Theory of Stochastic Processes
Vol. 12 (28), no. 3–4, 2006, pp. 88–113

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ROBUST ESTIMATION PROBLEMS FOR STOCHASTIC PROCESSES[1]

We deal with the problem of optimal linear estimation of the functional $A_L\vec{\xi} = \int_0^L \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ with the spectral density $F(\lambda)$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t \in R\backslash[0, L]$, where $\vec{\eta}(t)$ is uncorrelated with $\vec{\xi}(t)$ multidimensional stationary process with the spectral density $G(\lambda)$ (interpolation problem), and the problem of optimal linear estimation of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$, $t \geq 0$, from observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t < 0$ (extrapolation problem). Formulas are obtained for calculation the mean square errors and the spectral characteristics of the optimal estimates of the functionals under the condition that the spectral density matrix $F(\lambda)$ of the signal process $\vec{\xi}(t)$ and the spectral density matrix $G(\lambda)$ of the noise process $\vec{\eta}(t)$ are known. The least favorable spectral densities and the minimax spectral characteristics of the optimal estimates of the functionals are found for concrete classes $D = D_F \times D_G$ of spectral densities under the condition that spectral density matrices $F(\lambda)$ and $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of admissible spectral densities are given.

1. Introduction

Traditional methods of solution of the linear extrapolation, interpolation and filtering problems for stationary stochastic processes may be employed under the condition that spectral densities of processes are known exactly (see, for example, selected works of A. N. Kolmogorov (1992), survey article by T. Kailath (1974), books Yu. A. Rozanov (1990), N. Wiener (1966) and A. M. Yaglom (1987)). In practice, however, complete information on the spectral densities is impossible in most cases. To solve the problem one finds parametric or nonparametric estimates of the unknown spectral densities or selects these densities by other reasoning. Then applies the classical estimation method provided that the estimated or selected density is the true one. This procedure can result in a significant increasing of the

[1]Invited lecture.

2000 *Mathematics Subject Classification.* Primary 60G10, 62M20, 60G35, 93E10, 93E11.

Key words and phrases. Stationary stochastic process, interpolation, extrapolation, filtering, robust estimate, observations with noise, mean square error, least favorable spectral densities, minimax spectral characteristic.

value of error as K. S. Vastola and H. V. Poor (1983) have demonstrated with the help of some examples. This is a reason to search estimates which are optimal for all densities from a certain class of the admissible spectral densities. These estimates are called minimax since they minimize the maximal value of the error. A survey of results in minimax (robust) methods of data processing can be found in the paper by S. A. Kassam and H. V. Poor (1985). The paper by Ulf Grenander (1957) should be marked as the first one where the minimax approach to extrapolation problem for stationary processes was proposed. J. Franke (1984, 1985, 1991), J. Franke and H. V. Poor (1984) investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for various classes of densities. In the papers by M. P. Moklyachuk (1994, 1997, 1998, 2000, 2001), M. P. Moklyachuk and A. Yu. Masyutka (2005, 2006) the minimax approach to extrapolation, interpolation and filtering problems are investigated for functionals which depend on the unknown values of stationary processes and sequences.

In this article we considered the problem of estimation of the unknown value of the functional $A_L\vec{\xi} = \int_0^L \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$, $E\vec{\xi}(t) = 0$, with the spectral density matrix $F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^T$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t \in R\backslash[0, L]$, where $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$ is an uncorrelated with $\vec{\xi}(t)$ multidimensional stationary stochastic process with the spectral density matrix $G(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^T$ (interpolation problem), and the problem of optimal linear estimation of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$, $t \geq 0$, from observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t < 0$ (extrapolation problem). Formulas are obtained for calculation the mean square errors and the spectral characteristics of the optimal linear estimates of the functionals under condition that the spectral density matrices $F(\lambda)$ and $G(\lambda)$ are known. Formulas are proposed that determine the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functionals for concrete classes $D = D_F \times D_G$ of spectral densities under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of admissible spectral densities are given.

2. Hilbert space projection method of interpolation

Let $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$, $E\vec{\xi}(t) = 0$, $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$, $E\vec{\eta}(t) = 0$, be uncorrelated multidimensional stationary stochastic processes with the spectral density matrices $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$ and $G(\lambda)=\{g_{kl}(\lambda)\}_{k,l=1}^{T}$, which satisfy the minimality condition

$$\int_{-\infty}^{\infty} b(\lambda)(F(\lambda)+G(\lambda))^{-1}b^{*}(\lambda)d\lambda<\infty \tag{1}$$

for a nontrivial vector function of the exponential type $b(\lambda)=\int_0^L \vec{\alpha}(t)e^{it\lambda}dt$, where $\vec{\alpha}(t)=\{\alpha_k(t)\}_{k=1}^{T}$. Under this condition the error-free interpolation is impossible (see, for example, Yu. A. Rozanov (1990)).
Denote by $L_2(F)$ the Hilbert space of vector-valued functions $\varphi(\lambda)=\{\varphi_k(\lambda)\}_{k=1}^{T}$ which are square integrable with respect to measure with the density $F(\lambda)=\{f_{kl}(\lambda)\}_{k,l=1}^{T}$:

$$\int_{-\infty}^{\infty}\varphi(\lambda)F(\lambda)\varphi^{*}(\lambda)d\lambda=\int_{-\infty}^{\infty}\sum_{k,l=1}^{T}\varphi_k(\lambda)\overline{\varphi_l(\lambda)}\,f_{kl}(\lambda)d\lambda<\infty.$$

Denote by $L_2^{L-}(F)$ the subspace in $L_2(F)$, generated by functions $e^{it\lambda}\delta_k$, $\delta_k=\{\delta_{kl}\}_{l=1}^{T}$, $k=\overline{1,T}$, $t\in R\backslash[0,L]$, where $\delta_{kk}=1$, $\delta_{kl}=0$ for $k\neq l$. Any linear estimate $\hat{A}_L\vec{\xi}$ of the functional $A_L\vec{\xi}$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t\in R\backslash[0,L]$ is of the form

$$\hat{A}_L\vec{\xi}=\int_{-\infty}^{\infty}h(\lambda)\left(Z^{\xi}(d\lambda)+Z^{\eta}(d\lambda)\right)=\int_{-\infty}^{\infty}\sum_{k=1}^{T}h_k(\lambda)(Z_k^{\xi}(d\lambda)+Z_k^{\eta}(d\lambda)),$$

where $Z^{\xi}(\Delta)=\left\{Z_k^{\xi}(\Delta)\right\}_{k=1}^{T}$ and $Z^{\eta}(\Delta)=\{Z_k^{\eta}(\Delta)\}_{k=1}^{T}$ are orthogonal random measures of the stationary processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ correspondingly, $h(\lambda)=\{h_k(\lambda)\}_{k=1}^{T}$ is the spectral characteristic of the estimate $\hat{A}_L\vec{\xi}$. The function $h(\lambda)\in L_2^{L-}(F+G)$. The value of the mean square error $\Delta(h;F,G)$ of the estimate $\hat{A}_L\vec{\xi}$ is calculated by the formula

$$\Delta(h;F,G)=E\left|A_L\vec{\xi}-\hat{A}_L\vec{\xi}\right|^2=$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(A_L(\lambda)-h(\lambda)\right)F(\lambda)\left(A_L(\lambda)-h(\lambda)\right)^{*}d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(\lambda)\,G(\lambda)\,h^{*}(\lambda)d\lambda,$$

where

$$A_L(\lambda)=\int_0^L\vec{a}(t)e^{it\lambda}dt.$$

The spectral characteristic $h(F,G)$ of the optimal estimate of the functional $A_L\vec{\xi}$ in the case of given spectral density matrices $F(\lambda)$ and $G(\lambda)$ is determined by the extremum condition

$$\Delta(F,G) = \Delta(h(F,G);F,G) =$$

$$= \min_{h \in L_2^{L-}(F+G)} \Delta(h;F,G) = \min_{\hat{A}_L\vec{\xi}} E\left|A_L\vec{\xi} - \hat{A}_L\vec{\xi}\right|^2 . \tag{2}$$

The optimal estimate $\hat{A}_L\vec{\xi}$ is a solution of the extremum problem (2). It is determined by two conditions (see, for example, selected works of A. N. Kolmogorov (1992))

$$\hat{A}_L\vec{\xi} \in H\left[\xi_k(t) + \eta_k(t)\,,\; k = \overline{1,T}\,,\; t \in R\backslash[0,L]\right]; \tag{3}$$

$$A_L\vec{\xi} - \hat{A}_L\vec{\xi} \perp H\left[\xi_k(t) + \eta_k(t)\,,\; k = \overline{1,T}\,,\; t \in R\backslash[0,L]\right], \tag{4}$$

where $H\left[\xi_k(t) + \eta_k(t),\, k = \overline{1,T},\, t \in R\backslash[0,L]\right]$ is a subspace generated by the random variables $\left\{\xi_k(t) + \eta_k(t), k = \overline{1,T}, t \in R\backslash[0,L]\right\}$ in the Hilbert space H of the second order random variables with zero mean value. These conditions give a possibility to find the spectral characteristic $h(F,G)$ and the mean-square error $\Delta(F,G)$ of the optimal estimate of the functional $A_L\vec{\xi}$ under the condition that the spectral density matrices $F(\lambda), G(\lambda)$ are known and satisfy the minimality condition (1). In this case

$$h(F,G) = (A_L(\lambda)\,F(\lambda) - C_L(\lambda))\,(F(\lambda)+G(\lambda))^{-1} =$$

$$= A_L(\lambda) - (A_L(\lambda)\,G(\lambda) + C_L(\lambda))\,(F(\lambda)+G(\lambda))^{-1}, \tag{5}$$

$$\Delta(F,G) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (A_L(\lambda)\,G(\lambda) + C_L(\lambda))\,(F(\lambda)+G(\lambda))^{-1}\,F(\lambda)\times$$

$$\times(F(\lambda)+G(\lambda))^{-1}(A_L(\lambda)\,G(\lambda) + C_L(\lambda))^* d\lambda +$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty} (A_L(\lambda)\,F(\lambda) - C_L(\lambda))(F(\lambda)+G(\lambda))^{-1}G(\lambda)\,(F(\lambda)+G(\lambda))^{-1}\times$$

$$\times(A_L(\lambda)\,G(\lambda) + C_L(\lambda))^* d\lambda = \langle B_L c\,,\, c\rangle + \langle R_L a\,,\, a\rangle\,, \tag{6}$$

where

$$C_L(\lambda) = \int_0^L \vec{c}(t)e^{it\lambda}dt, \quad \vec{c}(t) = (B_L^{-1}D_L a)(t),\; 0 \le t \le L,$$

$$\langle B_L c, c\rangle = \int_0^L \sum_{k=1}^T (B_L c)_k(t)\,\overline{c_k(t)}\,dt,$$

$$\langle R_L a, a\rangle = \int_0^L \sum_{k=1}^T (R_L a)_k(t)\,\overline{a_k(t)}dt,$$

operators B_L, D_L, R_L are determined by the following relations

$$(B_L c)(t) = \frac{1}{2\pi}\int_0^L \int_{-\infty}^{\infty} \vec{c}(u)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}d\lambda du,$$

$$(D_L c)(t) = \frac{1}{2\pi}\int_0^L \int_{-\infty}^{\infty} \vec{c}(u)F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}d\lambda du,$$

$$(R_L c)(t) = \frac{1}{2\pi}\int_0^L \int_{-\infty}^{\infty} \vec{c}(u)F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i(u-t)\lambda}d\lambda du,$$

$$0 \le t \le L;$$

From the preceding formulas we can conclude that the following theorem holds true.

Theorem 1.1. *Let $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$, $E\vec{\xi}(t) = 0$, and $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$, $E\vec{\eta}(t) = 0$, be uncorrelated multidimensional stationary stochastic processes with the spectral density matrices*

$$F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^T \quad \text{and} \quad G(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^T$$

that satisfy the minimality condition (1). The value of the mean-square error $\Delta(h(F,G),F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear estimate of the functional $A_L\vec{\xi} = \int_0^L \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t \in R\backslash[0,L]$ are calculated by formulas (5), (6).

Corollary 1.1. *Let $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$ be a multidimensional stationary stochastic process with the spectral density matrix $F(\lambda) = \{f_{kl}(\lambda)\}_{k,l=1}^T$, that satisfies the minimality condition*

$$\int_{-\infty}^{\infty} b(\lambda)(F(\lambda))^{-1}b^*(\lambda)d\lambda < \infty$$

for a nontrivial vector function of the exponential type $b(\lambda) = \int_0^L \vec{\alpha}(t)e^{it\lambda}dt$. The value of the mean-square error $\Delta(F)$ and the spectral characteristic $h(F)$ of the optimal linear estimate of the functional $A_L\vec{\xi} = \int_0^L \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of the process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t)$ for $t \in R\backslash[0,L]$ can be calculated by the formulas

$$h(F) = A_L(\lambda) - C_L(\lambda)\,(F(\lambda))^{-1}, \tag{7}$$

$$\Delta(F) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_L(\lambda)\,(F(\lambda))^{-1}(C_L(\lambda))^* d\lambda = \langle B_L a,\, a\rangle = \langle c, a, \rangle \tag{8}$$

where

$$C_L(\lambda) = \int_0^L \vec{c}(t) e^{it\lambda} dt,$$

$$\vec{c}(t) = (B_L^{-1} a)(t), \quad 0 \le t \le L,$$

$$\langle c, a\rangle = \int_0^L \sum_{k=1}^T c_k(t)\,\overline{a_k(t)} dt,$$

$$(B_L a)(t) = \frac{1}{2\pi} \int_0^L \int_{-\infty}^{\infty} \vec{a}(u)(F(\lambda))^{-1} e^{i(u-t)\lambda} d\lambda du, \quad 0 \le t \le L.$$

Example 1. Consider the problem of estimation of the value of the functional $A_1\vec{\zeta} = \int_0^1 \vec{a}(t)\vec{\zeta}(t)dt$ based on observations of the process $\vec{\zeta}(t) = (\zeta_1(t), \zeta_2(t))$, $t \in R\backslash[0, 1]$, where $\zeta_1(t) = \xi(t)$ is a stationary stochastic process with the spectral density $f(\lambda)$, and $\zeta_2(t) = \xi(t) + \eta(t)$, where $\eta(t)$ is an uncorrelated with $\xi(t)$ stationary stochastic process with the spectral density $g(\lambda)$. In this case

$$F(\lambda) = \begin{pmatrix} f(\lambda) & f(\lambda) \\ f(\lambda) & f(\lambda) + g(\lambda) \end{pmatrix}.$$

The determinant $D = |F(\lambda)| = f(\lambda)\, g(\lambda)$, and the inverse matrix

$$F(\lambda)^{-1} = \begin{pmatrix} \frac{f(\lambda)+g(\lambda)}{f(\lambda)\, g(\lambda)} & \frac{-1}{g(\lambda)} \\ \frac{-1}{g(\lambda)} & \frac{1}{g(\lambda)} \end{pmatrix}.$$

Let $f(\lambda) = \frac{P_1}{\lambda^2 + \alpha_1^2}$, $g(\lambda) = \frac{P_2}{\lambda^2 + \alpha_2^2}$, $\vec{a}(t) = (1,\ 1)$. Then $A_1(\lambda) = \frac{e^{i\lambda}-1}{i\lambda}\vec{a}(t)$, and the function $\vec{c}(t) = (c_1(t),\ c_2(t))$ are determined by the equation $(B_1 c)(t) = \vec{a}(t)$, where

$$(B_1 c)(t) = \frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} \vec{c}(u)(F(\lambda))^{-1} e^{i(u-t)\lambda} d\lambda du.$$

We get the following system of integral equations with respect to $c_1(t)$ and $c_2(t)$

$$\frac{1}{2\pi} \int_0^1 \int_{-\infty}^{\infty} c_1(u) \frac{\alpha_1^2 + \lambda^2}{P_1}\, e^{i(u-t)\lambda} d\lambda du = 2,$$

$$\frac{1}{2\pi}\int_0^1\int_{-\infty}^{\infty}(c_2(u)-c_1(u))\frac{\alpha_2^2+\lambda^2}{P_1}\,e^{i(u-t)\lambda}d\lambda du=1.$$

Transform these integral equations to the differential ones and get

$$c_1(t)=P_1x(t),x''(t)-\alpha_1x(t)+2=0,x(0)=x(1)=0,$$

$$c_2(t)=c_1(t)+P_2y(t),y''(t)-\alpha_2y(t)+1=0,y(0)=y(1)=0.$$

Solutions to these equations are of the form

$$c_1(t)=\frac{2P_1}{\alpha_1^2}\left(\frac{e^{\alpha_1}-1}{e^{-\alpha_1}-e^{\alpha_1}}e^{-\alpha_1t}-\frac{e^{-\alpha_1}-1}{e^{-\alpha_1}-e^{\alpha_1}}e^{\alpha_1t}+1\right),$$

$$c_2(t)=c_1(t)+\frac{P_2}{\alpha_2^2}\left(\frac{e^{\alpha_2}-1}{e^{-\alpha_2}-e^{\alpha_2}}e^{-\alpha_2t}-\frac{e^{-\alpha_2}-1}{e^{-\alpha_2}-e^{\alpha_2}}e^{\alpha_2t}+1\right).$$

The spectral characteristic of the optimal estimate of the value $A_1\vec{\zeta}$ is of the form $h(F)=A_1(\lambda)-C_1(\lambda)\,F(\lambda)^{-1}=(h_1(\lambda),h_2(\lambda))$, where

$$h_1(\lambda)=(h_1-h_2)+(h_1-h_2)e^{i\lambda},\quad h_2(\lambda)=h_2+h_2e^{i\lambda},$$

$$h_1=\frac{2}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}(2-e^{\alpha_1}-e^{-\alpha_1}),h_2=\frac{1}{\alpha_2(e^{-\alpha_2}-e^{\alpha_2})}(2-e^{\alpha_2}-e^{-\alpha_2}).$$

The optimal estimate of the value $A_1\vec{\zeta}$ is of the form

$$\hat{A}_1\vec{\zeta}=(h_1-h_2)\zeta_1(0)+h_2\zeta_2(0)+(h_1-h_2)\zeta_1(1)+h_2\zeta_2(1).$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F)=\frac{4P_1}{\alpha_1^2}\left(1+\frac{2(1-e^{-\alpha_1})(e^{\alpha_1}-1)}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}\right)+\frac{P_2}{\alpha_2^2}\left(1+\frac{2(1-e^{-\alpha_2})(e^{\alpha_2}-1)}{\alpha_2(e^{-\alpha_2}-e^{\alpha_2})}\right).$$

Example 2. Consider the problem of estimation of the value $A_1\vec{\zeta}=\int_0^1\vec{a}(t)\vec{\zeta}(t)dt$, where $\vec{a}(t)=(1,\ 1-t)$. In this case

$$c_1(t)=P_1\left(Ae^{-\alpha_1t}+Be^{\alpha_1t}-\frac{1}{\alpha_1^2}t+\frac{2}{\alpha_1^2}\right),$$

$$A=\frac{1}{\alpha_1^2}\frac{2e^{\alpha_1}-1}{e^{-\alpha_1}-e^{\alpha_1}},\quad B=-\frac{1}{\alpha_1^2}\frac{2e^{-\alpha_1}-1}{e^{-\alpha_1}-e^{\alpha_1}};$$

$$c_2(t)=c_1(t)+P_2\left(Ce^{-\alpha_2t}+De^{\alpha_2t}-\frac{1}{\alpha_2^2}t+\frac{1}{\alpha_2^2}\right),$$

$$C=\frac{1}{\alpha_2^2}\frac{e^{\alpha_2}}{e^{-\alpha_2}-e^{\alpha_2}},\quad D=-\frac{1}{\alpha_2^2}\frac{e^{-\alpha_2}}{e^{-\alpha_2}-e^{\alpha_2}};$$

$$A_1(\lambda)=\left(\frac{e^{i\lambda}-1}{i\lambda},\ \frac{1}{\lambda^2}-\frac{1}{i\lambda}-\frac{e^{i\lambda}}{\lambda^2}\right).$$

The spectral characteristic of the optimal estimate is of the form

$h(F)=(h_1(\lambda),h_2(\lambda))$, $h_1(\lambda)=(h_1-h_2)+(h_3-h_4)e^{i\lambda}$, $h_2(\lambda)=h_2+h_4e^{i\lambda}$, where

$$h_1=\frac{2-2(e^{-\alpha_1}+e^{\alpha_1})}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}-\frac{1}{\alpha_1^2},\quad h_2=-\frac{1}{\alpha_2}\frac{e^{-\alpha_2}+e^{\alpha_2}}{e^{-\alpha_2}-e^{\alpha_2}}-\frac{1}{\alpha_2^2},$$

$$h_3=\frac{4-(e^{-\alpha_1}+e^{\alpha_1})}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}+\frac{1}{\alpha_1^2},\quad h_4=\frac{2}{\alpha_2(e^{-\alpha_2}-e^{\alpha_2})}+\frac{1}{\alpha_2^2}.$$

The optimal estimate of the value $A_1\vec{\zeta}$ is of the form

$$\hat{A}_1\vec{\zeta}=(h_1-h_2)\zeta_1(0)+(h_3-h_4)\zeta_1(1)+h_2\zeta_2(0)+h_4\zeta_2(1).$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F)=2P_1\left(\frac{1}{3\alpha_1^2}+A\left(\frac{1}{\alpha_1}+\frac{e^{-\alpha_1}}{\alpha_1^2}-\frac{1}{\alpha_1^2}\right)+B\left(-\frac{1}{\alpha_1}+\frac{e^{\alpha_1}}{\alpha_1^2}-\frac{1}{\alpha_1^2}\right)\right)+$$
$$+P_2\left(-\frac{1}{6\alpha_2^2}+C\left(\frac{1}{\alpha_2}+\frac{e^{-\alpha_2}}{\alpha_2^2}-\frac{1}{\alpha_2^2}\right)+D\left(-\frac{1}{\alpha_2}+\frac{e^{\alpha_2}}{\alpha_2^2}-\frac{1}{\alpha_2^2}\right)\right).$$

Example 3. Consider the problem of estimation of the value $A_1\vec{\zeta}=\int_0^1\vec{a}(t)\vec{\zeta}(t)dt$, where $\vec{a}(t)=(1,\ e^{\beta t})$. In this case

$$A_1(\lambda)=\left(\frac{e^{i\lambda}-1}{i\lambda},\frac{e^{i\lambda+\beta}-1}{i\lambda+\beta}\right),$$

$$c_1(t)=P_1\left(Ae^{-\alpha_1 t}+Be^{\alpha_1 t}+\frac{e^{\beta t}}{\alpha_1^2-\beta^2}+\frac{1}{\alpha_1^2}\right),$$

$$c_2(t)=c_1(t)+P_2\left(Ce^{-\alpha_2 t}+De^{\alpha_2 t}+\frac{e^{\beta t}}{\alpha_2^2-\beta^2}\right),$$

$$A=\frac{1}{e^{-\alpha_1}-e^{\alpha_1}}\left(\frac{e^{\alpha_1}-e^{\beta}}{\alpha_1^2-\beta^2}+\frac{e^{\alpha_1}-1}{\alpha_1^2}\right),$$

$$B=-\frac{1}{e^{-\alpha_1}-e^{\alpha_1}}\left(\frac{e^{-\alpha_1}-e^{\beta}}{\alpha_1^2-\beta^2}+\frac{e^{-\alpha_1}-1}{\alpha_1^2}\right),$$

$$C=\frac{1}{e^{-\alpha_2}-e^{\alpha_2}}\frac{e^{\alpha_2}-e^{\beta}}{\alpha_2^2-\beta^2},\quad D=-\frac{1}{e^{-\alpha_2}-e^{\alpha_2}}\frac{e^{-\alpha_2}-e^{\beta}}{\alpha_2^2-\beta^2}.$$

The spectral characteristic of the optimal estimate is of the form

$h(F)=(h_1(\lambda)\ ,\ h_2(\lambda))$, $h_1(\lambda)=(h_1-h_2)+(h_3-h_4)e^{i\lambda}$, $h_2(\lambda)=h_2+h_4e^{i\lambda}$, where

$$h_1=\frac{\alpha_1}{\alpha_1^2-\beta^2}\frac{2e^{\beta}-e^{-\alpha_1}-e^{\alpha_1}}{e^{-\alpha_1}-e^{\alpha_1}}+\frac{2-e^{-\alpha_1}-e^{\alpha_1}}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}+\frac{\beta}{\alpha_1^2-\beta^2},$$

$$h_3=\frac{\alpha_1}{\alpha_1^2-\beta^2}\frac{2-e^{\beta}(e^{-\alpha_1}+e^{\alpha_1})}{e^{-\alpha_1}-e^{\alpha_1}}+\frac{2-e^{-\alpha_1}-e^{\alpha_1}}{\alpha_1(e^{-\alpha_1}-e^{\alpha_1})}-\frac{\beta e^{\beta}}{\alpha_1^2-\beta^2},$$

$$h_2=\frac{\alpha_2(2e^{\beta}-e^{-\alpha_2}-e^{\alpha_2})}{(e^{-\alpha_2}-e^{\alpha_2})(\alpha_2^2-\beta^2)}+\frac{\beta}{\alpha_2^2-\beta^2},$$

$$h_4 = \frac{\alpha_2(2 - e^{\beta}e^{-\alpha_2} - e^{\alpha_2})}{(e^{-\alpha_2} - e^{\alpha_2})(\alpha_2^2 - \beta^2)} - \frac{\beta}{\alpha_2^2 - \beta^2}e^{\beta}.$$

The mean-square error of the optimal estimate can be calculated by the formula

$$\Delta(F) = 2P_1\left(\frac{1}{\alpha_1^2} - A\frac{e^{-\alpha_1} - 1}{\alpha_1} + B\frac{e^{\alpha_1} - 1}{\alpha_1} + \frac{e^{\beta} - 1}{\beta(\alpha_1^2 - \beta^2)}\right) +$$

$$+P_2\left(C\frac{e^{\beta-\alpha_2} - 1}{\beta - \alpha_2} + D\frac{e^{\beta+\alpha_2} - 1}{\beta + \alpha_2} + \frac{e^{2\beta} - 1}{2\beta(\alpha_2^2 - \beta^2)}\right).$$

3. Minimax-robust method of interpolation

Formulas (5)-(8) may be used to determine the mean-square error and the spectral characteristic of the optimal linear estimate of the functional $A_L\vec{\xi}$ when the spectral density matrices $F(\lambda)$ and $G(\lambda)$ of multidimensional stationary stochastic processes $\vec{\xi}(t)$ and $\vec{\eta}(t)$ are known. In the case where the spectral density matrices are unknown, but a set $D = D_F \times D_G$ of admissible spectral density matrices is given, the minimax-robust method of estimation of the unknown values of the functional $A_L\vec{\xi}$ is reasonable (see, for example, the survey article by S. A. Kassam and H. V. Poor (1985)). By means of this method it is possible to determine the estimate that minimizes the mean-square error for all spectral density matrices $F(\lambda)$ and $G(\lambda)$ from the class $D = D_F \times D_G$ simultaneously.

Definition 3.1. For a given class of spectral density matrices $D = D_F \times D_G$ spectral density matrices $F^0(\lambda) \in D_F,\ G^0(\lambda) \in D_G$ are called the least favorable for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if the following relation holds true

$$\Delta\left(F^0,\ G^0\right) = \Delta\left(h\left(F^0,\ G^0\right);\ F^0,\ G^0\right) = \max_{(F,\ G)\in D}\Delta\left(h\left(F,\ G\right);\ F,\ G\right).$$

Definition 3.2. For a given class of spectral density matrices $D = D_F \times D_G$ the spectral characteristic $h^0(\lambda)$ of the optimal linear estimate of the functional $A_L\vec{\xi}$ is called the minimax-robust if the conditions

$$h^0(\lambda) \in H_D = \bigcap_{(F,G)\in D} L_2^{L-}(F+G),$$

$$\min_{h\in H_D}\max_{(F,\ G)\in D}\Delta(h;\ F,\ G) = \max_{(F,\ G)\in D}\Delta\left(h^0;\ F,\ G\right).$$

are satisfied.

Taking into account relations (1)–(8), it is possible to verify the following propositions.

Proposition 3.1. *The spectral density matrices* $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ *are the least favorable in the class* $D = D_F \times D_G$ *for the optimal linear interpolation of the functional* $A_L\vec{\xi}$ *if the density matrix functions* $(F^0(\lambda) + G^0(\lambda))^{-1}$, $F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}$, $F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda)$ *determine* *operators B_L^0, D_L^0, R_L^0, which give solutions to the conditional extremum problem*

(9)
$$\max_{(F,\,G)\in D}\left(\left\langle D_L a, B_L^{-1} D_L a\right\rangle + \langle R_L a, a\rangle\right) = \left\langle D_L^0 a, (B_L^0)^{-1} D_L^0 a\right\rangle + \left\langle R_L^0 a, a\right\rangle.$$

The minimax-robust spectral characteristic $h^0 = h(F^0, G^0)$ of the optimal linear estimate of the functional $A_L\vec{\xi}$ can be calculated by formula (5) if the condition $h(F^0, G^0) \in H_D$ holds true.

Proposition 3.2. *The spectral density matrix $F^0(\lambda) \in D_F$ which satisfies the minimality condition is the least favorable in the class $D = D_F$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ based on observations of $\vec{\xi}(t)$, $t \in R\backslash[0, L]$, if the density matrix function $(F^0(\lambda))^{-1}$ determine the operator B_L^0 which gives a solution to the conditional extremum problem*

$$\max_{F\in D_F}\left\langle (B_L)^{-1} a, a\right\rangle = \left\langle (B_L^0)^{-1} a, a\right\rangle. \tag{10}$$

The minimax-robust spectral characteristic $h^0 = h(F^0)$ of the optimal linear estimate of the functional $A_L\vec{\xi}$ can be calculated by formula (7) if the condition $h(F^0) \in H_D$ holds true.

The least favorable spectral density matrices $F^0(\lambda) \in D$, $G^0(\lambda) \in D_G$ and the minimax-robust spectral characteristic $h^0 = h\left(F^0,\ G^0\right) \in H_D$ form a saddle point of the function $\Delta(h; F, G)$ on the set $H_D \times D$. The saddle point inequalities

$$\Delta\left(h^0; F, G\right) \le \Delta\left(h^0; F^0, G^0\right) \le \Delta\left(h; F^0, G^0\right),$$
$$\forall h \in H_D, \qquad \forall F \in D_F, \qquad \forall G \in D_G$$

hold true if $h^0 = h(F^0, G^0) \in H_D$ and (F^0, G^0) give a solution to the conditional extremum problem

$$\sup_{(F,G)\in D}\Delta\left(h\left(F^0, G^0\right); F, G\right) = \Delta\left(h\left(F^0, G^0\right); F^0, G^0\right), \tag{11}$$

where

$$\Delta\left(h\left(F^0, G^0\right); F, G\right) =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(A_L(\lambda)\, G^0(\lambda) + C_L^0(\lambda)\right)\left(F^0(\lambda) + G^0(\lambda)\right)^{-1} F(\lambda)\times$$
$$\times(F^0(\lambda) + G^0(\lambda))^{-1}(A_L(\lambda)\, G^0(\lambda) + C_L^0(\lambda))^* d\lambda +$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}(A_L(\lambda)\, F^0(\lambda) - C_L^0(\lambda))(F^0(\lambda) + G^0(\lambda))^{-1} G(\lambda)\times$$
$$\times(F^0(\lambda) + G^0(\lambda))^{-1}(A_L(\lambda)\, G^0(\lambda) + C_L^0(\lambda))^* d\lambda.$$

This conditional extremum problem is equivalent to the unconditional extremum problem

$$\Delta_D(F,G) = -\Delta(h(F^0,G^0);F,G) + \delta((F,G)|D) \to \inf, \tag{12}$$

where $\delta((F,G)|D)$ is the indicator function of the set $D = D_F \times D_G$. A solution to this problem is determined by the condition $0 \in \partial\Delta_D(F^0,G^0)$, where $\partial\Delta_D(F^0,G^0)$ is the subdifferential of the convex functional $\partial\Delta_D(F,G)$ at the point (F^0,G^0) (see, for example, B. N. Pshenichnyi (1982)).

The following propositions holds true.

Proposition 3.3. *Let $(F^0,\ G^0)$ be a solution to the conditional extremum problem (11). The spectral density matrices $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ are the least favorable in the class $D = D_F \times D_G$ and the spectral characteristic $h^0 = h(F^0,G^0)$ is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if the condition $h(F^0,G^0) \in H_D$ holds true.*

Proposition 3.4. *The spectral density matrix $F^0(\lambda) \in D_F$ which satisfies the minimality condition is the least favorable in the class $D = D_F$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ based on observations of $\vec{\xi}(t)$, $t \in R\backslash[0,L]$, if the density matrix function $F^0(\lambda)$ gives a solution to the conditional extremum problem*

$$\sup_{F\in D_F} \Delta\left(h\left(F^0\right);\ F\right) = \Delta\left(h\left(F^0\right);F^0\right), \tag{13}$$

where

$$\Delta\left(h\left(F^0\right);F\right) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} C_L^0(\lambda)(F^0(\lambda))^{-1}F(\lambda)(F^0(\lambda))^{-1}(C_L^0(\lambda))^{*}d\lambda.$$

The spectral characteristic $h^0 = h(F^0)$ is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if the condition $h(F^0) \in H_D$ holds true.

4. Least favorable spectral densities in the class $D_F^0 \times D_G^0$

Consider the problem of minimax estimation of the functional $A_L\vec{\xi}$ based on observations $\vec{\xi}(t) + \vec{\eta}(t)$, $t \in R\backslash[0,L]$ under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ of the multidimentional stationary processes $\vec{\xi}(t)$, $\vec{\eta}(t)$ are from the set of spectral density matrices $D_F^0 \times D_G^0$, where

$$D_F^0 = \left\{ F(\lambda) \left| \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(\lambda)d\lambda = P_1 \right. \right\},$$

$$D_G^0 = \left\{ G(\lambda) \left| \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} G(\lambda)d\lambda = P_2 \right. \right\}.$$

With the help of the Lagrange multipliers method we can find the following relations that determine the least favorable spectral density matrices $(F^0(\lambda), G^0(\lambda)) \in D_F^0 \times D_G^0$

$$A_L(\lambda)G^0(\lambda) + C_L^0(\lambda) = \vec{\alpha} \cdot (F^0(\lambda) + G^0(\lambda)), \tag{14}$$

$$A_L(\lambda)F^0(\lambda) - C_L^0(\lambda) = \vec{\beta} \cdot (F^0(\lambda) + G^0(\lambda)), \tag{15}$$

where $\vec{\alpha} = (\alpha_1, \ \dots \ , \alpha_T)$, $\quad \vec{\beta} = (\beta_1, \ \dots \ , \beta_T)$ are the Lagrange multipliers. It follows from these relations that the the following theorems hold true.

Theorem 4.1. *Let the spectral density matrices $F^0(\lambda) \in D_F^0$, $G^0(\lambda) \in D_G^0$ satisfy the minimality condition (1). These spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$ are the least favorable in the class $D = D_F^0 \times D_G^0$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if $F^0(\lambda)$, $G^0(\lambda)$ are solutions to the equations (14), (15) and determine a solution to the extremum problem (9). The spectral characteristic $h(F^0, G^0)$ calculated by the formula (5) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.*

Theorem 4.2. *Let the spectral density matrix $F(\lambda)$ be know and let spectral density matrices $F(\lambda)$, $G^0(\lambda) \in D_G^0$ satisfy the minimality condition (1). The spectral density matrix $G^0(\lambda)$ is the least favorable in the class D_G^0 for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if*

$$G^0(\lambda) = \max\left\{0, \ \vec{\alpha}^{-1} \cdot (A_L(\lambda)F(\lambda) - C_L^0(\lambda)) - F(\lambda)\right\},$$

where

$$\vec{\alpha}^{-1} = \left(\frac{\overline{\alpha_1}}{D}, \ \dots \ , \frac{\overline{\alpha_T}}{D}\right)^{\top}, \quad D = |\alpha_1|^2 + \ldots + |\alpha_T|^2,$$

and $F(\lambda)$, $G^0(\lambda)$ determine a solution to the extremum problem (9). The spectral characteristic $h(F, G^0)$ calculated by the formula (5) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.

Theorem 4.3. *Let the spectral density matrix $F^0(\lambda) \in D_F^0$ satisfies the minimality condition (1). The spectral density matrix $F^0(\lambda)$ is the least favorable in the class $D = D_F^0$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ based on observations of $\vec{\xi}(t)$ for $t \in R\backslash[0, L]$ if $\vec{\alpha} \cdot F^0(\lambda) = C_L^0(\lambda)$ and $F^0(\lambda)$ determine a solution to the extremum problem (10). The spectral characteristic $h^0 = h(F^0)$ calculated by the formula (7) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.*

5. Least favorable spectral densities in the class $D_V^U \times D_\varepsilon$

Consider the problem of minimax estimation of the functional $A_L\vec{\xi}$ based on observations $\vec{\xi}(t) + \vec{\eta}(t)$, $t \in R\backslash[0, L]$ under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ of the multidimensional stationary processes

$\vec{\xi}(t)$, $\vec{\eta}(t)$ are from the set of spectral density matrices $D_V^U \times D_\varepsilon$, where

$$D_V^U = \left\{ F(\lambda) \left| V(\lambda) \leq F(\lambda) \leq U(\lambda), \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) d\lambda = P_1 \right. \right\},$$

$$D_\varepsilon = \left\{ G(\lambda) \left| G(\lambda) = (1-\varepsilon) G_1(\lambda) + \varepsilon W(\lambda), \ \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\lambda) d\lambda = P_2 \right. \right\},$$

where $V(\lambda)$, $U(\lambda)$, $G_1(\lambda)$ are given fixed spectral density matrices, $W(\lambda)$ ia an unknown spectral density matrix, and expression $B(\lambda) \geq D(\lambda)$ means that $B(\lambda) - D(\lambda) \geq 0$ (positive definite matrix function). The set D_V^U describes the 'band' model of stochastic processes while the set D_ε describes the 'ε-contaminated' model of stochastic processes. For the set $D_V^U \times D_\varepsilon$ from the condition $0 \in \partial \Delta_D(F^0, G^0)$ we can get the following relations which determine the least favorable spectral density matrices

$$\vec{a}^0(\lambda) \vec{a}^0(\lambda)^* = \vec{\alpha} \cdot \vec{\alpha}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda); \tag{16}$$

$$\vec{b}^0(\lambda) \vec{b}^0(\lambda)^* = \vec{\beta} \cdot \vec{\beta}^* + \Gamma_3(\lambda), \tag{17}$$

where

$$\vec{a}^0(\lambda) = \left((A_L(\lambda) G^0(\lambda) + C_L^0(\lambda))(F^0(\lambda) + G^0(\lambda))^{-1} \right)^T,$$

$$\vec{b}^0(\lambda) = \left((A_L(\lambda) F^0(\lambda) - C_L^0(\lambda))(F^0(\lambda) + G^0(\lambda))^{-1} \right)^T.$$

The coefficients $\vec{\alpha} = (\alpha_1, \ldots, \alpha_T)^T$, $\vec{\beta} = (\beta_1, \ldots \beta_T)^T$ are determined by the conditions

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} F^0(\lambda) d\lambda = P_1, \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} G^0(\lambda) d\lambda = P_2. \tag{18}$$

The matrix functions $\Gamma_1(\lambda) \geq 0$, $\Gamma_2(\lambda) \geq 0$, $\Gamma_3(\lambda) \geq 0$ are determined by the conditions

$$V(\lambda) \leq F^0(\lambda) \leq U(\lambda), \quad G^0(\lambda) = (1-\varepsilon) G_1(\lambda) + \varepsilon W(\lambda), \tag{19}$$

$$\Gamma_1(\lambda) = 0 \text{ if } F^0(\lambda) \geq V(\lambda), \quad \Gamma_2(\lambda) = 0 \text{ if } F^0(\lambda) \leq U(\lambda), \tag{20}$$

$$\Gamma_3(\lambda) = 0 \text{ if } G^0(\lambda) \geq (1-\varepsilon) G_1(\lambda). \tag{21}$$

It follows from these relations that the the following theorems hold true.

Theorem 5.1. *Let the spectral density matrices $F^0(\lambda) \in D_V^U$, $G^0(\lambda) \in D_\varepsilon$ satisfy the minimality condition (1). These spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$ are the least favorable in the class $D_V^U \times D_\varepsilon$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if they satisfy conditions (16) – (21) and determine a solution to the extremum problem (9). The spectral* *characteristic $h(F^0, G^0)$ calculated by the formula (5) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.*

Theorem 5.2. *Let the spectral density matrix $F(\lambda)$ be known and let spectral density matrices $F(\lambda)$, $G^0(\lambda) \in D_\varepsilon$ satisfy the minimality condition (1). The spectral density matrix $G^0(\lambda)$ is the least favorable in the class D_ε for the optimal linear interpolation of the functional $A_L\vec{\xi}$ if*

$$G^0(\lambda) = \max\left\{(1-\varepsilon)G_1(\lambda), \vec{\alpha}^{-1}(A_L(\lambda)F(\lambda) - C_L^0(\lambda)) - G(\lambda)\right\}$$

and $(F(\lambda), G^0(\lambda))$ determine a solution to the extremum problem (9). The spectral characteristic $h(F, G^0)$ calculated by the formula (5) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.

Theorem 5.3. *Let the spectral density matrix $F^0(\lambda) \in D_V^U$ satisfies the minimality condition (1). This spectral density matrix $F^0(\lambda)$ is the least favorable in the class $D = D_V^U$ for the optimal linear interpolation of the functional $A_L\vec{\xi}$ based on observations of $\vec{\xi}(t)$ for $t \in R\backslash[0, L]$ if*

$$F^0(\lambda) = \max\left\{V(\lambda), \min\left\{U(\lambda), \vec{\alpha}^{-1}C_L^0(\lambda)\right\}\right\}$$

and $F^0(\lambda)$ determine a solution to the extremum problem (10). The spectral characteristic $h^0 = h(F^0)$ calculated by the formula (7) is minimax-robust for the optimal linear interpolation of the functional $A_L\vec{\xi}$.

6. Hilbert space projection method of extrapolation

Let the vector function $\vec{a}(t)$ which determines the functional $A\vec{\xi}$ satisfies conditions:

$$\int_0^\infty \sum_{k=1}^T |a_k(t)|\,dt < \infty, \quad \int_0^\infty t\sum_{k=1}^T |a_k(t)|^2 dt < \infty. \tag{22}$$

Under these conditions the functional $A\vec{\xi}$ has the second moment and the operator A defined below is compact.

Let the spectral density matrices

$$F(\lambda) = \{f_{kl}(\lambda)\}_{k,l=1}^T \quad \text{and} \quad G(\lambda) = \{g_{kl}(\lambda)\}_{k,l=1}^T$$

of uncorrelated multidimensional stationary stochastic processes $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$, $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$ satisfy the minimality condition (1), where $b(\lambda) = \int_0^\infty \vec{\alpha}(t)e^{it\lambda}dt$, $\vec{\alpha}(t) = \{\alpha_k(t)\}_{k=1}^T$, is a nontrivial vector function of the exponential type.

Denote by $L_2^-(F)$ the subspace in $L_2(F)$, generated by functions $e^{it\lambda}\delta_k$, $t < 0$, $\delta_k = \{\delta_{kl}\}_{l=1}^T$, $k = 1, \ldots, T$, where $\delta_{kk} = 1$, $\delta_{kl} = 0$ for $k \neq l$. Any linear linear extrapolation $\hat{A}\vec{\xi}$ of the functional $A\vec{\xi}$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t < 0$ is of the form

$$\hat{A}\vec{\xi} = \int_{-\infty}^{\infty} h(\lambda)\,(Z^\xi(d\lambda) + Z^\eta(d\lambda)) = \int_{-\infty}^{\infty} \sum_{k=1}^T h_k(\lambda)(Z_k^\xi(d\lambda) + Z_k^\eta(d\lambda)),$$

where $h(\lambda)=\{h_k(\lambda)\}_{k=1}^{T}$ is the spectral characteristic of the linear extrapolation $\hat{A}\vec{\xi}$. The function $h(\lambda)\in L_2^-(F+G)$. The value of the mean square error $\Delta(h;F,G)$ of the linear extrapolation $\hat{A}\vec{\xi}$ is calculated by the formula

$$\Delta(h;F,G)=E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^2=$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)-h(\lambda))\,F(\lambda)\,(A(\lambda)-h(\lambda))^*d\lambda+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(\lambda)\,G(\lambda)\,h^*(\lambda)d\lambda,$$

where

$$A(\lambda)=\int_0^{\infty}\vec{a}(t)e^{it\lambda}dt.$$

The spectral characteristic $h(F,G)$ of the optimal linear linear extrapolation of $A\vec{\xi}$ minimizes the mean square error

$$\Delta(F,G)=\Delta(h(F,G);F,G)$$

$$=\min_{h\in L_2^-(F+G)}\Delta(h;F,G)=\min_{\hat{A}\vec{\xi}}E\left|A\vec{\xi}-\hat{A}\vec{\xi}\right|^2. \tag{23}$$

With the help of the Hilbert space projection method proposed by A. N. Kolmogorov we can find a solution of the optimization problem (23):

$$h(F,G)=(A(\lambda)\,F(\lambda)-C(\lambda))\,(F(\lambda)+G(\lambda))^{-1}=$$

$$=A(\lambda)-(A(\lambda)\,G(\lambda)+C(\lambda))\,(F(\lambda)+G(\lambda))^{-1}, \tag{24}$$

$$\Delta(F,G)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)\,G(\lambda)+C(\lambda))\,(F(\lambda)+G(\lambda))^{-1}\,F(\lambda)\times$$

$$\times(F(\lambda)+G(\lambda))^{-1}(A(\lambda)\,G(\lambda)+C(\lambda))^*d\lambda+$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)\,F(\lambda)-C(\lambda))(F(\lambda)+G(\lambda))^{-1}G(\lambda)\,(F(\lambda)+G(\lambda))^{-1}\times$$

$$\times(A(\lambda)\,G(\lambda)+C(\lambda))^*d\lambda=\langle\mathbf{B}c\,,\,c\rangle+\langle\mathbf{R}a\,,\,a\rangle\,, \tag{25}$$

where

$$C(\lambda)=\int_0^{\infty}\vec{c}(t)e^{it\lambda}dt,\quad \vec{c}(t)=(\mathbf{B}^{-1}\mathbf{D}a)(t),$$

$$\langle\mathbf{B}c,c\rangle=\int_0^{\infty}\sum_{k=1}^{n}(\mathbf{B}c)_k(t)\,\overline{c_k(t)}\,dt,$$

$$\langle \mathbf{R}a, a\rangle = \int_0^\infty \sum_{k=1}^n (\mathbf{R}a)_k(t)\,\overline{a_k(t)}dt,$$

operators $\mathbf{B}, \mathbf{D}, \mathbf{R}$ are determined by the following relations

$$(\mathbf{B}c)(t) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty \vec{c}(u)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}d\lambda du,$$

$$(\mathbf{D}c)(t) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty \vec{c}(u)F(\lambda)(F(\lambda)+G(\lambda))^{-1}e^{i(u-t)\lambda}d\lambda du,$$

$$(\mathbf{R}c)(t) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^\infty \vec{c}(u)F(\lambda)(F(\lambda)+G(\lambda))^{-1}G(\lambda)e^{i(u-t)\lambda}d\lambda du.$$

From the preceding formulas we can conclude that the following theorem holds true.

Theorem 6.1. *Let $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$ and $\vec{\eta}(t) = \{\eta_k(t)\}_{k=1}^T$ be uncorrelated multidimensional stationary stochastic processes with the spectral density matrices $F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^T$ and $G(\lambda) = \{g_{ij}(\lambda)\}_{i,j=1}^T$ that satisfy the minimality condition (1). Let condition (22) be satisfied. The value of the mean-square error $\Delta(h(F,G),F,G)$ and the spectral characteristic $h(F,G)$ of the optimal linear extrapolation of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of the process $\vec{\xi}(t)$ $t \geq 0$, based on observations of the process $\vec{\xi}(t)+\vec{\eta}(t)$ for $t<0$ are calculated by formulas (24), (25).*

Corollary 6.1. *Let $\vec{\xi}(t) = \{\xi_k(t)\}_{k=1}^T$ be a multidimensional stationary stochastic process with the spectral density matrix $F(\lambda) = \{f_{kl}(\lambda)\}_{k,l=1}^T$, that satisfies the minimality condition*

$$\int_{-\infty}^\infty b(\lambda)(F(\lambda))^{-1}b^*(\lambda)d\lambda < \infty$$

for a nontrivial vector function of the exponential type $b(\lambda) = \int_0^\infty \vec{\alpha}(t)e^{it\lambda}dt$. Let condition (22) be satisfied. The value of the mean-square error $\Delta(F)$ and the spectral characteristic $h(F)$ of the optimal linear extrapolation of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(t)dt$ based on observations of the process $\vec{\xi}(t)$ for $t<0$ are calculated by the formulas

$$h(F) = A(\lambda) - C(\lambda)\,(F(\lambda))^{-1}, \tag{26}$$

$$\Delta(F) = \frac{1}{2\pi}\int_{-\infty}^\infty C(\lambda)\,(F(\lambda))^{-1}(C(\lambda))^*d\lambda = \langle \mathbf{B}a, a\rangle = \langle c, a\rangle \tag{27}$$

where $\vec{c}(t) = (\mathbf{B}^{-1}a)(t)$,

$$(\mathbf{B}a)(t) = \frac{1}{2\pi} \int\limits_{0}^{\infty} \int\limits_{-\infty}^{\infty} \vec{a}(u)(F(\lambda))^{-1} e^{i(u-t)\lambda} d\lambda du.$$

Let the process $\vec{\xi}(t)$ admits the canonical moving average representation

$$\vec{\xi}(t) = \int_{-\infty}^{t} d(t-u)\, d\vec{\varepsilon}(u), \tag{28}$$

where $d(s) = \{d_{ij}(s)\}_{i=\overline{1,T}}^{j=\overline{1,m}}$ is a matrix function and $\vec{\varepsilon}(u) = \{\varepsilon_k(u)\}_{k=1}^{m}$ is a multidimensional stationary stochastic process with uncorrelated increments. In this case the spectral density matrix $F(\lambda) = \{f_{ij}(\lambda)\}_{i,j=1}^{n}$ of the process $\vec{\xi}(t)$ admits the canonical factorization:

$$F(\lambda) = \varphi(\lambda)\, \varphi^{*}(\lambda), \quad \varphi(\lambda) = \int_{0}^{\infty} d(u) e^{-iu\lambda} d\lambda. \tag{29}$$

If the process $\vec{\xi}(t)$ admits the canonical moving average representation (28), then the optimal linear extrapolation of the functional $A\vec{\xi} = \int_0^{\infty} \vec{a}(t)\vec{\xi}(t)dt$ based on observations of the process $\vec{\xi}(t)$ for $t < 0$ is determined by the spectral characteristic $h(F) \in L_2^{-}(F)$ that minimizes the mean square error

$$\Delta(h(F), F) = \min_{h \in L_2^{-}(F)} \Delta(h, F) = \left\|\mathbf{A}d\right\|^{2}, \tag{30}$$

where

$$(\mathbf{A}d)(t) = \int_{0}^{\infty} \vec{a}(t+u) d(u)\, du, \quad \left\|\mathbf{A}d\right\|^{2} = \int_{0}^{\infty} \sum_{k=1}^{m} \left|(Ad)_k(t)\right|^{2} dt.$$

Note, that $\left\|\mathbf{A}d\right\|^{2} < \infty$ under condition (22). The spectral characteristic $h(F)$ is calculated by the formula

$$h(F) = A(\lambda) - r(\lambda)\psi(\lambda), \quad r(\lambda) = \int_{0}^{\infty} (\mathbf{A}d)(t) e^{it\lambda} dt. \tag{31}$$

Here $\psi(\lambda) = \{\psi_{ij}(\lambda)\}_{i=\overline{1,m}}^{j=\overline{1,T}}$ is a matrix function which satisfies the equation

$$\psi(\lambda)\varphi(\lambda) = I_m,$$

where I_m is the identity matrix of order m.

For the functional $A_L\vec{\xi} = \int_0^{L} \vec{a}(t)\vec{\xi}(t)d(t)$ the value of the mean square error and the spectral characteristics of the optimal linear extrapolation are determined by the following formulas

$$\Delta_L(h(F), F) = \left\|\mathbf{A}_L d\right\|^{2}, \tag{32}$$

$$h(F) = A_L(\lambda) - r_L(\lambda)\psi(\lambda), \tag{33}$$

where

$$\mathbf{A}_L d(t) = \int_0^{L-t} \vec{a}(t+u)d(u)\,du, \quad \|\mathbf{A}_T d\|^2 = \int_0^L \sum_{k=1}^m |(\mathbf{A}_L d)_k(t)|^2 dt,$$

$$A_L(\lambda) = \int_0^L \vec{a}(t)e^{it\lambda}dt, \quad r_L(\lambda) = \int_0^L (\mathbf{A}_L d)(t)e^{it\lambda}dt.$$

As a corollary we can get the following formulas for calculation the mean square error of the optimal linear extrapolation $\hat{\xi}_k(L)$ of the unknown values $\xi_k(L)$, $k = 1, \ldots, T$:

$$E\left|\xi_k(L) - \hat{\xi}_k(L)\right|^2 = \int_0^L \sum_{l=1}^m |d_{kl}(t)|^2 dt. \tag{34}$$

The following theorem holds true.

Theorem 6.2. *Let $\vec{\xi}(t) = \{\xi_k(u)\}_{k=1}^n$ be a stationary stochastic process that admits the canonical moving average representation (28) with the spectral density matrix $F(\lambda)$ that admits the canonical factorization (29) and let condition(22) be satisfied. The value of the mean-square error $\Delta(h(F), F)$ of the optimal linear extrapolation of the functional $A\vec{\xi}$ from observations of the process $\vec{\xi}(t)$ for $t < 0$ is calculated by formula (30) (by formula (32) if the functional $A_L\vec{\xi}$ is estimated). The spectral characteristics $h(F)$ of the optimal linear extrapolation can be calculated by formula (31) (by formula (33) if the functional $A_L\vec{\xi}$ is estimated).*

7. Minimax-robust method of extrapolation

Taking into account relations (22)–(27), we can verify the following propositions.

Proposition 7.1. *The spectral density matrices $F^0(\lambda) \in D_F$, $G^0(\lambda) \in D_G$ are the least favorable in the class $D = D_F \times D_G$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ if the density matrix functions $(F^0(\lambda) + G^0(\lambda))^{-1}$, $F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}$, $F^0(\lambda)(F^0(\lambda)+G^0(\lambda))^{-1}G^0(\lambda)$ determine operators $\mathbf{B}^0$, $\mathbf{D}^0$, $\mathbf{R}^0$, which give solutions to the conditional extremum problem*

$$\max_{(F,\,G)\in D} \left(\left\langle \mathbf{D}a, \mathbf{B}^{-1}\mathbf{D}a\right\rangle + \langle \mathbf{R}a, a\rangle\right) = \left\langle \mathbf{D}^0 a, (\mathbf{B}^0)^{-1}\mathbf{D}^0 a\right\rangle + \left\langle \mathbf{R}^0 a, a\right\rangle. \tag{35}$$

The minimax-robust spectral characteristic $h^0 = h(F^0, G^0)$ of the optimal linear extrapolation of the functional $A\vec{\xi}$ is calculated by formula (24) if the condition $h(F^0, G^0) \in H_D$ holds true.

Proposition 7.2. *The spectral density matrix $F^0(\lambda) \in D_F$ which satisfies the minimality condition is the least favorable in the class $D = D_F$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ based on observations of*

$\vec{\xi}(t)$, $t<0$, *if the density matrix function* $(F^0(\lambda))^{-1}$ *determine the operator* $\mathbf{B}^0$ *which gives a solution to the conditional extremum problem*

$$\max_{F\in D_F}\left\langle(\mathbf{B})^{-1}a,a\right\rangle=\left\langle(\mathbf{B}^0)^{-1}a,a\right\rangle. \tag{36}$$

The minimax-robust spectral characteristic $h^0=h(F^0)$ *of the optimal linear extrapolation of the functional* $A\vec{\xi}$ *is calculated by formula (26) if the condition* $h(F^0)\in H_D$ *holds true.*

The least favorable spectral density matrices $F^0(\lambda)\in D$, $G^0(\lambda)\in D_G$ and the minimax-robust spectral characteristic $h^0=h\left(F^0,\ G^0\right)\in H_D$ form a saddle point of the function $\Delta(h;F,G)$ on the set $H_D\times D$. The saddle point inequalities hold true if $h^0=h(F^0,G^0)\in H_D$ and (F^0,G^0) gives a solution to the conditional extremum problem

$$\sup_{(F,G)\in D}\Delta\left(h\left(F^0,G^0\right);F,G\right)=\Delta\left(h\left(F^0,G^0\right);F^0,G^0\right), \tag{37}$$

where

$$\Delta\left(h\left(F^0,G^0\right);F,G\right)=$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(A(\lambda)\,G^0(\lambda)+C^0(\lambda)\right)\left(F^0(\lambda)+G^0(\lambda)\right)^{-1}F(\lambda)\times$$
$$\times(F^0(\lambda)+G^0(\lambda))^{-1}(A(\lambda)\,G^0(\lambda)+C^0(\lambda))^{*}d\lambda+$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}(A(\lambda)\,F^0(\lambda)-C^0(\lambda))(F^0(\lambda)+G^0(\lambda))^{-1}G(\lambda)\times$$
$$\times(F^0(\lambda)+G^0(\lambda))^{-1}(A(\lambda)\,G^0(\lambda)+C^0(\lambda))^{*}d\lambda.$$

This conditional extremum problem is equivalent to the unconditional extremum problem

$$\Delta_D(F,G)=-\Delta(h(F^0,G^0);F,G)+\delta((F,G)|D)\to\inf, \tag{38}$$

where $\delta((F,G)|D)$ is the indicator function of the set $D=D_F\times D_G$.

The following propositions hold true.

Proposition 7.3. *Let* $(F^0,\ G^0)$ *be a solution to the conditional extremum problem (37). The spectral density matrices* $F^0(\lambda)\in D_F$, $G^0(\lambda)\in D_G$ *are the least favorable in the class* $D=D_F\times D_G$ *and the spectral characteristic* $h^0=h(F^0,G^0)$ *is minimax-robust for the optimal linear extrapolation of the functional* $A\vec{\xi}$ *if the condition* $h(F^0,G^0)\in H_D$ *holds true.*

Proposition 7.4. *The spectral density matrix* $F^0(\lambda)\in D_F$ *is the least favorable in the class* D_F *for the optimal linear extrapolation of the functional* $A\vec{\xi}$ *based on observations of* $\vec{\xi}(t)$, $t<0$, *if* $F^0(\lambda)$ *admits the canonical factorization*

$$F^0(\lambda)=\left[\int_0^{\infty}d^0(t)e^{-it\lambda}dt\right]\cdot\left[\int_0^{\infty}d^0(t)e^{-it\lambda}dt\right]^{*}, \tag{39}$$

where $d^0(t)$ is a solution to the conditional extremum problem

$$(40)\quad \|Ad\|^2 \to \max, \quad F(\lambda) = \left[\int_0^\infty d(t)e^{-it\lambda}dt\right] \cdot \left[\int_0^\infty d(t)e^{-it\lambda}dt\right]^* \in D.$$

The process $\vec{\xi}(t)$ in this case admits the canonical moving average representation

$$\vec{\xi}(t) = \int_{-\infty}^{t} d^0(t-u)d\vec{\varepsilon}(u). \tag{41}$$

The minimax spectral characteristic $h^0 = h(F^0)$ is calculated by the formula (31) under the condition $h(F^0) \in H_D$.

Proposition 7.5. *The spectral density matrix $F^0(\lambda) \in D_F$ is the least favorable in the class D_F for the optimal linear extrapolation of the functional $A_L\vec{\xi}$ based on observations of $\vec{\xi}(t)$, $t < 0$, if $F^0(\lambda)$ admits the canonical factorization*

$$F^0(\lambda) = \left[\int_0^L d^0(t)e^{-it\lambda}dt\right] \cdot \left[\int_0^L d^0(t)e^{-it\lambda}dt\right]^*, \tag{42}$$

where $d^0(t)$, $0 \le t \le L$, is a solution to the conditional extremum problem

$$(43)\quad \|A_L d\|^2 \to \max, \quad F(\lambda) = \left[\int_0^L d(t)e^{-it\lambda}dt\right] \cdot \left[\int_0^L d(t)e^{-it\lambda}dt\right]^* \in D.$$

The process $\vec{\xi}(t)$ in this case admits the canonical moving average representation

$$\vec{\xi}(t) = \int_{t-L}^{t} d^0(t-u)d\vec{\varepsilon}(u). \tag{44}$$

The minimax spectral characteristic $h^0 = h(F^0)$ is calculated by the formula (33) under the condition $h(F^0) \in H_D$.

8. Least favorable spectral densities in the class $D_V^U \times D_\varepsilon$

Consider the problem of minimax extrapolation of the functional $A\vec{\xi}$ based on observations $\vec{\xi}(t) + \vec{\eta}(t)$, $t < 0$, under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ of the multidimensional stationary processes $\vec{\xi}(t)$, $\vec{\eta}(t)$ are from the set of spectral density matrices $D_V^U \times D_\varepsilon$, where

$$D_V^U = \left\{ F(\lambda) \left| V(\lambda) \le F(\lambda) \le U(\lambda), \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)d\lambda = P_1 \right. \right\},$$

$$D_\varepsilon = \left\{ G(\lambda) \left| G(\lambda) = (1-\varepsilon)G_1(\lambda) + \varepsilon W(\lambda), \ \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\lambda)d\lambda = P_2 \right. \right\},$$

where $V(\lambda)$, $U(\lambda)$, $G_1(\lambda)$ are given fixed spectral density matrices, $W(\lambda)$ ia an unknown spectral density matrix, and expression $B(\lambda) \ge D(\lambda)$ means that $B(\lambda) - D(\lambda) \ge 0$ (positive definite matrix function). For the set

$D_V^U \times D_\varepsilon$ from the condition $0 \in \partial\Delta_D(F^0, G^0)$ we can get the following relations which determine the least favorable spectral density matrices

$$\vec{a}^0(\lambda)\vec{a}^0(\lambda)^* = \vec{\alpha} \cdot \vec{\alpha}^* + \Gamma_1(\lambda) + \Gamma_2(\lambda); \tag{45}$$

$$\vec{b}^0(\lambda)\vec{b}^0(\lambda)^* = \vec{\beta} \cdot \vec{\beta}^* + \Gamma_3(\lambda), \tag{46}$$

where

$$\vec{a}^0(\lambda) = \left((A(\lambda)G^0(\lambda) + C^0(\lambda))(F^0(\lambda) + G^0(\lambda))^{-1}\right)^T,$$

$$\vec{b}^0(\lambda) = \left((A(\lambda)F^0(\lambda) - C^0(\lambda))(F^0(\lambda) + G^0(\lambda))^{-1}\right)^T.$$

The coefficients $\vec{\alpha} = (\alpha_1, \ldots, \alpha_T)^T$, $\vec{\beta} = (\beta_1, \ldots \beta_T)^T$ are determined by the conditions

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F^0(\lambda)d\lambda = P_1, \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} G^0(\lambda)d\lambda = P_2. \tag{47}$$

The matrix functions $\Gamma_1(\lambda) \geq 0,\ \Gamma_2(\lambda) \geq 0,\ \Gamma_3(\lambda) \geq 0$ are determined by the conditions

$$V(\lambda) \leq F^0(\lambda) \leq U(\lambda), \quad G^0(\lambda) = (1-\varepsilon)G_1(\lambda) + \varepsilon W(\lambda), \tag{48}$$

$$\Gamma_1(\lambda) = 0 \text{ if } F^0(\lambda) \geq V(\lambda), \quad \Gamma_2(\lambda) = 0 \text{ if } F^0(\lambda) \leq U(\lambda), \tag{49}$$

$$\Gamma_3(\lambda) = 0 \text{ if } G^0(\lambda) \geq (1-\varepsilon)G_1(\lambda). \tag{50}$$

From these relations we can conclude that the following theorems hold true.

Theorem 8.1. *Let the spectral density matrices $F^0(\lambda) \in D_V^U$, $G^0(\lambda) \in D_\varepsilon$ satisfy the minimality condition (1). Let condition (22) be satisfied. These spectral density matrices $F^0(\lambda)$, $G^0(\lambda)$ are least favorable in the class $D_V^U \times D_\varepsilon$ for the optimal linear extrapolation of the functional $A\vec{\xi}$ if they satisfy conditions (45)–(50) and determine a solution to the extremum problem (35). The spectral characteristic $h(F^0, G^0)$ calculated by the formula (24) is minimax-robust for the optimal linear extrapolation of the functional $A\vec{\xi}$.*

Theorem 8.2. *Let the spectral density matrix $F(\lambda)$ be known and let spectral density matrices $F(\lambda)$, $G^0(\lambda) \in D_\varepsilon$ satisfy the minimality condition (1). Let condition (22) be satisfied. The spectral density matrix $G^0(\lambda)$ is least favorable in the class D_ε for the optimal linear extrapolation of the functional $A\vec{\xi}$ if*

$$G^0(\lambda) = \max\left\{(1-\varepsilon)G_1(\lambda), \vec{\alpha}^{-1}(A(\lambda)F(\lambda) - C^0(\lambda)) - G(\lambda)\right\}$$

and $(F(\lambda), G^0(\lambda))$ determine a solution to the extremum problem (35). The spectral characteristic $h(F, G^0)$ calculated by the formula (24) is minimax-robust for the optimal linear extrapolation of the functional $A\vec{\xi}$.

9. Least favorable spectral densities in the class D_0.

Consider the problem of minimax extrapolation of the functional $A\vec{\xi}$ based on observations $\vec{\xi}(t)$ for $t<0$ under the condition that spectral density matrix $F(\lambda)$ is from the set of spectral density matrices

$$D_0 = \left\{ F(\lambda) \colon \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) d\lambda = P \right\}.$$

With the help of the Lagrange multipliers method we can find that solutions to the conditional extremum problem (36) that determine the least favorable density matrix $F^0(\lambda) \in D_0$ of the maximal rank is of the form

$$F^0(\lambda) = \vec{\beta} \left[\int_0^{\infty} (Ad)(t) e^{it\lambda} dt \right] \cdot \left[\int_0^{\infty} (Ad)(t) e^{it\lambda} dt \right]^* \vec{\beta}^*. \tag{51}$$

The unknown $\vec{\beta} = (\beta_1, \ldots, \beta_n)^\top$ and $\{d(t),\, t \geq 0\}$ are determined by the canonical factorization (39) of the density matrix $F^0(\lambda)$ and the condition $F^0(\lambda) \in D_0$.
For solutions $d = d(t)$, $t \geq 0$ to the system of equations

$$(\mathbf{A}d)(t) = \vec{c}\, d^*(t), \quad t \geq 0, \quad \vec{c} = (c_1, \ldots, c_n), \tag{52}$$

such that

$$\frac{1}{2\pi} \int_0^{\infty} d(t) d^*(t)\, dt = P, \tag{53}$$

the following equality holds true

$$\begin{aligned} F(\lambda) &= \left[\int_0^{\infty} d(t) e^{-it\lambda} dt \right] \left[\int_0^{\infty} d(t) e^{-it\lambda} dt \right]^* \\ &= \vec{\beta} \left[\int_0^{\infty} (\mathbf{A}d)(t) e^{it\lambda} dt \right] \left[\int_0^{\infty} (\mathbf{A}d)(t) e^{it\lambda} dt \right]^* \vec{\beta}^*. \end{aligned} \tag{54}$$

Denote by ν_0^P the maximal value of $\|\mathbf{A}d\|^2$, where $d = \{d(t),\, t \geq 0\}$ are solutions to equation (52) that satisfy condition (53) and determine the canonical factorization (39) of the density matrix of the maximal rank $F(\lambda)$, $F(\lambda) \in D_0$. Denote by μ_0^P the maximal value of $\|\mathbf{A}d\|^2$, where $d = \{d(t),\, t \geq 0\}$ satisfy condition (53) and determine the canonical factorization (39) of the density matrix $F^0(\lambda) \in D_0$. If there exists a solution $d^0 = \{d^0(t),\, t \geq 0\}$ to the equation (52) that satisfy condition (53) and such that $\nu_0^P = \mu_0^P = \|\mathbf{A}d^0\|^2$, then the least favorable in D_0 is the density matrix of the maximal rank (39). The stationary process $\vec{\xi}(t)$ in this case admits the moving average representation (28). The minimax (robust) spectral characteristics of the optimal linear extrapolationof the functional $A\vec{\xi}$

is calculated by formula (31) since the functions $A(\lambda)$ and $r(\lambda)$ are bounded and $h(F^0) \in H_D$.

The following theorem holds true.

Theorem 9.1. *If there exists a solution $d^0 = \{d^0(t), t \geq 0\}$ to equation (52) that satisfy condition (53) and such that $\nu_0^P = \mu_0^P = \|Ad^0\|^2$, then the least favorable in D_0 for the optimal linear extrapolation of the functional $A\vec{\xi}$ is the density matrix of the maximal rank (39). If $\nu_0^P < \mu_0^P$, then the least favorable in D_0 density matrix is determined by conditions (39), (40). The corresponding stationary process $\vec{\xi}(t)$ in this case admits the moving average representation (28). The minimax spectral characteristics $h(F)$ of the optimal linear extrapolation is calculated by formula (31).*

For the functional $A_L\vec{\xi}$ the density matrix (51) is of the form

$$F^0(\lambda) = \vec{\beta}\left[\int_0^L (A_Ld)(t)e^{it\lambda}dt\right]\left[\int_0^L (A_Ld)(t)e^{it\lambda}dt\right]^* \vec{\beta}^*. \tag{55}$$

In this case the equality holds true

$$\begin{aligned} r_L(\lambda)r_L^*(\lambda) &= \left[\int_0^L (A_Ld)(t)e^{it\lambda}dt\right]\left[\int_0^L (A_Ld)(t)e^{it\lambda}dt\right]^* = \\ &= \left[\int_0^L (\tilde{A}_Ld)(t)e^{it\lambda}dt\right]\left[\int_0^L (\tilde{A}_Ld)(t)e^{it\lambda}dt\right]^*, \end{aligned}$$

where

$$(\tilde{A}_Ld)(t) = \int_0^t \vec{a}(L-t+u)d(u)du.$$

For these reasons for all solutions $d = \{d(t), 0 \leq t \leq L\}$ to equations

$$(A_Ld)(t) = \vec{c}d^*(t), \quad 0 \leq t \leq L, \quad \vec{c} = (c_1, \ldots, c_T), \tag{56}$$

$$(\tilde{A}_Ld)(t) = \vec{b}d^*(t), \quad 0 \leq t \leq L, \quad \vec{b} = (b_1, \ldots, b_T), \tag{57}$$

such that

$$\int_0^L d(t)d^*(t)dt = P, \tag{58}$$

the equality holds true

$$F(\lambda) = \left[\int_0^L d(t)e^{-it\lambda}dt\right]\left[\int_0^L d(t)e^{-it\lambda}dt\right]^* = \vec{\beta}\, r_L(\lambda)r_L^*(\lambda)\vec{\beta}^*.$$

Denote by ν_0^{LP} the maximal value of $\|A_Ld\|^2 = \left\|\tilde{A}_Ld\right\|^2$, where $d = \{d(t), 0 \leq t \leq L\}$ are solutions to equations (56), (57) that satisfy condition (58) and determine the canonical factorization (29) of the density matrix $F^0(\lambda)$.

Denote by μ_0^{LP} the maximal value of $\|A_L d\|^2$, where $d = \{d(t),\, 0 \le t \le L\}$ satisfy condition (58) and determine the canonical factorization (29) of the density matrix $F^0(\lambda)$ with $F^0(\lambda)$ of the form (55). If there exists a solution $d^0 = \{d^0(t),\, 0 \le t \le L\}$ to equation (56), or equation (57), that satisfy condition (58) and such that $\nu_0^{LP} = \mu_0^{LP} = \|A_L d^0\|^2$, then the least favorable in D_0 is the density matrix

$$F^0(\lambda) = \left[\int_0^L d^0(t)e^{-it\lambda}dt\right]\left[\int_0^L d^0(t)e^{-it\lambda}dt\right]^*. \tag{59}$$

The following theorem holds true.

Theorem 9.2. *If there exists a function $d^0 = \{d^0(t),\, 0 \le t \le L\}$, that satisfy condition (58) and such that $\nu_0^{LP} = \mu_0^{LP} = \|A_L d^0\|^2$, then the least favorable in D_0 for the optimal linear estimation of the functional $A_L\vec{\xi}$ is the density matrix (59). The corresponding stationary process $\vec{\xi}(t)$ in this case admits the moving average representation (44). If $\nu_0^{LP} < \mu_0^{LP}$, then the density matrix (55), that admits the canonical factorization (42), is the least favorable in D_0. The vector $\vec{\beta}$ and the function $d^0 = \{d^0(t),\, 0 \le t \le L\}$ are determined by conditions (43), (58). The minimax spectral characteristics $h(F)$ of the optimal linear estimate of the functional $A_L\vec{\xi}$ is calculated by formula (33).*

Example 4. Consider the problem for the functional

$$A_1\vec{\xi} = \int_0^1 \vec{a}(t)\vec{\xi}(t)dt,$$

where $\vec{\xi}$ is a two-dimensional stochastic process. The least favorable in D_0 for the optimal linear estimation of the functional $A_1\vec{\xi}$ is the density matrix

$$F(\lambda) = \left[\int_0^1 d(t)e^{-it\lambda}dt\right]\cdot\left[\int_0^1 d(t)e^{-it\lambda}dt\right]^*,$$

where the matrix function $d = \{d_{kj}(t),\, 0 \le t \le 1; k, j = 1, 2\}$ is a solution to the conditional extremum problem

$$\int_0^1\int_0^1\int_0^{\min(x,y)} \vec{a}(y)d(y-u)d^*(x-u)\vec{a}^*(x)dydx \to \max,$$

$$\int_0^1 d(t)d^*(t)dt = P.$$

The corresponding process $\vec{\xi}(t)$ in this case admits the canonical moving average representation of the form

$$\vec{\xi}(t) = \int_{t-1}^{t} d(t-u)d\vec{\varepsilon}(u).$$

For more results on minimax-robust extrapolation of multidimensional stationary processes in the case of observations without noise see article [16]. For the corresponding results for multidimensional stationary sequences see article [15].

10. Conclusions

We propose formulas for calculation the mean square errors and the spectral characteristic of the optimal linear estimate of the unknown value of the functional $A_L\vec{\xi} = \int_0^L \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t \in R\backslash[0, L]$, where $\vec{\eta}(t)$ is uncorrelated with $\vec{\xi}(t)$ multidimensional stationary process, and formulas for calculation the mean square error and the spectral characteristic of the optimal linear extrapolation of the unknown value of the functional $A\vec{\xi} = \int_0^\infty \vec{a}(t)\vec{\xi}(t)dt$ which depends on the unknown values of a multidimensional stationary stochastic process $\vec{\xi}(t)$ based on observations of the process $\vec{\xi}(t) + \vec{\eta}(t)$ for $t < 0$ under the condition that spectral density matrices $F(\lambda)$ and $G(\lambda)$ of the signal process $\vec{\xi}(t)$ and the noise process $\vec{\eta}(t)$ are known exactly. Formulas are proposed that determine the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functionals for some classes $D = D_F \times D_G$ of spectral density matrices under the condition that spectral density matrices $F(\lambda)$, $G(\lambda)$ are not known, but classes $D = D_F \times D_G$ of admissible spectral matrices are given.

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