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ON UNIFORM CONVERGENCE OF WAVELET EXPANSIONS OF SOME RANDOM PROCESSES

In the paper there are found conditions for uniform convergence with probability one of wavelet expansion of g-sub-Gaussian random processes under additional condition for norm of such process

1. Introduction

It this paper I proceed with research presented in [1] and derive conditions for uniform convergence of wavelet expansions of g-sub-Gaussian random processes on the finite interval in case when norm τ_g of such process $X = \{X(t), t \in R\}$ increases for positive t.

2. Main results

Definition 1.[2] Let $g = \{g(x), x \in R\}$ be a continuous even convex function; g is called an N-function if $g(0) = 0$, $g(x) > 0$ as $x \neq 0$ and $\lim\limits_{x\to 0} \frac{g(x)}{x} = 0$, $\lim\limits_{x\to\infty} \frac{g(x)}{x} = \infty$.

Condition Q. [3] An N-function g satisfies condition Q if $\liminf\limits_{x\to 0} \frac{g(x)}{x^2} = C > 0$. It may happen that $C = \infty$.

Definition 2. [2, 3] Let g be an N-function, which satisfies condition Q. Let $\{\Omega, L, P\}$ be a standard probability space. A random variable $\xi = \{\xi(\omega), \omega \in \Omega\}$ belongs to the space $\mathrm{Sub}_g(\Omega)$ (is g-sub-Gaussian) if $E\xi = 0$, $E\exp\{\lambda\xi\}$ exists for all $\lambda \in R$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in R : E\exp\{\lambda\xi\} \le \exp\{g(a\lambda)\}$.

The space $\mathrm{Sub}_g(\Omega)$ is a Banach space with respect to the norm

$$\tau_g(\xi) = \frac{\sup\limits_{\lambda\neq 0} g^{(-1)}(\ln E\exp\{\lambda\xi\})}{\lambda}.$$

Definition 3. [2] A random process $\{X(t), t \in T\}$ belongs to the space $\mathrm{Sub}_g(\Omega)$ (is g-sub-Gaussian) if the random variable $X(t) \in \mathrm{Sub}_g(\Omega)$ for all $t \in T$.

Let $\varphi = \{\varphi(x), x \in R\}$ be an f-wavelet and $\psi = \{\psi(x), x \in R\}$ be the m-wavelet, which corresponds to φ.

Define a family of functions $\{\varphi_{jk}, \psi_{jk}, j \in Z, k \in Z\}$ in the following way: $\varphi_{jk}(x) = 2^{j/2} \cdot \varphi(2^j x - k)$, $\psi_{jk}(x) = 2^{j/2} \cdot \psi(2^j x - k)$.

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It is know that the family of functions $\{\varphi_{0k},\ \psi_{jk},\ j=0,1,\ldots k\in Z\}$ is an orthonormal basis in $L_2(R)$.

Definition 4. [1] Let φ be an f-wavelet (ψ be an m-wavelet). The **assumption S** holds for φ (or ψ) if there exists a function $\Phi=\{\Phi(x),\ x\geq 0\}$ such that $\Phi(x)$ decreases, $|\varphi(x)|\leq\Phi(|x|)$ (or $\psi(x)\leq\Phi(|x|)$) almost everywhere and $\int\limits_R \Phi(|x|)\,dx<\infty$.

The following theorem is a particular case of the theorem 4.1 from the paper [1].

Theorem 1. *Let $X=\{X(t),\ t\in R\}$ be a separable g-sub-Gaussian random process, $B_l=[a_l,\ a_{l+1}]$, $a_{l+1}-a_l=e$, $l\in Z$, $a_l\to+\infty$ as $l\to+\infty$, $a_l\to-\infty$ as $l\to-\infty$. Assume that there exists an increasing continuous function $\sigma=\{\sigma(h),\ h>0\}$ such that $\sup\limits_{|t-s|\leq h}\tau_g(X(t)-X(s))\leq\sigma(h)$. Let $c=\{c(t),\ t\in R\}$ be a continuous even positive function such that for sufficiently large x we have: $c(ax)\leq c(x)A(a)$, $A(a)\in(0;\infty)$. Denote $\delta_l=\sup\limits_{t\in B_l}(c(t))^{-1}$, $\chi_l=\sup\limits_{t\in B_l}\tau_g(X(t)-X(a_{l+1}))$, $Z_l=\tau_g(X(a_{l+1}))$, $l\in Z$. Assume that for any $\varepsilon>0$:*

$$\int_0^{\varepsilon} a_g\left(\ln\left(\left(2\sigma^{(-1)}(u)\right)^{-1}+1\right)\right)du<\infty, \tag{1}$$

and

$$\sum_{l\in Z}\delta_l Z_l<\infty, \tag{2}$$

$$\sup_{l\in Z}\frac{\chi_l}{Z_l}\leq\beta<\infty, \tag{3}$$

$$\sum_{l\in Z}\delta_l\int_0^{\chi_l} a_g\left(\ln\left(\frac{a_{l+1}-a_l}{2\sigma^{(-1)}(u)}+1\right)\right)du<\infty, \tag{4}$$

where $a_g(x)=\frac{x}{g^{(-1)}(x)}$. Let φ be an f-wavelet and ψ be the m-wavelet, which corresponds to φ, and suppose that the assumption S holds for φ and ψ with respect to a function Φ and

$$\int_R c(x)\Phi(|x|)\,dx<\infty. \tag{5}$$

Then with probability one there exist

$$a_{0k}=\int_R X(t)\overline{\varphi_{0k}(t)}dt \text{ and } b_{jk}=\int_R X(t)\psi_{jk}(t)\,dt,\quad k\in Z,\ j=\overline{0,+\infty}$$

and wavelet expansion $X_m(t) = \sum\limits_{k\in Z} \alpha_{0k}\varphi_{0k}(x) + \sum\limits_{j=0}^{m-1}\sum\limits_{k\in Z}\beta_{jk}\psi_{jk}(x)$ *converges to* $X(t)$ *as* $m\to\infty$ *uniformly on each interval* $[a,\, b]$ *with probability one* $(-\infty < a < b < +\infty)$.

Theorem 2. *Let the assumptions (1) and (2) of the Theorem 1 hold and assume that*

$$\sum_{l\in Z}\delta_l\chi_l\, a_g\left(\ln\left(1+(a_{l+1}-a_l)\right)\right) < \infty, \tag{6}$$

$$\sum_{l\in Z}\delta_l\int\limits_0^{\chi_l} a_g\left(\ln\left(\left(2\sigma^{(-1)}(u)\right)^{-1}+1\right)\right)du < \infty. \tag{7}$$

Also suppose that $\tau_g(X(t)) = \tau_g(X(-t)) > 0$, $t\neq 0$, *and norm* $\tau_g(X(t))$ *increases as* $t>0$.

Then the assertion of the Theorem 1 follows.

Proof. It follows from Lemma 2.2.3 of the book [2] that the function $a_g(x) = \frac{x}{g^{(-1)}(x)}$ increases as $x>0$. If $x>0$ and $y>0$ then

$$a_g(x+y) = \frac{x+y}{g^{(-1)}(x+y)} = \frac{x}{g^{(-1)}(x+y)} + \frac{y}{g^{(-1)}(x+y)} \le$$

$$\le \frac{x}{g^{(-1)}(x)} + \frac{y}{g^{(-1)}(y)} = a_g(x) + a_g(y).$$

Therefore

$$\int\limits_0^{\chi_l} a_g\left(\ln\left(\frac{a_{l+1}-a_l}{2\sigma^{(-1)}(u)}+1\right)\right)du \le$$

$$\int\limits_0^{\chi_l} a_g\left(\ln\left(1+a_{l+1}-a_l\right)\right) + \ln\left(1+\left(2\sigma^{(-1)}(u)\right)^{-1}\right)du \le$$

$$\le \chi_l a_g\left(\ln\left(1+a_{l+1}-a_l\right)\right) + \int\limits_0^{\chi_l} a_g\left(\ln\left(1+\left(2\sigma^{(-1)}(u)\right)^{-1}\right)\right)du$$

and the assumption (4) follows from (6) and (7).

Since $\sup\limits_{l\in Z}\frac{\chi_l}{Z_l} = \sup\limits_{l>0}\frac{\chi_l}{Z_l}$, then

$$\tau_g\left(X(t) - X(a_{l+1})\right) \le \tau_g(X(t)) + \tau_g\left(X(a_{l+1})\right) \le 2\tau_g(a_{l+1})$$

for any $t\in B_l$, $l>0$. Therefore $\frac{\chi_l}{Z_l}\le 2$ and assumption (3) holds true.

Example 1. Let the assumptions of the Theorem 2 hold true for the function $\sigma(u) = \frac{c}{\left(\ln\left(1+\frac{1}{2u}\right)\right)^{\gamma}}$, where $c>0, \gamma>0$. Then $\sigma^{(-1)}(u) =$

$\frac{1}{2\left(\exp\left\{\left(\frac{c}{t}\right)^{1/\gamma}\right\}-1\right)}$ and

$$\int\limits_{0}^{\chi_l} a_g\left(\ln\left(1+\left(2\sigma^{(-1)}\left(u\right)\right)^{-1}\right)\right)\,du=\int\limits_{0}^{\chi_l} a_g\left(\left(\frac{c}{u}\right)^{1/\gamma}\right)du. \tag{8}$$

Since $a_g\left(\left(\frac{c}{u}\right)^{1/\gamma}\right)=\frac{\left(\frac{c}{u}\right)^{1/\gamma}}{g^{(-1)}\left(\left(\frac{c}{u}\right)^{1/\gamma}\right)}\leq\frac{\left(\frac{c}{u}\right)^{1/\gamma}}{g^{(-1)}\left(\left(\frac{c}{\chi_l}\right)^{1/\gamma}\right)}$, as $u<\chi_l$ then

$$\int\limits_{0}^{\chi_l} a_g\left(\ln\left(1+\left(2\sigma^{(-1)}\left(u\right)\right)^{-1}\right)\right)\,du\leq\frac{c^{1/\gamma}}{g^{(-1)}\left(\left(\frac{c}{\chi_l}\right)^{1/\gamma}\right)}\cdot\frac{\chi_l^{1-\frac{1}{\gamma}}}{\left(1-\frac{1}{\gamma}\right)}$$

and assumption (7) holds true if

$$\sum_{l\in Z}\delta_l\chi_l^{1-\frac{1}{\gamma}}\left(g^{(-1)}\left(\left(\frac{c}{\chi_l}\right)^{1/\gamma}\right)\right)^{-1}<\infty. \tag{9}$$

If $g\left(x\right)=\left|x\right|^{\alpha}$, $1<\alpha\leq 2$, then $a_g\left(\left(\frac{c}{u}\right)^{1/\gamma}\right)=\left(\frac{c}{u}\right)^{\frac{1}{\gamma}-\frac{1}{\gamma\alpha}}$ and if $\gamma>1-\frac{1}{\alpha}$ then $\int\limits_{0}^{\chi_l} a_g\left(\ln\left(\left(2\sigma^{(-1)}\left(u\right)\right)^{-1}+1\right)\right)du=\frac{c^{\frac{1}{\gamma}-\frac{1}{\gamma\alpha}}\chi_l^{\left(1-\frac{1}{\gamma}+\frac{1}{\gamma\alpha}\right)}}{\left(1-\frac{1}{\gamma}+\frac{1}{\gamma\alpha}\right)}$. Thus assumption (7) holds true if

$$\sum_{l\in Z}\delta_l\chi_l^{\left(1-\frac{1}{\gamma}+\frac{1}{\gamma\alpha}\right)}<\infty. \tag{10}$$

Theorem 3. *Let $X=\{X\left(t\right),\,t\in R\}$ be a separable g-sub-Gaussian random process, where $g\left(x\right)=\left|x\right|^{\alpha}$, $1<\alpha<2$; $X\left(t\right)=X\left(-t\right)$ with probability one; $B_l=\left[a_l,a_{l+1}\right]$, $l=0,1,2,...$, $a_0=0$, $a_{l+1}-a_l>e$, $a_l\to\infty$, $l\to\infty$, and*

$$\sup_{|t-s|\leq h}\tau_g\left(X\left(t\right)-X\left(s\right)\right)\leq\frac{c}{\left(\ln\left(1+\frac{1}{2u}\right)\right)^{\gamma}}, c>0,\ \gamma>1-\frac{1}{\alpha}.$$

Let $\tau_g\left(X\left(t\right)\right)$ increase as $t>0$ and

$$\sum_{l=0}^{\infty}\delta_l Z_l<\infty, \tag{11}$$

$$\sum_{l=0}^{\infty}\delta_l\chi_l\left(\ln\left(1+\left(a_{l+1}-a_l\right)\right)\right)^{1-\alpha}<\infty, \tag{12}$$

$$\sum_{l=0}^{\infty}\delta_l\chi_l^{1-\frac{1}{\gamma}+\frac{1}{\gamma\alpha}}<\infty. \tag{13}$$

Then with probability one $X_m\left(t\right)\to X\left(t\right)$ as $m\to\infty$ uniformly on each bounded interval $[a,\,b]$.

Theorem 3 follows from Example 1 and Theorem 2.

Remark 1. Since $\chi_l \leq 2Z_l$ then from the assumption

$$\sum_{l=0}^{\infty} \delta_l Z_l \left(\ln\left(1 + (a_{l+1} - a_l)\right)\right)^{1-\alpha} < \infty$$

the assumptions (11)–(13) follow, if $\chi_l > c > 0$.

If $a_l = e^l$ and $\tau_g(X(t)) = t$ then $c(t) = t \cdot (\ln t)^{\beta}$, $t > 1$, and $\Phi(|t|) = \frac{1}{|t|(\ln|t|)^{\beta+v}}$, $v > 1$, $|t| > 1$.

Conclusions. In the paper there are found conditions for uniform convergence with probability one of the wavelet expansion of g-sub-Gaussian random process such that $\tau_g(X(t))$ increases for $t > 0$.

I plan to obtain similar results for random processes from $\mathrm{Sub}_g(\Omega)$ such that

$$\tau_g(X(t) - X(s)) \leq c \cdot |t - s|^{\alpha}.$$

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