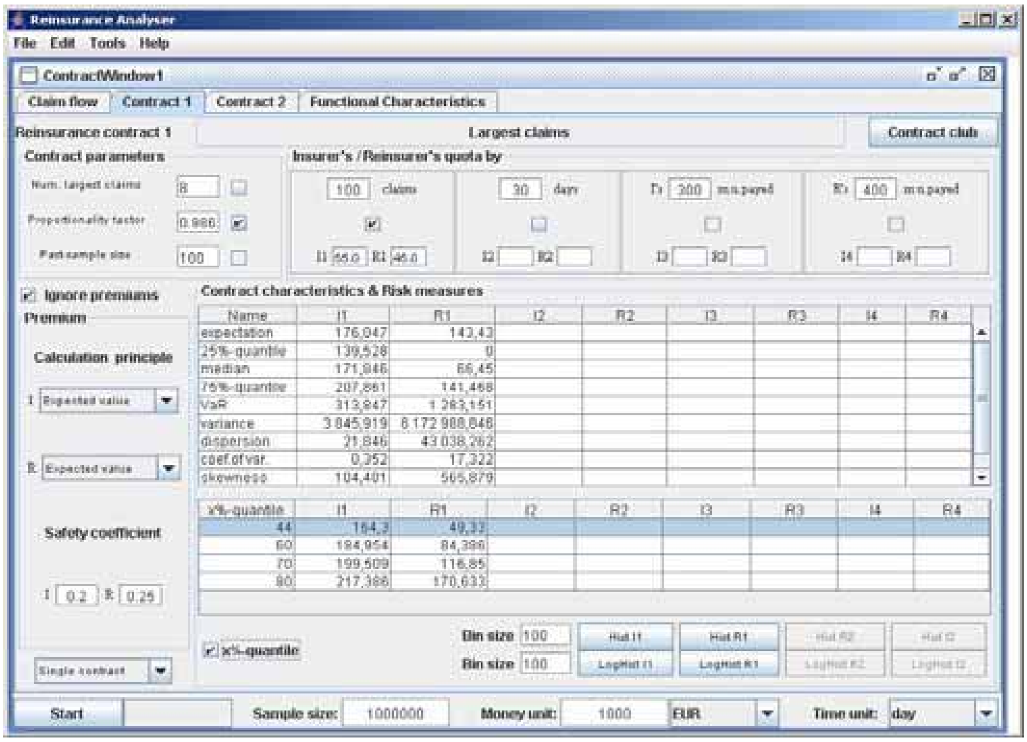


FIGURE 17. Estimation of the proportionality factor c of the LC[100, 8, c], given the average reinsurer's quota load $Q_R = 45\%$.

At the same time, the distribution obtained under the LC-contract is much more asymmetric. In particular, the left slope of the histogram is extremely steep so that its left foot is situated far from zero (one can see a gap between the ordinate axis and the histogram), while the histogram based on the excess-of-loss is symmetric in the neighbourhood of its peak and its left slope is smoothly approaching zero. Furthermore, in contrast with the excess-of-loss, the distribution regarding the LC-contract is bimodal, indicating the underlying claim size mixture, and the expectation is lying between the peak values.

The experimental studies on the comparison of a pair of reinsurance contracts by means of the ReAn program showed that, despite the infinite variance of the claim size distribution, the ratio of the estimate of the variance-based risk measure of one contract to the estimate of the corresponding variance-based risk measure of the other is stable.

This result suggests that the ratios mentioned above can still be used for the comparison of the riskiness of reinsurance contracts. An example of such a comparison made for the excess-of-loss and the largest claims reinsurance contract defined by (4) was given above. As seen in Figure 20, the ratios, denoted by $R1/R2$, of the estimates of all risk measures of the

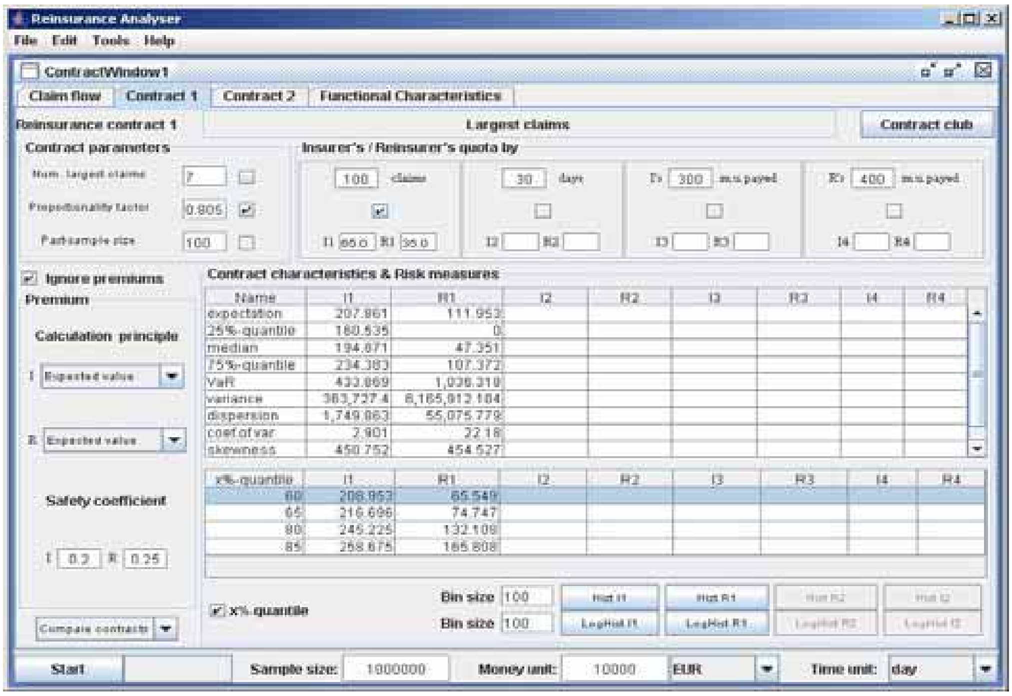


FIGURE 18. Comparison of LC[l, r, c] and Excess-of-loss [M]. Reinsurer's quota load $Q_R = 35\%$. Contract 1: LC[100; 7; 0.805], $c = 0.805$ is estimated.

reinsured amount are less than 1. Hence, from the reinsurer's point of view, the application of the LC-contract to the given claim flows is less risky than the application of the excess-of-loss.

8. Conclusions and prospective developments

I. In the present paper we have proposed an approach to analyse and compare the riskiness of different reinsurance treaties. The key idea is to use Monte Carlo simulations to calibrate the treaties so that their reinsurer's quota loads are equal, and to estimate the *ratios* of the corresponding risk measures of the treaties.

The approach based on Monte Carlo methods allows to cope with the mathematical complexity of the largest claims, ECOMOR, and other extreme value reinsurance contracts which makes the last more attractive and accessible to the practitioners.

II. The experimental software Reinsurance Analyser based on the approach indicated above can fruitfully be used for experimental studies on the application of deferent reinsurance contracts to a variety of claims flows. As such, the program may be appreciated as a flexible and handy tool for comparing the riskiness of reinsurance contracts in the interactive regime.

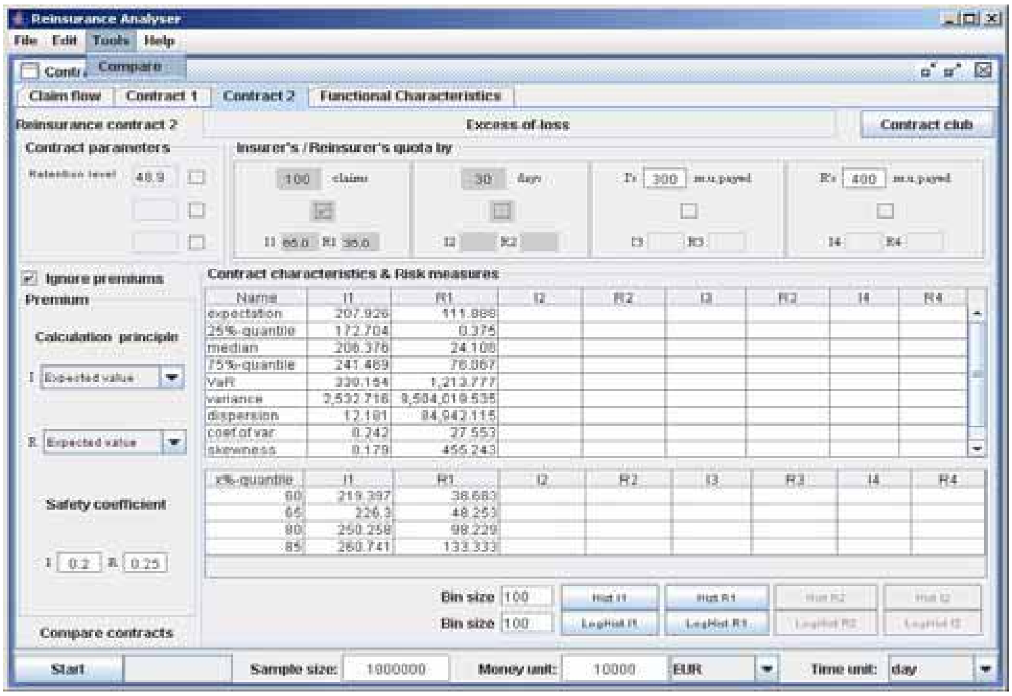
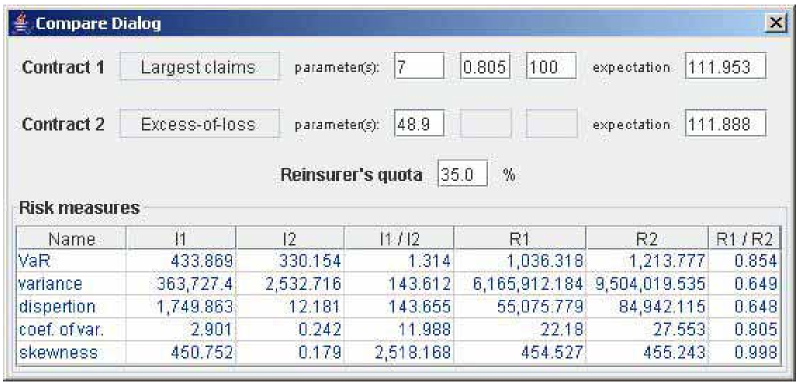


FIGURE 19. Comparison of LC[l, r, c] and Excess-of-loss [M]. Reinsurer's quota load $Q_R = 35\%$. Contract 2: Excess-of-loss [48.9], $M = 48.9$ is estimated.

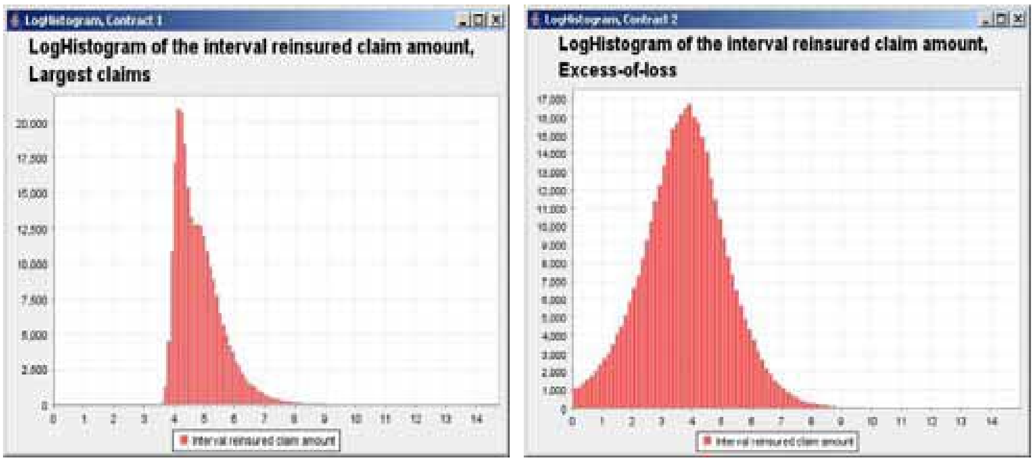


Name	I1	I2	I1 / I2	R1	R2	R1 / R2
VaR	433.869	330.154	1.314	1,036.318	1,213.777	0.854
variance	363,727.4	2,532.716	143.612	6,165,912.184	9,504,019.535	0.649
dispertion	1,749.863	12.181	143.655	55,075.779	84,942.115	0.648
coef. of var.	2.901	0.242	11.988	22.18	27.553	0.805
skewness	450.752	0.179	2,518.168	454.527	455.243	0.998

FIGURE 20. Comparison of LC[l, r, c] and Excess-of-loss [M]. The Compare Dialog.

It can be used by actuaries and experts in reinsurance business for analysis of treaties of high complexity.

III. Experimental studies performed by means of ReAn showed that the ratios of the estimates of the variance-based risk measures are stable even in the case when the variance of the underlying distribution is infinite. It allowed us to use the ratios in question for the comparison of the pair of



(a) LC-contract. (b) Excess-of-loss.

FIGURE 21. Comparison of LC[l, r, c] and Excess-of-loss [M]. Log-histograms.

reinsurance treaties. In particular, it was established that the LC-contract may be less risky to the reinsurer than the excess-of-loss when the contracts are applied to the claim process with infinite variance.

In conclusion, we would like to outline possible directions in which further research may develop.

(i) The asymptotic stability of the estimates of Q_R-type characteristics regarding the EV[l]-type reinsurance contracts needs to be investigated. The case when the underlying claim size and inter-claim interval distributions are heavy-tailed is of special interest.

(ii) The stability of the ratios of the risk measure estimates, established in the experimental studies, requires a deeper theoretical investigation in case of heavy-tailed claim size distribution with infinite second moment.

(iii) The influence of the parameters of heavy-tailed distributions involved for modelling claim flows on the stability of the estimate of the average reinsurer's quota load Q_R needs further analysis. In particular, theoretical results on the simulation sample size N needed to acquire a given relative precision for the estimate pointed out above should be derived. A facility for regulating the lower bound of the sample size (defined by the user in the ReAn program) will subsequently be added.

(iv) More complicated claim flow models allowing the dependence between claim sizes and inter-claim intervals can be investigated. The Reinsurance Analyser will be implemented with algorithms for modelling such claim flows. Furthermore, an algorithm for modelling claim flows by means of parametric re-sampling from real-life historical data needs to be elaborated and implemented into the program.

(v) Alternative approaches for balancing and comparing reinsurance treaties can be studied.

(vi) The lists of reinsurance treaties needs to be extended and new risk measures (e.g., inter-quartile range, reduction effect, etc.) have to be introduced. Finally, the full-scale development of the ReAn program including a well-developed Help system, is envisaged in the future.

References

1. Albrecher, H., Teugels, J.L. *Asymptotic analysis of a measure of variation*, Theory of Probability and Mathematical Statistics, **74**, (2006), 1–9.
2. Berliner, B. *Correlations between excess-of-loss reinsurance covers and reinsurance of the n largest claims*, Astin Bulletin, **6**, 3, (1972), 260–275.
3. Daykin, C.D., Pentikäinen, T., Pesonen, M. *Practical Risk Theory for Actuaries*, Chapman & Hall, (1993).
4. Denuit, M., Vermandele, C. *Optimal reinsurance and stop-loss order*, Insur. Math. Econ., **22**, 3, (1998), 229-233.
5. Feller, W. *An Introduction to Probability Theory and its Applications*, **II**, Wiley Series in Probability and Statistics, Wiley, (1966).
6. Gajek, L., Zagrodny, D. *Insurer's optimal reinsurance strategies*, Insur. Math. Econ., **27**, 1, (2000), 105-112.
7. Hesselager, O. *Some results on optimal reinsurance in terms of the adjustment coefficient*, Scand. Actuarial J., (1990), 80-95.
8. Kaluszka, M. *Optimal reinsurance under mean-variance premium principles*, Insur. Math. Econ., **28**, 1, (2001), 61-67.
9. Kaluszka, M. *Mean-variance optimal reinsurance arrangements*, Scand. Actuarial J., (2004), 28–41.
10. Kremer, E. *Rating of largest claims and ECOMOR reinsurance treaties for large portfolios*, Astin Bulletin, **13**, 1, (1982), 47–56.
11. Kremer, E. *An asymptotic formula for the net premium of some reinsurance treaties*, Scand. Actuarial J., (1984), 11-22.
12. Kremer, E. *The asymptotic efficiency of largest claimsreinsurance treaties*, Astin Bulletin, **20**, 2, (1990) , 11–22.
13. Kremer, E. *The limit-equivalence of theexcess-of-loss and largest claims reinsurance treaty*, J. Bl. Dtsch. Ges. Versicherungsmath, **20**, 3, (1991), 329-336.
14. Ladoucette S., Teugels J. *Analysis of risk measures for reinsurance layers*, Insurance: Mathematics & Economics, **38(3)**, (2006a), 630–639.
15. Ladoucette, S., Teugels J.L. *Reinsurance of large claims*, J. Comp. Appl. Math., **186**, (2006b), 163-190.
16. Pesonen, M. *Optimal reinsurances*, Scand. Actuarial J., (1984), 65-90.
17. Rolski, T., Schmidli, H., Schmidt, V., Teugels, J.L. *Stochastic Processes for Insurance and Finance*. John Wiley & Sons, (1999).
18. Teugels, J.L. *Reinsurance. Actuarial Aspects*. EURANDOM Report 2003-006, Technical University of Eindhoven, The Netherlands, (2003).
19. Thépaut, A. *Une nouvelle forme de réassurance: le traité d'excédent du coût moyen relatif (ECOMOR)*, Bull. Trim. Inst. Actu. Français, **49**, (1950), 273–343.

20. Shurenkov, V.M. *Ergodic Markov Processes. Probability Theory and Mathematical Statistics*, Nauka, Moscow, (1989).
21. Silvestrov, D.S. *Remarks on the strong law of large numbers for accumulation processes*, Theory Probab. Math. Stat., **22**, (1980a) , 131–143.
22. Silvestrov, D.S. *Semi-Markov Processes with Discrete State Space.* Library for an engineer in reliability, Sovetskoe Radio, Moscow, (1980b).
23. Verlaak, R., Beirlant, J. *Optimal reinsurance programs: An optimal combination of several reinsurance protections on a heterogeneous insurance portfolio*, Insur. Math. Econ., **33**, 2, (2003), 381-403.

Department of Mathematics and Physics, Mälardalen University, Box 883, SE-721 23 Västerås, Sweden
E-mail address: dmitrii.silvestrov@mdh.se

Katholieke Universiteit Leuven, B-3001 Leuven (Heverlee), Belgium
E-mail address: jef.teugels@wis.kuleuven.be

Department of Mathematics and Physics, Mälardalen University, Box 883, SE-721 23 Västerås, Sweden
E-mail address: vicamasol@bigmir.net

Department of Mathematics and Physics, Mälardalen University, Box 883, SE-721 23 Västerås, Sweden
E-mail address: anatoliy.malyarenko@mdh.se