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A MULTIDIMENSIONAL MODEL OF THE DIFFUSION PROCESS WITH MEMBRANE WHOSE PROPERTIES ARE DESCRIBED BY A GENERAL WENTZEL BOUNDARY CONDITION

By methods of the general potential theory, a semigroup of operators is constructed that describes a multidimensional diffusion process with membrane located on a given hypersurface, whose properties are determined by a general Wentzel boundary condition.

Let, in the Euclidean space R^n, $n \geq 2$, a hypersurface S be given which subdivides the space into two domains: $\mathcal{D}_1$ and $\mathcal{D}_2$ so that $\overline{\mathcal{D}}_l = \mathcal{D}_l \cup S$, $l = 1, 2$, $R^n = \mathcal{D}_1 \cup \mathcal{D}_2 \cup S$.

In the domains $\mathcal{D}_l$, $l = 1, 2$, noninterrupting diffusion processes are given that are ruled by generating differential operators $\mathcal{L}_l$, $l = 1, 2$:

$$\mathcal{L}_l = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}^{(l)}(x) D_{ij} + \sum_{i=1}^{n} a_i^{(l)}(x) D_i, \quad x \in \mathcal{D}_l, \quad l = 1, 2, \tag{1}$$

where $a_{ij}^{(l)}(x)$ and $a_i^{(l)}(x)$, $i, j = 1, \ldots, n$, are bounded continuous functions on R^n (the numbers $a_{ij}^{(l)}(x)$ form, for every $x \in R^n$, a symmetric nonnegatively determined matrix $A_l(x)$, and the numbers $a_i^{(l)}(x)$, $i = 1, \ldots, n$, determine an n-dimensional vector $a_l(x)$), $D_i \equiv \frac{\partial}{\partial x_i}$, $D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, n$.

A problem is posed to describe a sufficiently general class of continuous and noninterrupting Feller processes in R^n such that their generating differential operator in $\mathcal{D}_l$, $l = 1, 2$, coincides with $\mathcal{L}_l$. This problem is also called a problem on the glueing of two diffusion processes or a problem on the construction of mathematical models of a diffusion process in R^n, in which a membrane on S is located (see [1]).

In order to solve this problem, we apply analytic methods, i.e. the construction of this process is realized with the aid of an operator semigroup T_t, $t \geq 0$, which is determined, in its turn, for every bounded measurable function $\varphi(x)$, $x \in R^n$, by means of a solution $u(x, t) = T_t\varphi(x)$ of the following adjunction problem for the linear parabolic second-order equation with discontinuous coefficients:

$$L_l u(t, x) \equiv \mathcal{L}_l\, u(t, x) - D_t u(t, x) = 0, \qquad t > 0, \quad x \in \mathcal{D}_l, \quad l = 1, 2, \tag{2}$$

$$u(0, x) = \varphi(x), \qquad x \in R^n, \tag{3}$$

$$u(t, x-) = u(t, x+), \qquad t \geq 0, \quad x \in S, \tag{4}$$

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$$(5)\quad L_0u(t,x) \equiv \frac{1}{2}\sum_{i,j=1}^{n}\beta_{ij}(x)\delta_i\delta_j u(t,x) - \sum_{i=1}^{n}\beta_i^{(1)}(x)D_iu(t,x-)+ \\ +\sum_{i=1}^{n}\beta_i^{(2)}(x)D_iu(t,x+) - \sigma(x)D_tu(t,x) = 0, \quad t>0, \quad x\in S,$$

where $D_t \equiv \frac{\partial}{\partial t}$, δ_i, $i=1,\dots,n$, is a tangent differential operator on S, i.e. $\delta_i = \sum_{k=1}^{n}\tau_{ik}D_k$, $\tau_{ik} = \delta_i^k - \nu_i\nu_k$, δ_i^k is the Kronecker symbol, $\nu_1(x),\dots,\nu_n(x)$, are the coordinates of the unit normal vector $\nu(x)$ to S at x, directed inwards the domain $\mathcal{D}_2$, $\sigma(x)$, $\beta_{ij}(x)$, $\beta_i^{(l)}(x)$, $i,j=1,\dots,n$, $l=1,2$, are bounded continuous functions on S such that $\sigma(x)\ge 0$, the matrix $B(x) = (\beta_{ij}(x))_{i,j=1}^n$ is symmetric and nonnegatively determined, the vector fields $\beta_l(x) = (\beta_1^{(l)}(x),\dots,\beta_n^{(l)}(x))$, $l=1,2$, satisfy the property

$$(6)\qquad (\beta_l(x),\nu(x))\ge 0,\quad l=1,2,\qquad \sum_{l=1}^{2}(\beta_l(x),\nu(x))>0.$$

In addition, by $u(t,x-)$, $D_iu(t,x-)$ and $u(t,x+)$, $D_iu(t,x+)$, $i=1,\dots,n$, we denote non-tangent limits of the functions $u(t,z)$, $D_iu(t,z)$, $i=1,\dots,n$, as z approaches a point $x\in S$ from the side of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

In problem (2)–(5), the adjunction condition (4) means that the searched processes will be Feller ones, and the adjunction condition (5) corresponds to the general Wentzel boundary condition for multidimensional diffusion processes [2]. Note that different partial cases of the formulated problem that were investigated by means of analytic methods in the assumption that the operator L_0 in (5) contains only the derivatives with respect to spatial variables are exposed in monographs [1, 3] (see also [4, 5]). Here, it was shown, in particular, that the obtained processes can be treated as the generalized diffusion in the sense of M.I. Portenko [1]. The multidimensional problem of the glueing of diffusion processes was studied also by means of the methods of stochastic analysis (see, e.g., papers [6, 7]).

In this paper, problem (2)–(5) is studied in the case where the boundary differential operator L_0 in (5) satisfies the condition of uniform parabolicity, and the common boundary S of the domains $\mathcal{D}_l$, $l=1,2$, is an elementary surface that belongs to the Hölder class $H^{2+\lambda}$, $\lambda\in(0;1)$. The existence of a classical solution of this problem was first obtained by the method of boundary integral conditions. Here, we prove that, for the constructed process, the diffusion local characteristics of motion – the displacement vector and the diffusion matrix – exist in the usual sense (in the sense of A.N. Kolmogorov) and are piecewise differentiable functions defined on R^n. Remark that, in this assumption, a parabolic boundary-value problem with the Wentzel boundary condition was investigated earlier in [8, 9]. In both papers, the general scheme of the proof of its classical solvability in the Hölder spaces is based on applying the method of expansion in a parameter.

Below, we will use the following notations:
$x=(x',x_n)=(x_1,\dots,x_{n-1},x_n)$ is a point in R^n; $x'=(x_1,\dots,x_{n-1})$ is a point in R^{n-1} (sometimes, by x', we denote a point of the form $(x_1,\dots,x_{n-1},0)$); $|x|^2=\sum_{i=1}^n x_i^2$; $|x'|^2=\sum_{i=1}^{n-1}x_i^2$; $(x,y)=\sum_{i=1}^n x_iy_i$; $(x',y')=\sum_{i=1}^{n-1}x_iy_i$; $R_+^n=\{x\in R^n|x_n>0\}$; $R_-^n=\{x\in R^n|x_n<0\}$; $(t,x)=(t,x',x_n)$ is a point in R^{n+1}; T is a given positive number; $R_T^{n+1}=(0,T)\times R^n$, $R_T^n=(0,T)\times R^{n-1}$; $\mathcal{D}_T^{(l)}=(0,T)\times\mathcal{D}_l$, $l=1,2$; $\Sigma=[0,T]\times S$; D_t^r and D_x^p are the symbols of the partial derivative of order r with respect to t and arbitrary partial derivative of order p with respect to x, where r and p are integer nonnegative numbers; $\nabla=(D_1,\dots,D_n)$; $\nabla'=(D_1,\dots,D_{n-1})$; $C(\Omega)$ is the set of continuous in Ω functions, where Ω is the domain in R^{n+1}; $C^{1,2}(\Omega)$ is the set of functions

continuous in Ω that have derivatives $D_t^r D_x^p$ continuous in R^{n+1}, where $2r+p \le 2$; $\mathbf{B}(R^n)$ is the Banach space of all bounded real-valued measurable functions on R^n, with the norm $||\varphi|| \equiv \sup_{x\in R^n} |\varphi(x)|$; $H^{(k+\lambda)/2,\, k+\lambda}_{t,\quad x}(\overline{\mathcal{D}}_T^{(l)}) \equiv H^{(k+\lambda)/2,\, k+\lambda}(\overline{\mathcal{D}}_T^{(l)})$, $H^{(k+\lambda)/2,\, k+\lambda}_{t,\quad x}(\Sigma) \equiv H^{(k+\lambda)/2,\, k+\lambda}(\Sigma)$, $H^{k+\lambda}(R^n)$, $H^{k+\lambda}(S)$ ($k = 0, 1, 2$, $\lambda \in (0,1)$), as in [10, p.16] denote the corresponding Hölder spaces; $\underset{0}{H}^{(k+\lambda)/2,\, k+\lambda}(\Sigma)$ is the subspace of functions from $H^{(k+\lambda)/2,\, k+\lambda}(\Sigma)$, that, together with the admissible derivative with respect to the time variable, vanishes at $t = 0$; by $||u||_{H^{(k+\lambda)/2,\, k+\lambda}(\overline{\mathcal{D}}_T^{(l)})}$, $||u||_{H^{(k+\lambda)/2,\, k+\lambda}(\Sigma)}$, $||u||_{H^{k+\lambda}(R^n)}$, $||u||_{H^{k+\lambda}(S)}$, we denote the norms of functions u in the corresponding Hölder spaces; C and c are positive constants independent of (t, x), we are not interested in their specific values.

We assume, in addition, that, for the coefficients of the operators $\mathcal{L}_l$, $l = 1, 2$, and L_0 from (1) and (5), the following conditions hold:

(A) the matrix $A_l(x)$, $x \in R^n$, is uniformly nondegenerated, $a_{ij}^{(l)}$, $a_i^{(l)} \in H^\lambda(R^n)$, $i, j = 1, \ldots, n$, $l = 1, 2$;

(B) the matrix $B(x)$, $x \in S$, is uniformly nondegenerated, $\sigma \equiv 1$, β_{ij}, $\beta_i^{(1)}, \beta_i^{(2)} \in H^\lambda(S)$, $i, j = 1, \ldots, n$.

We will also assume that the initial function φ from (3) is sufficiently smooth on R^n, and the following agreement conditions hold for it:

$$(7)\quad \frac{1}{2}\sum_{i,j=1}^{n} \beta_{ij}(x)\delta_i\delta_j\varphi(x) + (\beta_2(x) - \beta_1(x), \nabla\varphi(x)) - \mathcal{L}_l\varphi(x) = 0, \quad x \in S, \quad l = 1, 2.$$

Let $G_l(t, x, y)$ ($t > 0$, $x, y \in R^n$), $l = 1, 2$, be a fundamental solution (f.s.) for Eq. (2) see formula (11.13) from [10, p. 409]:

$$G_l(t,x,y) = G_{0l}(t,x,y) + G_{1l}(t,x,y), \qquad l = 1, 2,$$

where

$$G_{0l}(t,x,y) = G_{0l}(t,x-y) = G_{0l}(t,x'-y',x_n-y_n) = \\ = (2\pi t)^{-\frac{n}{2}}(\det A_l(y))^{-\frac{1}{2}} \exp\left\{ -\frac{\left(A_l^{-1}(y)(x-y), x-y\right)}{2t} \right\}, \qquad l = 1, 2,$$

G_{1l} is an integral term which has a "weaker" singularity than G_{0l} as $t \to 0+$, and $G_l \equiv 0$, for $t \le 0$.

Theorem 1. *If $\varphi \in H^{2+\lambda}(R^n)$, an elementary surface S belongs to the class $H^{2+\lambda}$ and conditions (A), (B), (6), (7) hold, then there exists a unique solution $u(t,x) = u_l(t,x)$, $t > 0$, $x \in \mathcal{D}_l$, $l = 1, 2$, of problem (2)–(5), moreover,*

$$(8)\qquad u_l \in H^{(2+\lambda)/2,\, 2+\lambda}\left(\overline{\mathcal{D}}_T^{(l)}\right), \qquad l = 1, 2,$$

the estimate

$$(9)\qquad \sum_{l=1}^{2} ||u_l||_{H^{(2+\lambda)/2,\, 2+\lambda}(\overline{\mathcal{D}}_T^{(l)})} \le C\,||\varphi||_{H^{2+\lambda}(R^n)},$$

holds, and the solution is of the form

$$(10)\quad u_l(t,x) = \int_{R^n} G_l(t,x,y)\varphi(y)dy + \int_0^t d\tau \int_S G_l(t-\tau,x,y)V_l(\tau,y)d\sigma_y, \\ t > 0, x \in \mathcal{D}_l, \quad l = 1, 2,$$

where V_l, $l=1,2$, is a solution of some system of second-order Volterra integral equations.

Proof. We will seek a solution of problem (2)–(5) in the form (10) with unknown densities V_l, $l=1,2$, that will be determined from the adjunction conditions (4), (5). The first and second terms on the right-hand side of equality (10) will be denoted by $u_{0l}(t,x)$ and $u_{1l}(t,x)$, $l=1,2$, respectively. The former will be called a Poisson potential, and the latter the simple layer potential.

First, we transform equality (5) by separating its tangential and normal components in the expressions that contain the derivatives with respect to the spatial variables. To this end, we use the relation

$$\sum_{i=1}^{n}\beta_i^{(l)}(x)D_iu_l(t,x)=\sum_{i=1}^{n}\beta_i^{(l)}(x)\tilde{\delta}_i^{(l)}u(t,x)+\gamma_{0l}(x)\frac{\partial u_l(t,x)}{\partial N_l(x)},\ t>0,\ x\in S, \tag{11}$$

where $\tilde{\delta}_i^{(l)}=D_i-\frac{\nu_i}{(N_l,\nu)}\sum_{k=1}^{n}N_{kl}D_k$, $i=1,\dots,n$, $l=1,2$, are tangential differential operators on S, $N_l(x)=(N_{1l}(x),\dots,N_{nl}(x))$, $l=1,2$, are the conormal vectors at $x\in S$, i.e. $N_l(x)=A_l(x)\nu(x)$, $l=1,2$,

$$\gamma_{0l}(x)=\frac{(\beta_l(x),\nu(x))}{(N_l(x),\nu(x))},\quad l=1,2.$$

Taking into account (11) and assumptions on the function $\sigma(x)$, one can write condition (5) in the form

$$\begin{aligned}L_0'u(t,x)\equiv\frac{1}{2}\sum_{i,j=1}^{n}\beta_{ij}(x)\delta_i\delta_ju(t,x)-\sum_{i=1}^{n}\beta_i^{(1)}(x)\tilde{\delta}_i^{(1)}u(t,x)+\\+\sum_{i=1}^{n}\beta_i^{(2)}(x)\tilde{\delta}_i^{(2)}u(t,x)-D_tu(t,x)=-\Theta_0(t,x),\quad t>0,\quad x\in S,\end{aligned} \tag{12}$$

where

$$\Theta_0(t,x)=\sum_{l=1}^{2}(-1)^l\gamma_{0l}(x)\frac{\partial u_l(t,x)}{\partial N_l(x)}.$$

Equality (12) can be regarded as an autonomous parabolic equation in the domain $t>0$, $x\in S$. If we assume *a priori* that $V_l\in H^{\lambda/2,\lambda}(\Sigma)$, $l=1,2$, then in this equation, as it follows from the condition of the theorem and properties of the parabolic potentials (see [10]), its coefficients and the right-hand side belong to the space $H^{\lambda/2,\lambda}(\Sigma)$.

Find an integral representation for the solution of Eq. (12) that satisfies the initial condition

$$u(0,x)=\varphi(x),\qquad x\in S. \tag{13}$$

To this end, we use the condition of the theorem that S is an elementary surface of the class $H^{2+\lambda}$. This means that $S=\{x\in R^n\,|\,x_n=F(x')\}$, $\mathcal{D}_1=\{x\in R^n\,|\,x_n<F(x')\}$, $\mathcal{D}_2=\{x\in R^n\,|\,x_n>F(x')\}$, where the function $F(x')$, $x'\in R^{n-1}$, satisfies the condition

$$F\in H^{2+\lambda}(R^{n-1}).$$

Let us make the following convention: the direct values of an arbitrary function $f(t,x)$ on $(0,\infty)\times S$ (i.e., when $(t,x)\equiv(t,x',F(x'))$) are denoted by $\overline{f}(t,x')$. Remark that, under the conditions of our assumptions, the components of the normal vector $\nu(x)$, $x\in S$, can be determined by the equalities

$$\nu_i(x)=\overline{\nu}_i(x')=-\frac{F_i(x')}{\sqrt{1+\sum_{j=1}^{n-1}F_j^2(x')}},\quad i=1,\dots,n-1,$$

$$\nu_n(x) = \overline{\nu}_n(x') = \frac{1}{\sqrt{1 + \sum_{j=1}^{n-1} F_j^2(x')}}, \quad \text{where } F_j(x') = D_j F(x'), \quad j = 1, \dots, n-1,$$

and the simple layer potentials u_{1l}, $l = 1, 2$, from (10) can be represented in the form

$$u_{1l}(t, x) = \int_0^t d\tau \int_{R^{n-1}} G_l(t-\tau, x, y)|_{y_n = F(y')} \hat{V}_l(\tau, y') dy', \qquad l = 1, 2,$$

where we put $\hat{V}_l(\tau, y') = \overline{V}_l(\tau, y')\overline{\nu}_n^{-1}(y')$, $l = 1, 2$.

Let us consider the following transformation of the variables:

$$(14) \qquad (t, x) \to (t, z), \quad z_i = x_i, \quad i = 1, \dots, n-1, \quad z_n = x_n - F(x').$$

It transforms the domains $\mathcal{D}_1$ and $\mathcal{D}_2$ into semilayers $\tilde{\mathcal{D}}_1 = R^n_-$ and $\tilde{\mathcal{D}}_2 = R^n_+$, respectively, the elementary surface S into the hyperplane $\tilde{S} = R^{n-1}$, and the side boundary Σ into the boundary $\tilde{\Sigma} = [0, T] \times R^{n-1}$. Let $w(t, z) = u(t, x(z)) = u(t, z', z_n + F(z'))$. Then $\overline{w}(t, z') = w(t, z', 0)$, $u(t, x) = w(t, z(x)) = w(t, x', x_n - F(x'))$, $\overline{u}(t, x') = \overline{w}(t, z')$, and Eq. (12) in new variables will take the form:

$$(15) \quad \overline{L}_0' \overline{w} \equiv \frac{1}{2} \sum_{k,l=1}^{n-1} \overline{\tilde{\beta}}_{kl}(z') D_{kl} \overline{w} + \sum_{k=1}^{n-1} \overline{\tilde{\beta}}_k(z') D_k \overline{w} - D_t \overline{w} = -\overline{\Theta}_0(t, z'),$$

$$(t, z') \in (0, \infty,) \times R^{n-1},$$

where

$$\overline{\tilde{\beta}}_{kl}(z') = \sum_{i,j=1}^{n} \overline{\beta}_{ij}(z') \overline{\tau}_{ik}(z') \overline{\tau}_{jl}(z'), \quad k, l = 1, \dots, n-1,$$

$$\overline{\tilde{\beta}}_k(z') = \overline{\beta}_k^{(2)}(z') - \overline{\beta}_k^{(1)}(z') + \overline{\gamma}_{01}(z') \overline{N}_k^{(1)}(z') - \overline{\gamma}_{02}(z') \overline{N}_k^{(2)}(z') + \\ + \frac{1}{2} \sum_{i,j=1}^{n} \overline{\beta}_{ij}(z') \delta_i(\overline{\tau}_{jk}(z')), \quad k = 1, \dots n-1.$$

In Eq. (15), its coefficients belong to the class $H^{\lambda}(R^{n-1})$, and its right-hand side $\Theta_0 \in H^{\lambda/2, \lambda}(\overline{R}^n_T)$. In addition, the matrix $\overline{\tilde{B}}_{n-1}(z') = \left(\overline{\tilde{\beta}}_{kl}(z')\right)_{k,l=1}^{n-1}$ is positive definite, symmetric, and uniformly nondegenerated. This means that, for the operator $\overline{L}_0'$, there exists an f.s. which will be denoted by $\overline{g}(t, z', v')$ ($t > 0$, $z', v' \in R^{n-1}$). With the aid of the f.s. $\overline{g}$ and the transformation of variables (14), we determine the function $\Gamma(t, x, y) = \overline{g}(t, z'(x), v'(y))\nu_n(y)$, where $t > 0$, $x, y \in S$. On the base of the properties of the f.s. $\overline{g}$, we are able to prove that $\Gamma(t, x, y)$, as a function of the arguments $(t, x) \in (0, \infty) \times S$, satisfies the equation $L_0'\Gamma = 0$, and, for an arbitrary function $\varphi(x)$ continuous and bounded on S, we have

$$\lim_{t \downarrow 0} \int_S \Gamma(t, x, y) \varphi(y) d\sigma_y = \varphi(x).$$

The function Γ will be called the fundamental solution of the operator L_0' from (12).

It is easy to verify that one can extend onto the f.s. Γ, mutatis mutandis, the other known properties of the f.s. $\overline{g}$ (see [10, Ch. IV]). In particular, for the function $\Gamma(t, x, y)$ and its admissible derivatives with respect to the variables t and x, the estimate

$$|D_t^r D_x^p \Gamma(t, x, y)| \le C\, t^{-\frac{(n-1)+2r+p}{2}} \exp\left\{ -c \frac{|x-y|^2}{t} \right\},$$

holds, where $2r + p \le 2$, $t \in (0, T]$, $x, y \in S$.

Another consequence of properties of the f.s. Γ and generated potentials is an integral representation of the solution of the Cauchy problem (12), (13) defined by the formula

$$u(t,x)=\int_S \Gamma(t,x,y)\varphi(y)d\sigma_y+\int_0^t d\tau\int_S \Gamma(t-\tau,x,y)\Theta_0(\tau,y)d\sigma_y,\qquad (t,x)\in(0,\infty)\times S. \tag{16}$$

Therefore, we have two different expressions for the function $u(t,x)$ on $(0,\infty)\times S$: relation (10), in which one has to assume that $(t,x)\in(0,\infty)\times S$, and relation (16). Making their right-hand sides equal, we obtain the following system of integral equations for the unknown functions V_l, $l=1,2$:

$$\int_0^t d\tau\int_S G_l(t-\tau,x,y)V_l(\tau,y)d\sigma_y+\sum_{i=1}^{2}\int_0^t d\tau\int_S K_i(t-\tau,x,y)V_i(\tau,y)d\sigma_y= \\ =\psi_l(t,x),\qquad l=1,2,\quad (t,x)\in(0,\infty)\times S, \tag{17}$$

where

$$K_i(t-\tau,x,y)=\Gamma(t-\tau,x,y)\gamma_{0i}(y)+\int_\tau^t ds\int_S \Gamma(t-s,x,z)(-1)^{i-1}\gamma_{0i}(z)\times \\ \times\frac{\partial G_i(s-\tau,z,y)}{\partial N_i(z)}d\sigma_z,\quad i=1,2,$$

$$\psi_l(t,x)=\int_S \Gamma(t,x,y)\varphi(y)d\sigma_y-u_{0l}(t,x)+ \\ +\sum_{i=1}^{2}(-1)^i\int_0^t d\tau\int_S \Gamma(t-\tau,x,y)\gamma_{0i}(y)\frac{\partial u_{0i}(\tau,y)}{\partial N_i(y)}d\sigma_y,\quad l=1,2,$$

and, in addition, for the kernels $K_i(t-\tau,x,y)$, $i=1,2$, if $0\le\tau<t\le T$, $x,y\in S$, the estimate

$$|K_i(t-\tau,x,y)|\le C\,(t-\tau)^{-\frac{n-1}{2}}\exp\left\{-c\,\frac{|x-y|^2}{t-\tau}\right\}$$

holds. Whence, taking into account the conditions of Theorem 1 and the properties of potentials, we also have $\psi_l\in \underset{0}{H}^{(2+\lambda)/2,\,2+\lambda}(\Sigma)$, $l=1,2$.

The system of equations (17) is a system of Volterra first-kind integral equations. For its regularization, we introduce integro-differential operators $\mathcal{E}_l$, $l=1,2$, that act by the rule

$$\mathcal{E}_l(t,x)\psi=\sqrt{\frac{2}{\pi}}\left\{\frac{\partial}{\partial t}\int_0^t (t-\tau)^{-\frac12}d\tau\int_S H_l(\hat t-\tau,x,y)\nu_n(y)\psi(\tau,y)d\sigma_y\right\}\Bigg|_{\hat t=t}, \\ l=1,2,\quad (t,x)\in(0,\infty)\times S. \tag{18}$$

Here, by $H_l(t-\tau,x,y)=\overline{H}_l(t-\tau,x',y')$ $(0\le\tau<t\le T,\ x',y'\in R^{n-1})$, $l=1,2$, we denote the f.s. of the operator with Hölder coefficients

$$L_{n-1}^{(l)}\equiv\frac12\sum_{i,j=1}^{n-1}\overline{h}_{ij}^{(l)}(x')D_{ij}-D_t,\quad l=1,2,$$

where $\overline{h}^{(l)}_{ij}(x')$, $i,j=1,\dots,n-1$, $l=1,2$, can be expressed via the elements of the matrices $A_l(x)$, $l=1,2$, $x\in S$, from (1) by means of the formulas

$$\overline{h}^{(l)}_{ij}(x')=\overline{\overline{a}}^{(l)}_{ij}(x')-\frac{\overline{\overline{a}}^{(l)}_{in}(x')\,\overline{\overline{a}}^{(l)}_{jn}(x')}{\overline{\overline{a}}^{(l)}_{nn}(x')},\qquad i,j=1,\dots,n-1,\quad l=1,2,$$

$$\overline{\overline{a}}^{(l)}_{ij}(x')=\overline{a}^{(l)}_{ij}(x'),\qquad i,j=1,\dots,n-1,\quad l=1,2,$$

$$\overline{\overline{a}}^{(l)}_{in}(x')=\overline{\overline{a}}^{(l)}_{ni}(x')=\overline{a}^{(l)}_{in}(x')-\sum_{k=1}^{n-1}\overline{a}^{(l)}_{ik}(x')F_k(x'),\qquad i=1,\dots,n-1,\quad l=1,2,$$

$$\overline{\overline{a}}^{(l)}_{nn}(x')=\overline{a}^{(l)}_{nn}(x')-2\sum_{k=1}^{n-1}\overline{a}^{(l)}_{kn}(x')F_k(x')+\sum_{k,m=1}^{n-1}\overline{a}^{(l)}_{km}(x')F_k(x')F_m(x'),\quad l=1,2.$$

It is worth to note that the integro-differential operator (18) was first introduced by O.A. Baderko in [11] and was applied therein as a regularizator of the first kind that arises in solving the first boundary-value problem by the potential method. From the results of that paper, it follows that $\mathcal{E}_l$, $l=1,2$, are bounded operators acting from $\underset{0}{H}{}^{(1+\lambda)/2,\,1+\lambda}(\Sigma)$ into $\underset{0}{H}{}^{\lambda/2,\,\lambda}(\Sigma)$ and having bounded inverse operators. In our case, we establish that the operators, $\mathcal{E}_l$, $l=1,2$, map the space $\underset{0}{H}{}^{(2+\lambda)/2,\,2+\lambda}(\Sigma)$ into the space $\underset{0}{H}{}^{(1+\lambda)/2,\,1+\lambda}(\Sigma)$.

Applying $\mathcal{E}_l$, $l=1,2$, to both sides of the corresponding equations of system (17), we transform this system to the equivalent second kind Volterra equations

$$V_l(t,x)+\sum_{i=1}^{2}\int_0^t d\tau\int_S K_{li}(t-\tau,x,y)V_i(\tau,y)d\sigma_y=\Phi_l(t,x),\tag{19}$$
$$(t,x)\in(0,\infty)\times S,\quad l=1,2,$$

where

$$\Phi_l(t,x)=(A_l(x)\nu(x),\nu(x))^{\frac{1}{2}}\mathcal{E}_l(t,x)\psi_l,\quad l=1,2.$$

For the kernels K_{li}, $l=1,2$, $i=1,2$, in every domain of the form $0\le\tau<t\le T$, $x,y\in S$, the estimate

$$|\,K_{li}(t-\tau,x,y)\,|\le C\,(t-\tau)^{-\frac{n+1-\lambda}{2}}\exp\left\{-c\,\frac{|x-y|^2}{t-\tau}\right\},\tag{20}$$

holds, and the functions Φ_l, $l=1,2$, satisfy the condition

$$\Phi_l\in\underset{0}{H}{}^{\lambda/2,\,\lambda}(\Sigma).\tag{21}$$

Inequality (20) means that the kernels K_{li}, $l=1,2$, $i=1,2$, possess weak singularities. Solving the system of equations (19) by the method of successive approximations, we find V_l, $l=1,2$. One can additionally verify that V_l, $l=1,2$, belong to class (21). Whence we conclude that there exists a classical solution of problem (2)–(5), and it can be represented in form (10).

Finishing the proof of Theorem 1, we have only to check correctedness of condition (8) and estimate (9) and to verify the statement of the theorem on the uniqueness of the obtained solution of problem (2)–(5). To this end, one has only to remark that, under conditions of Theorem 1, every function u_l, $l=1,2$, constructed by formulas (10), (19) can be considered as a unique solution from class (8) of the corresponding parabolic first boundary-value problem:

$$L_lu_l(t,x)=0,\quad (t,x)\in(0,\infty)\times\mathcal{D}_l,\ l=1,2,$$

$$u_l(0,x) = \varphi(x), \quad x \in \mathcal{D}_l,\ l = 1,2,$$
$$u_l(t,x) = v(t,x), \quad (t,x) \in (0,\infty) \times S,\ l = 1,2,$$

under the agreement conditions

$$\varphi(x) = v(0,x), \quad x \in S,$$
$$\left.\frac{\partial u_l(t,x)}{\partial t}\right|_{t=0} = \left.\frac{\partial v(t,x)}{\partial t}\right|_{t=0}, \quad x \in S,\ l = 1,2,$$

where the function $v(x,t)$ is determined by means of relationship (16). In addition, on the base of the conditions $\varphi \in H^{2+\lambda}(R^n)$, $V_l \in \underset{0}{H}{}^{\lambda/2,\,\lambda}(\Sigma)$, $l = 1,2$, and properties of the potentials, we have $v \in H^{(2+\lambda)/2,\,2+\lambda}(\Sigma)$, and the inequality

$$||\, v\,||_{H^{(2+\lambda)/2,\,2+\lambda}(\Sigma)} \le C\,[\,||\varphi||_{H^{2+\lambda}(S)} + ||\Theta_0||_{H^{\lambda/2,\,\lambda}(\Sigma)}] \le C\,||\varphi||_{H^{2+\lambda}(R^n)}$$

holds. Theorem 1 is proved.

Remark 1. If one does not require in Theorem 1 that the initial function satisfy the agreement conditions (7), then one can only claim that there exists a unique solution of problem (2)–(5) from the class

$$u_l \in C^{1,\,2}((0,T] \times \overline{\mathcal{D}}_l) \cap C(\overline{R}^{n+1}_T), \qquad l = 1,2.$$

Remark 2. The assertion of Theorem 1 remains true also in the case where the boundary S of the domains $\mathcal{D}_l$, $l = 1,2$, is an arbitrary bounded hypersurface of the class $H^{2+\lambda}$ of the general form. Here, the construction of the f.s. Γ for the operator L_0' from (12) and the construction of regularizators $\mathcal{E}_l$, $l = 1,2$, for the system of equations (17) can be realized on the base of the "partition of unity" subordinated to a covering of the boundary layer and the results obtained in the process of studying problem (2)–(5) in the considered case.

Let us proceed with the construction of the required process. It can be realized by the scheme used for the construction of similar processes in [12]. Namely, the family of operators T_t, $t \ge 0$, generating the required process can be determined by formulas (10), (19). First, using the equality

$$\begin{aligned}
&\sqrt{\frac{\pi}{2}}\mathcal{E}_l(t,x)\psi_l \\
&\quad = -\frac{1}{2}\int_0^{t/2} (t-\tau)^{-\frac{3}{2}} d\tau \int_{R^{n-1}} \overline{H}_l(t-\tau, x', y')\overline{\psi}_l(\tau, y')\, dy' \\
&\qquad - \frac{1}{2}\int_{t/2}^{t} (t-\tau)^{-\frac{3}{2}} d\tau \int_{R^{n-1}} \overline{H}_l(t-\tau, x', y') \\
&\qquad\qquad \times [\overline{\psi}_l(\tau, y') - \overline{\psi}(\tau, x') - (y'-x', \nabla'\overline{\psi}_l(\tau, x'))]\, dy' \\
&\qquad + \frac{1}{2}\int_{t/2}^{t} (t-\tau)^{-\frac{3}{2}} [\overline{\psi}_l(t,x') - \overline{\psi}(\tau,x')]\, d\tau + (2t)^{-\frac{1}{2}}\overline{\psi}_l(t,x'), \\
&\qquad\qquad l = 1,2, \quad (t,x) \in (0,\infty) \times S,
\end{aligned}$$

by means of elementary estimates for the f.s. (see [10, Ch. IV, § 13]) we prove that, under the condition $\varphi \in \mathbf{B}(R^n)$, the functions Φ_l from (19) and consequently V_l, $l = 1,2$, admit the estimate

$$|V_l(t,x)| \le C\,||\,\varphi\,||\,t^{-\frac{1}{2}}, \qquad l = 1,2, \quad (t,x) \in (0,T] \times S. \tag{22}$$

Taking into account (22) and (10), we easily obtain the inequality $|T_t\varphi(x)| \le C\,||\,\varphi\,||$ which holds in every domain of the form $(t,x) \in (0,T] \times R^n$ with some constant C. The latter inequality means that the operators T_t, $t > 0$, can be applied to the functions from the space $\mathbf{B}(R^n)$. Further, analogously as in [1, 3, 12], we check that the family of operators T_t satisfies the following properties: 1) $||\,T_t\,|| \le 1$ for all $t \ge 0$; 2) for all $t \ge 0$, $s \ge 0$, the relations $T_{t+s} = T_t T_s$ hold; 3) if $\varphi(x) \ge 0$ for all $t \ge 0$, $x \in R^n$, then $T_t\varphi(x) \ge 0$ for all $t \ge 0$, $x \in R^n$; 4) if $\varphi_n \in \mathbf{B}(R^n)$ for $n = 1, 2, \ldots$, $\sup ||\varphi_n|| < \infty$ and for all $x \in R^n$, we have $\lim\limits_{n\to\infty} \varphi_n(x) = \varphi(x)$, then, for all $t \ge 0$, $x \in R^n$, the relation $\lim\limits_{n\to\infty} T_t\varphi_n(x) = T_t\varphi(x)$ holds. Whence (see [1, 3]) we conclude that the semigroup of operators T_t, $t > 0$, which is constructed by formulas (10), (19) determines some non-interrupting Feller process in R^n. Denoting its transition probability by $P(t, x, dy)$, we can represent the function $T_t\varphi(x)$ in the form

$$T_t\varphi(x) = \int_{R^n} \varphi(y) P(t, x, dy), \quad t > 0, \quad x \in R^n.$$

Finally, the existence of the integral representation for $T_t\varphi(x)$ allows us to establish, by means of immediate calculations,the following relations for the transition probability $P(t, x, dy)$:

$$\text{(23)} \qquad a) \qquad \sup_{x\in R^n} \int_{R^n} |y - x|^4 \, P(t, x, dy) \le C\, t^2, \qquad t \in (0, T],$$

b) for all $x \in R^n$, $\Theta \in R^n$, there exist the limits

$$\text{(24)} \qquad \lim_{t\downarrow 0} \frac{1}{t} \int_{R^n} (y - x, \Theta)\, P(t, x, dy) = (\tilde{a}(x), \Theta),$$

$$\text{(25)} \qquad \lim_{t\downarrow 0} \frac{1}{t} \int_{R^n} (y - x, \Theta)^2\, P(t, x, dy) = (\tilde{A}(x)\Theta, \Theta),$$

where

$\tilde{a}(x) = \begin{cases} (a_l(x), \Theta) & x \in \mathcal{D}_l, \quad l = 1, 2, \\ (\tilde{\beta}_0(x), \Theta), & x \in S, \end{cases}$

$\tilde{\beta}_0(x) = (\tilde{\beta}_1^{(0)}(x), \ldots, \tilde{\beta}_{n-1}^{(0)}(x), \tilde{\beta}_n^{(0)}(x))$,

$\tilde{\beta}_k^{(0)}(x) = \beta_k^{(2)}(x) - \beta_k^{(1)}(x) + \frac{1}{2}\sum_{i,j=1}^n \beta_{ij}(x)\delta_i(\tau_{jk}(x)), \qquad k = 1, \ldots, n-1,$

$\tilde{\beta}_n^{(0)}(x) = L_0' F(x') + \sum_{i=1}^2 (-1)^i \gamma_{0i}(x) N_{ni}(x), \qquad x \in S,$

$\tilde{A}(x) = \begin{cases} (A_l(x)\Theta, \Theta), & x \in \mathcal{D}_l, \quad l = 1, 2, \\ (\tilde{B}_0(x)\Theta, \Theta), & x \in S, \end{cases}$

$\tilde{B}_0(x) = \left(\tilde{\beta}_{km}^{(0)}(x)\right)_{k,m=1}^n,$

$\tilde{\beta}_{km}^{(0)}(x) = \tilde{\beta}_{km}(x) = \sum_{i,j=1}^{n-1} \beta_{ij}(x)\tau_{ik}(x)\tau_{jm}(x), \qquad k, m = 1, \ldots n-1,$

$\tilde{\beta}_{kn}^{(0)}(x) = \tilde{\beta}_{nk}^{(0)}(x) = \sum_{m=1}^{n-1} \tilde{\beta}_{km}(x) F_m(x'), \quad k = 1, \ldots, n-1,$

$\tilde{\beta}_{nn}^{(0)}(x) = \sum_{k,m=1}^{n-1} \tilde{\beta}_{km}(x) F_k(x') F_m(x').$

On the base of inequality (23), we conclude that the trajectories of the constructed process can be assumed continuous, and equalities (24), (25) mean that this process is a diffusion one, in the sense of A.N. Kolmogorov's definition. For it, the transition vector equals $\tilde{a}(x)$ and the diffusion matrix equals $\tilde{A}(x)$.

We formulate the obtained results in the following form.

Theorem 2. *Let, for the coefficients of the operators $\mathcal{L}_l$, $l = 1, 2$, from (1) and L_0 from (5), conditions (A), (B), and (6) hold. Then the solution of the parabolic adjunction problem (2)–(5), constructed in Theorem 1 uniquely determines the operator semigroup* *that describes the diffusion process in R^n, $n \geq 2$, with discontinuous diffusion coefficients determined by relations (24), (25).*

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