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VERTICAL AND HORIZONTAL FLUID QUEUES IN HEAVY AND LOW TRAFFIC

The paper considers vertical and horizontal fluid queueing systems with consecutive service. The workload processes in these systems satisfy the Langevin equations with Poisson input. The objective is to investigate the main stationary characteristics in heavy and low traffic.

Introduction

Investigating the fluid queues became popular during the last decade. Contrary to ordinary queueing system, the input flow to a fluid queue has continuous structure. A fluid queue is the input-output system, whose input flow consists of substance flowing into a reservoir and flowing out from it with random speed. Fluid models play an important role in analyzing the operating characteristics of high-speed networks and repetition work systems when a huge amount of small tasks are processed. They are actively used in the sphere of telecommunications. Studying the fluid queues with priorities is motivated by their usefulness in analyzing the effectiveness of ATM-commutators and IP-routers which support classes of traffic with different qualities of service [6]. Involving the fluid queues with priorities is effective for checking the overload in modern high-speed integrated networks, such as Internet. Fluid queues are also used in the dam theory and in transport systems for simulating the flow of transport facilities on crossroads.

1. Statement of the problem

The matters of investigation are two fluid systems which will be referred to as vertical and horizontal. The choice of these names is explained by the peculiarities of their functioning.

Vertical system is a fluid system with consecutive service consisting of n servers. The input flow to the first server is given by a generalized Poisson process $z_1(t)$ with parameter λ and jumps $\eta_1^1 = \eta_1,\ \eta_1^2, \dots, \eta_1^i, \dots$. The output from any server constitutes the input flow to the next server. The service speed on the k-th server is proportional to the value of incomplete work on this server. Any demand entering the system has to pass through all servers. To be served, the i-th demand needs η_1^i units of work. If $x_k(t)$, $k = \overline{1,\ n}$, denotes the value of incomplete work on the k-th server at the moment t, then it serves with the speed $\mu_k x_k(t)$, $k = \overline{1, n}$, and the vector of incomplete work

$$x(t) = (x_1(t),\ x_2(t), \dots, x_n(t))^T \in \mathbb{R}R^n$$

in this system satisfies the Langevin equation

$$dx(t) = Ax(t)\,dt + dz(t), \tag{1}$$

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where $z(t) = (z_1(t),\ 0, \ldots, 0)^T \in \mathbb{R}^n$,

$$A = \begin{Vmatrix} -\mu_1 & 0 & 0 & \ldots & 0 & 0 \\ \mu_1 & -\mu_2 & 0 & \ldots & 0 & 0 \\ 0 & \mu_2 & -\mu_3 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \mu_{n-1} & -\mu_n \end{Vmatrix},$$

$\mu_k > 0,\ k = \overline{1, n}$.

Horizontal systems only differ from vertical in the matrices A. For horizontal systems, processes $x_i(t)$ satisfy the system of differential equations

$$dx_1(t) = -\mu_1 (x_1(t) - x_2(t))\, dt + dz_1(t),$$
$$dx_k(t) = -\mu_k (x_k(t) - x_{k+1}(t))\, dt + \mu_{k-1} (x_{k-1}(t) - x_k(t))\, dt,$$
$$k = 2, \ldots, n-1,$$
$$dx_n(t) = -\mu_n x_n(t)\, dt + \mu_{n-1} (x_{n-1}(t) - x_n(t))\, dt,$$

where $\mu_k > 0, \quad k = 1, 2, \ldots, n$.

Hence, the vector $x(t)$ of incomplete work in this system satisfies the Langevin equation (1) with the same process $z(t)$, but with the matrix

$$A = \begin{Vmatrix} -\mu_1 & \mu_1 & \ldots & 0 & 0 & 0 \\ \mu_1 & -(\mu_1 + \mu_2) & \ldots & 0 & 0 & 0 \\ 0 & \mu_2 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \mu_{n-2} & -(\mu_{n-2} + \mu_{n-1}) & \mu_{n-1} \\ 0 & 0 & \ldots & 0 & \mu_{n-1} & -(\mu_{n-1} + \mu_n) \end{Vmatrix},$$

where $\mu_k > 0,\ k = \overline{1, n}$.

We establish conditions for the stationary regimes of these systems to exist and investigate the behavior of their stationary characteristics in heavy ($\lambda \to \infty$) and low ($\lambda \to 0$) traffic.

2. Conditions of existence of stationary regimes

It follows from [1] that the process $x(t)$ satisfying Eq. (1) possesses a limit distribution as $t \to \infty$ which does not depend on the initial value $x_0 = x(0)$ if and only if
a) all eigenvalues of A have negative real parts,
b) $\mathbb{E}(\ln \eta_1;\ \eta_1 > 1) < \infty$.
If these conditions are satisfied, then the limit distribution is the unique stationary distribution of $x(t)$ with the characteristic function

$$\Xi_\xi(s) = \exp\{-\lambda \int_0^\infty (1 - \varphi(\exp\{uA^T\}s))du\}, \tag{3}$$

where $\varphi(s) = \mathbb{E}\{\exp\ i(s,\ \eta)\}$, $s = (s_1, s_2, \ldots, s_n)$, $\eta = (\eta_1, 0, \ldots, 0)$.

Note that, for a vertical system, condition a) is equivalent to

$$\mu_1 > 0,\ \mu_2 > 0, \ldots,\ \mu_n > 0.$$

3. Behavior of the stationary characteristics of a vertical system in heavy traffic

Let the stationary distribution of the process

$$x(t) = (x_1(t),\ x_2(t), \dots, x_n(t))^T$$

coincide with the distribution of the vector $\xi = (\xi_1,\ \xi_2, \dots,\ \xi_n)^T$. Let also

$$F(y_1, y_2, \dots, y_n)$$

be the distribution function of $\eta = (\eta_1, 0, \dots,\ 0)^T$, $\mathbb{E}\eta = (\mathbb{E}\eta_1,\ 0, \dots, 0)^T$, $\mathbb{E}\eta_1 = m_1$.

Lemma 1. *If $\mathbb{E}\eta_1 = m_1 < \infty$, then $\lambda^{-1}(\xi_1,\ \xi_2, \dots,\ \xi_n)^T$ converges in probability to $m_1\left(\mu_1^{-1},\ \mu_2^{-1}, \dots, \mu_n^{-1}\right)^T$ as $\lambda \to \infty$.*

D. ue to condition a),

$$\int_0^\infty \exp\left\{uA^T\right\}du = -\left(A^T\right)^{-1},$$

and (3) implies

$$\lim_{\lambda\to\infty} \Xi_{\lambda^{-1}\xi}(s) = \exp\left\{i\left(s_1 m_1\mu_1^{-1} + s_2 m_1 \mu_2^{-1} + \dots + s_n m_1 \mu_n^{-1}\right)\right\}.$$

The following lemma is obvious.

Lemma 2. *If $\mathbb{E}\eta_1 = m_1 < \infty$, then $\lambda^{-1}(z_1(t),\ 0, \dots, 0)^T$ converges in probability to $(m_1 t,\ 0, \dots, 0)^T$ as $\lambda \to \infty$.*

Assume that a demand entering the system at the moment t_0 needs y units of work to be served and that

$$x(t_0) = (x_1(t_0), x_2(t_0), \dots, x_n(t_0)) = (x_1^* + y,\ x_2^*, \dots, x_n^*).$$

Denote, by T, the time to complete servicing this demand by all servers and, by W_i, the time that passes till the moment when this demand enters the i-th server ($i = \overline{1,n}$). Introduce the process $\beta_i(t)$ which is equal to the amount of work executed by the i-th server at the moment $t \geq t_0$ (without loss of generality, put $t_0 = 0$). Then $d\beta_i(t) = \mu_i x_i(t)\,dt$, $\beta_i(0) = 0$. It follows from the definition of $\beta_i(t)$, that

$$\{T > t\} = \{x_1^* + x_2^* + \dots + x_n^* + y > \beta_n(t)\},$$
$$\{W_i > t\} = \{x_1^* + x_2^* + \dots + x_i^* > \beta_i(t)\}.$$

Rewriting system (1) in the form

$$\begin{cases} dx_1(t) &= dz_1(t) - \mu_1 x_1(t)\,dt, \\ dx_2(t) &= \mu_1 x_1(t)\,dt - \mu_2 x_2(t)\,dt, \\ \dots & \\ dx_n(t) &= \mu_{n-1}x_{n-1}(t)\,dt - \mu_n x_n(t)\,dt \end{cases}$$

and adding all equations of this system, we obtain

$$\mu_n x_n(t)\,dt = dz_1(t) - dx_1(t) - dx_2(t) - \dots - dx_n(t),$$

$$d\beta_n(t) = dz_1(t) - dx_1(t) - dx_2(t) - \dots - dx_n(t),$$

$$\int_0^t d\beta_n(u) = \int_0^t dz_1(u) - \int_0^t dx_1(u) - \int_0^t dx_2(u) - \dots - \int_0^t dx_n(u),$$

$$\begin{aligned}\beta_n(t) &= z_1(t) - x_1(t) + x_1(0) - x_2(t) + x_2(0) - \dots - x_n(t) + x_n(0) \\ &= z_1(t) - x_1(t) - x_2(t) - \dots - x_n(t) + x_1^* + x_2^* + \dots + x_n^* + y.\end{aligned}$$

Hence,

$$\mathrm{P}\{T>t\}=\mathrm{P}\{x_1(t)+x_2(t)+\cdots+x_n(t)>z_1(t)\}. \tag{4}$$

In a similar manner, we get

$$\mathrm{P}\{W_i>t\}=\mathrm{P}\{x_1(t)+x_2(t)+\cdots+x_i(t)>z_1(t)+y\}. \tag{5}$$

If the system operates in a stationary regime, we use notations T^s and W_i^s for the analogs of the above-mentioned characteristics. Their distributions are given by formulas (4) and (5), respectively, provided that the distribution of $x(t)=(x_1(t),\ x_2(t),\ldots,x_n(t))^T$ is stationary and coincides with that of $\xi=(\xi_1,\ \xi_2,\ldots,\ \xi_n)^T$ [2].

Theorem 1. *If* $\mathbb{E}\eta_1=m_1<\infty$*, then*

$$\left(\mu_1^{-1}+\mu_2^{-1}+\cdots+\mu_n^{-1}\right)^{-1}T^s \overset{\text{w.}}{\Rightarrow} 1$$

as $\lambda\to\infty$*.*

E. quality (4) implies

$$\mathrm{P}\{T^s>t\}=\mathrm{P}\{\xi_1+\xi_2+\cdots+\xi_n>z_1(t)\}=\mathrm{P}\left\{\lambda^{-1}(\xi_1+\xi_2+\cdots+\xi_n)>\lambda^{-1}z_1(t)\right\}.$$

Lemmas 1 and 2 give

$$\mathrm{P}\{T^s>t\}\overset{\text{w.}}{\Rightarrow}\mathrm{P}\left\{m_1\mu_1^{-1}+m_1\mu_2^{-1}+\cdots+m_1\mu_n^{-1}>m_1t\right\}$$

$$=\mathrm{P}\left\{\mu_1^{-1}+\mu_2^{-1}+\cdots+\mu_n^{-1}>t\right\}=\begin{cases}1, & t<t^1\\ 0, & t\geq t^1\end{cases}$$

as $\lambda\to\infty$, where $t^1=\mu_1^{-1}+\mu_2^{-1}+\cdots+\mu_n^{-1}$. This suffices to prove the theorem.

Theorem 2. *If* $\mathbb{E}\eta_1=m_1<\infty$*, then*

$$\left(\mu_1^{-1}+\mu_2^{-1}+\cdots+\mu_i^{-1}\right)^{-1}W_i^s \overset{\text{w.}}{\Rightarrow} 1$$

as $\lambda\to\infty$*.*

Theorems 1 and 1 allow us to investigate the following characteristics:
time T_i spent by a demand in the system till completing the servicing at the i-th server,
time T_i' spent by a demand in the system since it enters the i-th server till the moment of its output from the system,
time V_i spent by a demand at the i-th server,
time K_i spent by a demand in the queue to the i-th server,
time S_i of servicing a demand by the i-th server provided that the system operates in the stationary regime.

The following results are given in [3].

Theorem 3. *If* $\mathbb{E}\eta_1=m_1<\infty$*, then*

$$\left(\mu_i^{-1}+\cdots+\mu_n^{-1}\right)^{-1}T_i' \overset{\text{w.}}{\Rightarrow} 1$$

as $\lambda\to\infty$*.*

Theorem 4. *If* $\mathbb{E}\eta_1=m_1<\infty$*, then*

$$\mu_iK_i \overset{\text{w.}}{\Rightarrow} 1$$

as $\lambda\to\infty$*.*

Theorem 5. *If* $\mathbb{E}\eta_1 = m_1 < \infty$, *then*

$$\mu_i V_i \overset{\text{w.}}{\Rightarrow} 1$$

as $\lambda \to \infty$.

Denote $\exp\{tA\} = \|a_{ij}(t)\|_{i,j=1}^n$. The following results indicate the speed of convergence of T^s and W_n^s to $\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1}$.

Depending on the values of $\mu_k,\ k = \overline{1,n}$, consider the following cases:

1. $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$. In this case $a_{ni}\,(t) = \frac{1}{(n-i)!}\mu^{n-i}t^{n-i}\exp\{-\mu t\}$, $i = 1,\ldots,n$, $a_{i1}\,(t) = \frac{1}{(i-1)!}\mu^{i-1}t^{i-1}\exp\{-\mu t\}$, $i = 1,\ldots,n$.

2. All $\mu_k,\ k = \overline{1,n}$, are different. In this case $a_{i\,j+1}\,(t) = 0$, $i = 1,\ldots,n-1$, $j = i,\ldots n-1$, $a_{i\,i}\,(t) = \exp\{-\mu_i t\}$, $i = 1,\ldots,n$, $a_{i\,1}\,(t) = \sum_{k=1}^{i} c_{k1}^i \exp\{-\mu_k t\}$, $i = 2,\ldots,n$,

$$a_{nj}\,(t) = \sum_{k=j}^{n} c_{kj}^n \exp\{-\mu_k t\}, \qquad j = 1,\ldots,n-1$$

(here, c_{kj}^i is the coefficient of $\exp\{-\mu_k t\}$ in the i-th row and the j-th column of the matrix $\exp\{tA\}$).

3. The characteristic polynomial of A has the form

$$\chi\,(\gamma) = (\gamma + \mu_1)^{r_1}\,(\gamma + \mu_2)^{r_2}\ldots(\gamma + \mu_s)^{r_s},$$

$\mu_i \neq \mu_j$, $i \neq j$, $r_1 + r_2 + \cdots + r_s = n$ (we do not indicate the elements of $\exp\{tA\}$, since we do not consider this case for brevity).

In case 1, we have the following results.

Theorem 6. *Let* $F\,(x)$ *belong to the domain of attraction of a stable law with exponent* α, $1 < \alpha \leq 2$. *Then there exists a function* $f\,(\lambda) > 0$ *regularly varying with exponent* $\frac{1}{\alpha} - 1$ *such that the distributions of the variables* $\left(T^s - \frac{n}{\mu}\right)\frac{1}{f(\lambda)}$ *and* $\left(W_n^s - \frac{n}{\mu}\right)\frac{1}{f(\lambda)}$ *weakly converge as* $\lambda \to \infty$ *to the distribution of the sum of independent random variables* w_1 *and* w_2 *with the characteristic functions*

$$\chi_{w_1} = \exp\left\{\int_0^{\infty} \ln\chi\left(s_1 \sum_{k=1}^{n}\sum_{j=1}^{n-k+1} \frac{\left(n\mu^{-1}\right)^{n-k+1-j}\mu^{n-j}}{(n-k+1-j)!\,(k-1)!}u^{k-1}\exp\{-(n+\mu u)\}\right)du\right\}$$

and

$$\chi_{w_2} = \exp\left\{\int_0^{\frac{n}{\mu}} \ln\chi\left(s_1 \sum_{k=1}^{n} \frac{\mu^{n-k}u^{n-k}}{(n-k)!}\exp\{-\mu u\} - 1\right)du\right\},$$

where

$$\chi\,(s_1) = \exp\left\{-\left|s_1\right|^{\alpha}\left(1 - i\frac{s_1}{|s_1|}\mathrm{tg}\,\frac{\pi\alpha}{2}\right)\right\}.$$

This distribution is stable with exponent α.

The following theorem considers the case $m_1 = \infty$.

Theorem 7. *Let* $F\,(x)$ *belong to the domain of attraction of a stable law with exponent* α, $0 < \alpha < 1$. *Then*

$$\lim_{\lambda\to\infty} P\{T^s < t\} = \lim_{\lambda\to\infty} P\{W_n^s < t\} = P\{v_1 + v_2 < 0\},$$

where the random variables v_1 *and* v_2 *are independent and have Laplace transforms* $\exp\left\{-s_1^\alpha \int_0^\infty q_1^\alpha(u)\,du\right\}$ *and* $\exp\left\{-s_1^\alpha \int_0^\infty q_2^\alpha(u)\,du\right\}$*, respectively,*

$$q_1(u) = \sum_{k=1}^{n} \sum_{j=1}^{n-k+1} \frac{t^{n-k+1-j}\mu^{n-j}}{(n-k+1-j)!\,(k-1)!} u^{k-1} \exp\{-\mu(t-u)\},$$

$$q_2(u) = \sum_{k=1}^{n} \frac{\mu^{n-k}u^{n-k}}{(n-k)!} \exp\{-\mu u\} - 1.$$

Further, we denote $\frac{1}{\mu_1} + \cdots + \frac{1}{\mu_n} = c$.

In case 2, we have the following results.

Theorem 8. *Let* $F(x)$ *belong to the domain of attraction of a stable law with exponent* α, $1 < \alpha \le 2$. *Then there exists a function* $f(\lambda) > 0$ *regularly varying with exponent* $\frac{1}{\alpha} - 1$ *such that the distributions of the variables* $(T^s - c)\frac{1}{f(\lambda)}$ *and* $(W_n^s - c)\frac{1}{f(\lambda)}$ *weakly converge as* $\lambda \to \infty$ *to the distribution of the sum*

$$\sum_{k=1}^{n} \frac{c_{k1}^n}{\mu_k} \exp\{-\mu_k c\}\, v_1 + \sum_{k=2}^{n} \frac{c_{k2}^n}{\mu_k} \exp\{-\mu_k c\} \sum_{j=1}^{2} c_{j1}^2 v_j + \dots$$

$$+ \sum_{k=n-1}^{n} \frac{c_{kn-1}^n}{\mu_k} \exp\{-\mu_k c\} \sum_{j=1}^{n-1} c_{j1}^{n-1} v_j + \frac{1}{\mu_n} \exp\{-\mu_k c\} \sum_{j=1}^{n} c_{j1}^n v_j + \sum_{k=1}^{n} \frac{c_{k1}^n}{\mu_k} w_k.$$

This distribution is stable with exponent α. *The sum consists of summands involving the independent random variables* v_j, w_j, $j = 1, \dots, n$ *with the characteristic functions*

$$\chi^{\frac{1}{\alpha\mu_j}}(s_1), \exp\left\{|s_1|^\alpha \left(1 + i\frac{s_1}{|s_1|}\mathrm{tg}\,\frac{\pi\alpha}{2}\right) \int_0^c (\exp\{-\mu_j u\} - 1)^\alpha\, du\right\}, j = 1, \dots, n,$$

respectively.

Theorem 9. *Let* $F(x)$ *belong to the domain of attraction of a stable law with exponent* α, $0 < \alpha < 1$, *then* $\lim_{\lambda\to\infty} P\{T^s < t\} = \lim_{\lambda\to\infty} P\{W_n^s < t\} =$

$$P\left\{\sum_{k=1}^{n} \frac{c_{k1}^n}{\mu_k} \exp\{-\mu_k t\}\, w_1^* + \sum_{k=2}^{n} \frac{c_{k2}^n}{\mu_k} \exp\{-\mu_k t\} \sum_{j=1}^{2} c_{j1}^2 w_j^* + \dots + \right.$$

$$\left. \sum_{k=n-1}^{n} \frac{c_{kn-1}^n}{\mu_k} \exp\{-\mu_k t\} \sum_{j=1}^{n-1} c_{j1}^{n-1} w_j^* + \frac{1}{\mu_n} \exp\{-\mu_k t\} \sum_{j=1}^{n} c_{j1}^n w_j^* + \sum_{k=1}^{n} \frac{c_{k1}^n}{\mu_k} v_k^* < 0 \right\},$$

where random variables w_j^*, v_j^*, $j = 1, \dots, n$, *are independent and have the Laplace transforms* $\exp\left\{-\frac{s_1^\alpha}{\mu_j \alpha}\right\}$, $\exp\left\{-s_1^\alpha \int_0^t (1 - \exp\{-\mu_j u\})^\alpha du\right\}$, $j = 1, \dots, n$, *respectively.*

4. Behavior of the stationary characteristics of a horizontal system in heavy traffic

Assume that a demand entering a horizontal system at the moment t_0 needs y units of work to be served. Let S^s denote the time to complete servicing this demand by all servers and let V_n^s be the time that passes till the moment when this demand enters the n-th server provided that the system operates in the stationary regime.

As earlier, $F(y_1, y_2, \dots, y_n)$ is the distribution function of $\eta = (\eta_1, 0, \dots, 0)^T$,

$$\mathbb{E}\eta = (\mathbb{E}\eta_1, 0, \dots, 0)^T,$$

$\mathbb{E}\eta_1 = m_1$, $\xi = (\xi_1,\ \xi_2, \ldots,\ \xi_n)^T$ is a vector, whose distribution is stationary for $x(t)$.

The main results can be formulated as the following statements [5].

Lemma 3. *If* $\mathbb{E}\eta_1 = m_1 < \infty$, *then* $\lambda^{-1}(\xi_1,\ \xi_2, \ldots,\ \xi_n)^T$ *converges in probability to* $m_1 \left(\sum_{i=1}^n \mu_i^{-1},\ \sum_{i=2}^n \mu_i^{-1}, \ldots, \mu_n^{-1}\right)^T$ *as* $\lambda \to \infty$.

Theorem 10. *If* $\mathbb{E}\eta_1 = m_1 < \infty$, *then*

$$\left(\mu_1^{-1} + 2\mu_2^{-1} + \cdots + n\mu_n^{-1}\right)^{-1} S^s \stackrel{w.}{\Rightarrow} 1$$

as $\lambda \to \infty$.

Theorem 11. *If* $\mathbb{E}\eta_1 = m_1 < \infty$, *then*

$$\left(\mu_1^{-1} + 2\mu_2^{-1} + \cdots + n\mu_n^{-1}\right)^{-1} V_n^s \stackrel{w.}{\Rightarrow} 1$$

as $\lambda \to \infty$.

Investigating the speed of convergence of S^s and V_n^s to $\mu_1^{-1} + 2\mu_2^{-1} + \cdots + n\mu_n^{-1}$ turned out to be a troublesome problem. The proofs of limiting theorems contain cumbersome expressions even in the case of two servers with $\mu_1 = \mu_2 = \mu$, and the theorems themselves have unattractive forms. At the same time, changing the processes $x(t)$ and $z(t)$ by some linear transformations $\tilde{x}(t)$ and $\tilde{z}(t)$ allows us to establish the limit theorems in the n-dimensional case.

Consider the case where the matrix A is similar to a diagonal matrix D, i.e.

$$A = TDT^{-1},$$

where $D = \|\delta_{ij}\lambda_i\|_{i,j=1}^n$, λ_i, $i = 1, \ldots, n$ are the eigenvalues of A (real and negative) and T is a non-singular matrix, $T = \|\, t_{ij}\|_{i,j=1}^n$, $T^{-1} = \left\|\, t'_{ij}\right\|_{i,j=1}^n$.

If $\tilde{x}(t) = (\tilde{x}_1(t),\ \tilde{x}_2(t), \ldots, \tilde{x}_n(t))^T = T^{-1}x(t)$, then

$$d\tilde{x}(t) = D\tilde{x}(t)\,dt + d\tilde{z}(t),$$

where

$$\tilde{z}(t) = (\tilde{z}_1(t),\ \tilde{z}_2(t), \ldots, \tilde{z}_n(t))^T = T^{-1}z(t) = (t'_{11}z_1(t),\ t'_{21}z_1(t), \ldots, t'_{n1}z_1(t))^T.$$

Lemma 4. *If* $\mathbb{E}\eta_1 = m_1 < \infty$, *then* $\lambda^{-1}\tilde{\xi} = \lambda^{-1}T^{-1}\xi = \lambda^{-1}\left(\tilde{\xi}_1,\ \tilde{\xi}_2, \ldots,\ \tilde{\xi}_n\right)^T$ *converges in probability to* $m_1\left(\sum_{j=1}^n t'_{1j}\sum_{i=j}^n \mu_i^{-1},\ \sum_{j=1}^n t'_{2j}\sum_{i=j}^n \mu_i^{-1}, \ldots, \sum_{j=1}^n t'_{nj}\sum_{i=j}^n \mu_i^{-1}\right)^T$ *as* $\lambda \to \infty$.

Denote $c = \sum_{j=1}^n t'_{j1}\lambda_j^{-1}\sum_{k=1}^n t_{kj}\left(\mu_n \sum_{j=1}^n t_{nj}t'_{j1}\lambda_j^{-1}\right)^{-1}$.

Theorem 12. *Let* $F(x)$ *belong to the domain of attraction of a stable law with exponent* α, $1 < \alpha \le 2$. *Then there exists a function* $f(\lambda) > 0$ *regularly varying with exponent* $\frac{1}{\alpha} - 1$ *such that the distributions of the variables* $(S^s - c)\frac{1}{f(\lambda)}$ *and* $(V_n^s - c)\frac{1}{f(\lambda)}$ *weakly converge as* $\lambda \to \infty$ *to the distribution of the sum*

$$\sum_{j=1}^n t'_{j1}\left(\sum_{k=1}^n t_{kj} - \mu_n t_{nj}\lambda_j^{-1}(\exp\{\lambda_j t\} - 1)\right)v_j + \mu_n \sum_{j=1}^n t_{nj}t'_{j1}\lambda_j^{-1}w_j,$$

This distribution is stable with exponent α. *The sum consists of summands involving the independent random variables* v_j, w_j, $j = 1, \ldots, n$ *with characteristic functions*

$$\chi^{-\frac{1}{\alpha\lambda_j}}(s_1), \exp\left\{|s_1|^\alpha\left(1 + i\frac{s_1}{|s_1|}\mathrm{tg}\,\frac{\pi\alpha}{2}\right)\int_0^c (1 - \exp\{\lambda_j u\})^\alpha\, du\right\}, j = 1, \ldots, n,$$

respectively, where $\chi(s_1) = \exp\left\{-|s_1|^\alpha \left(1 - i\frac{s_1}{|s_1|}\mathrm{tg}\,\frac{\pi\alpha}{2}\right)\right\}$.

Theorem 13. *If* $F(x)$ *belongs to the domain of attraction of a stable law with exponent* $\alpha,\quad 0<\alpha<1$, *then*

$$\lim_{\lambda\to\infty} P\{S^s < t\} = \lim_{\lambda\to\infty} P\{V_n^s < t\}$$

$$= P\left\{\sum_{j=1}^n t'_{j1}\left(\sum_{k=1}^n t_{kj} - \mu_n t_{nj}\lambda_j^{-1}(\exp\{\lambda_j t\}-1)\right)v_j^* + \mu_n\sum_{j=1}^n t_{nj}t'_{j1}\lambda_j^{-1}w_j^* < 0\right\},$$

where the random variables v_j^*, w_j^*, $j=1,\dots,n$, *are independent and have the Laplace transforms* $\exp\left\{\frac{s_1^\alpha}{\lambda_j\alpha}\right\}$, $\exp\left\{-s_1^\alpha\int_0^t(1-\exp\{\lambda_j u\})^\alpha du\right\}$, $j=1,\dots,n$, *respectively.*

5. Limit theorems for the solution to the Langevin equation in low traffic

Consider Eq. (1) in a wider case where a matrix A has the form $A = UJU^{-1}$, J is a Jordan matrix, $U = \|u_{ij}\|_{i,\,j=1}^n$ is a non-singular matrix, and

$$z(t) = (z_1(t),\ z_2(t),\dots,z_n(t))^T \in \mathbb{R}^n$$

is a generalized Poisson process with parameter λ and jumps $\eta^1,\ \eta^2,\dots,\eta^j,\dots$. In this section, we investigate the limit behavior, as $\lambda\to 0$, of $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_n)^T = U^{-1}x(\cdot,\lambda)$ provided that $x(\cdot,\lambda)$ is in a stationary regime.

Denote $\tilde{\eta}^j = (\tilde{\eta}_1^j,\dots,\tilde{\eta}_n^j)^T = U^{-1}\eta^j$, $j=1,2,\dots$, $p_r = \mathrm{P}\left\{\tilde{\eta}_r^j = 0\right\}$, $p_r^+ = \mathrm{P}\{\tilde{\eta}_r^j > 0\}$, $\operatorname{sgn} z = (\operatorname{sgn} z_1,\dots,\operatorname{sgn} z_n)^T$, if $z = (z_1,\dots,\ z_n)^T\in\mathbb{R}^n$, $J = \{J_1,\dots,J_m\}$, where J_i is a Jordan cell of dimension k_i related to the eigenvalue λ_i, $i = \overline{1,m}$, of A (some λ_i can coincide), $\sum_{i=1}^m k_i = l_m$, $m = \overline{1,n}$, $l_n = n$.

Since the components of $\tilde{x} = (\tilde{x}_1,\dots,\tilde{x}_n)^T$ are determined by Jordan cells, we restrict ourselves by the description of the part of $\tilde{x}$ which corresponds to J_i. Denote it by $(\tilde{x}_{l_{i-1}+1},\tilde{x}_{l_{i-1}+2},\dots,\tilde{x}_{l_i})^T$. Let also $(\tilde{\eta}_{l_{i-1}+1}^j,\tilde{\eta}_{l_{i-1}+2}^j,\dots,\tilde{\eta}_{l_i}^j)^T$, $j=1,\ 2,\dots$, be the part of $\tilde{\eta}^j$ which corresponds to J_i, $A_1 = \{\tilde{\eta}_{l_i}^1 \neq 0\}$, $B_1 = \{\tilde{\eta}_{l_i-1}^1 \neq 0\}$, $\mathrm{P}\{A_1\} = p$, $\mathrm{P}\{B_1\} = q$. (All processes and random variables are supposed to be determined on the same probability space.)

Consider the following cases:

I. $\lambda_i < 0$. If this is the case, then $\tilde{x}_{l_{i-1}+1},\dots,\tilde{x}_{l_i}$ are real.

II. $\lambda_i = a_i + ib_i$, $(a_i<0,\ b_i\neq 0)$. If this is the case, then $\tilde{x}_{l_{i-1}+1},\dots,\tilde{x}_{l_i}$ are complex. So, $\tilde{x}_{l_{i-1}+1} = \left|\tilde{x}_{l_{i-1}+1}\right|\exp\left\{i\varphi_{l_{i-1}+1}\right\}$,..., $\tilde{x}_{l_i} = |\tilde{x}_{l_i}|\exp\{i\varphi_{l_i}\}$, where $\varphi_{l_{i-1}+1} = \arg\tilde{x}_{l_{i-1}+1},\ \dots,\varphi_{l_i} = \arg\tilde{x}_{l_i}$, $\varphi_{l_{i-1}+1},\dots,\varphi_{l_i}\in(0\ ,\ 2\pi)$.

Let us introduce the notation $\nu_i = \lambda\lambda_i^{-1}$, if λ_i is real, and $\kappa_i = \lambda a_i^{-1}$, if $\lambda_i = a_i+ib_i$. In the rest of the paper, α stays for a random variable which does not depend on other variables and is uniformly distributed on (0, 1).

In case I, we have the following theorems [4].

Theorem 14. *If* $p_{l_i} = 0$, *then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i},\dots,|\tilde{x}_{l_i}|^{-\nu_i},\ \operatorname{sgn}(\tilde{x}_{l_{i-1}+1},\dots,\tilde{x}_{l_i})\right)$$

weakly converges as $\lambda\to 0$ *to the distribution of* $(\alpha,\dots,\alpha,\ \operatorname{sgn}(\tilde{\eta}_{l_i}^1,\dots,\tilde{\eta}_{l_i}^1))$.

Theorem 15. *If $0 < p_{l_i} < 1$, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i}, \ldots, \left|\tilde{x}_{l_i}\right|^{-\nu_i}, \operatorname{sgn} \tilde{x}_{l_{i-1}+1}, \ldots, \operatorname{sgn}\tilde{x}_{l_i} \right)$$

weakly converges as $\lambda \to 0$ to the distribution of $\left(\alpha^{\frac{1}{p}}, \ldots, \alpha^{\frac{1}{p}}, \gamma, \ldots, \gamma\right)$, where γ takes values 1 and -1 with probabilities $p^+_{l_i}$ and $1 - p^+_{l_i}$, respectively.

Theorem 16. *If $p_{l_i} = 1$, $p_{l_i-1} = 0$, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i}, \ldots, \left|\tilde{x}_{l_i-1}\right|^{-\nu_i}, \left|\tilde{x}_{l_i}\right|, \operatorname{sgn}\tilde{x}_{l_{i-1}+1}, \ldots, \operatorname{sgn}\tilde{x}_{l_i-1}, \operatorname{sgn}\tilde{x}_{l_i}\right)$$

weakly converges as $\lambda \to 0$ to the distribution of $\left(\alpha, \ldots, \alpha, 0, \operatorname{sgn}\tilde{\eta}^1_{l_i-1}, \ldots, \operatorname{sgn}\tilde{\eta}^1_{l_i-1}, 0\right)$.

Theorem 17. *If $p_{l_i} = 1$, $0 < p_{l_i-1} < 1$, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\nu_i}, \ldots, \left|\tilde{x}_{l_i-1}\right|^{-\nu_i}, \left|\tilde{x}_{l_i}\right|, \operatorname{sgn}\tilde{x}_{l_{i-1}+1}, \ldots, \operatorname{sgn}\tilde{x}_{l_i-1}, \operatorname{sgn}\tilde{x}_{l_i}\right)$$

weakly converges as $\lambda \to 0$ to the distribution of $\left(\alpha^{\frac{1}{q}}, \ldots, \alpha^{\frac{1}{q}}, 0, \gamma, \ldots, \gamma, 0\right)$, where γ takes values 1 and -1 with probabilities $p^+_{l_i-1}$ and $1 - p^+_{l_i-1}$, respectively.

In case II, the following theorems are proved [4].

Theorem 18. *If $p_{l_i} = 0$, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\kappa_i}, \ldots, \left|\tilde{x}_{l_i}\right|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \ldots, \varphi_{l_i}\right)$$

weakly converges as $\lambda \to 0$ to the distribution of $(\alpha, \ldots, \alpha, \beta, \ldots, \beta)$, where β is uniformly distributed on $(0, 2\pi)$.

Theorem 19. *If $0 < p_{l_i} < 1$, then the distribution of*

$$\left(\left|\tilde{x}_{l_{i-1}+1}\right|^{-\kappa_i}, \ldots, \left|\tilde{x}_{l_i}\right|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \ldots, \varphi_{l_i}\right)$$

weakly converges as $\lambda \to 0$ to the distribution of $\left(\alpha^{\frac{1}{p}}, \ldots, \alpha^{\frac{1}{p}}, \beta, \ldots, \beta\right)$, where β is uniformly distributed on $(0,\ 2\pi)$.

These results can be applied in a natural way to vertical and horizontal systems. Note that a vertical system only uses Theorems 14–17, since its matrix A possesses only real eigenvalues.

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