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ON THE MARKOV PROPERTY OF STRONG SOLUTIONS TO SDE WITH GENERALIZED COEFFICIENTS

We present the complete proof of the Markov property of the strong solution to a multidimensional stochastic differential equation, whose coefficients involve the local time on a hyperplane of the unknown process.

1. Introduction.

We consider one class of stochastic differential equations (SDE) with the local time on a hyperplane of the unknown process included into the drift and martingale part. It is shown that this equation has the strong unique solution, and this solution is a Markov process. The processes constructed may be described as a generalized diffusion process in the sense of Portenko (see, for example, [1]) with the drift vector and the diffusion matrix of the general form.

One of the most known examples of a process described by an SDE with its coefficients containing the local time of the unknown process is the skew Brownian motion introduced in [2] (Section 4.2, Problem 1). In 1981, J.M. Harrisson and L.A. Shepp (see [3]) proved that the skew Brownian motion may be constructed as the strong solution to the SDE of the form $dx(t) = qd\eta_t + dw(t)$, where $q \in [-1, 1]$ is a given parameter and $\{\eta_t\}$ is the local time at 0 of the process $\{x(t)\}$. One-dimensional equations with the drift of a more general form were considered by J.-F. Le Gall, M. Barlow, K. Burdzy, H. Kaspi, A. Mandelbaum (see [4,5]). The main result of those papers is the theorem on the existence and uniqueness of a strong solution to the corresponding SDE.

The characterization of the obtained strong solution (say, as a Markov process or a generalized diffusion process) is a separate complicated problem. J.M. Harrisson and L.A. Shepp in their paper, while stating that the skew Brownian motion obtained as the strong solution to the SDE is a Markov process, referred to the corresponding results for the standard SDE in [6]. Since this question is rather delicate, let us discuss it in more details. The scheme of proof in [6] (or, in more generality, in [7], Chapter 6) consists of two steps. The first step contains the construction of a modification of the strong solution, jointly measurable w.r.t. starting point, and the Wiener noise. The second one, namely the proof of the Markov property, essentially uses the fact that the increments of a Wiener process are independent. The first step requires the proof of the additional fact that the solution is continuous in probability as a function of the starting point. This proof is non-trivial even in the simplest case of a skew Brownian motion (see [8] for a general result), and the arguments of J.M. Harrisson and L.A. Shepp are incomplete at this point. Notice that A.M. Kulik (see [9]) gave a sketch of the proof of the Markov property in the case of the one-dimensional equation considered by J.-F. Le Gall, which takes this difficulty into account. In the multidimensional case, we meet another difficulty

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caused by the fact that the martingale noise does not necessarily possess independent increments. The main purpose of this paper is to give the complete proof of the Markov property for the strong solution to an SDE involving the local time of the unknown process in the most general multidimensional case.

2. Construction of the process.

Let S be a hyperplane in $\Re^d$ orthogonal to a fixed unit vector $\nu \in \Re^d : S = \left\{x \in \Re^d | (x,\nu) = 0\right\}$. By π_S and L, we denote, respectively, the operator of orthogonal projection on S and the one-dimensional subspace of $\Re^d$, generated by ν.

We consider a Wiener process $\{w(t)\}$ in $\Re^d$ and a filtration $\mathcal{F}_t^w = \sigma\{w(u), 0 \le u \le t\}$, $t \ge 0$. We denote $w(t) = (w^1(t), w^S(t))$, where $w^1(t) = (w(t),\nu)$, $w^S(t) = \pi_S w(t)$. For a given parameter $q \in [-1,1]$ and the initial point $x_0^1 \in L$, we construct the skew Brownian motion (see [3]), i.e. a pair of $\{\mathcal{F}_t^w\}$-adapted processes $\{(x^1(t), \eta_t)\}$, where $\{\eta_t\}$ is the local time at 0 of $\{x_1(t)\}$,

$$\eta_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1\!\!\mathrm{I}_{\{|x^1(\tau)| \le \varepsilon\}} d\tau,$$

such that $\{x^1(t)\}$ and $\{\eta_t\}$ satisfy the equality

$$x^1(t) = x_0^1 + q\eta_t + w^1(t)$$

for all $t \ge 0$.

Let us assume that there exists one more Wiener process $\{\widetilde{w}(t)\}$ in S that does not depend on $\{w(t)\}$. Consider a new process $\zeta(t) = \widetilde{w}(\eta_t), t \ge 0$. We shall deal with the SDE involving the processes $\{w(t)\}$ and $\{\zeta(t)\}$ as a martingale part of the equation. Therefore, we have to construct a filtration such that $\{\zeta(t)\}$ is the square integrable martingale and $\{w(t)\}$ is the Wiener process with respect to this filtration. Remark that the process $\{\eta_t\}$ is not a stopping time w.r.t. $\{\mathcal{F}_t^w\}$, and therefore the classical results cannot be applied here.

For $t \ge 0$, we consider

$$\widetilde{\mathcal{F}}_t = \sigma\left\{\{\widetilde{w}(s) \in \Gamma\} \bigcap \{\eta_t \ge s\}, s \ge 0, \Gamma \in \mathcal{B}_S\right\},$$

where $\mathcal{B}_S$ is the Borel σ-algebra on S. We put

$$\mathcal{M}_t = \mathcal{F}_t^w \bigvee \widetilde{\mathcal{F}}_t.$$

The following three lemmas show that $\{\mathcal{M}_t\}$ is the required filtration.

Lemma 1. *$\{\mathcal{M}_t\}$ is a filtration.*

Proof. We have to prove that $\mathcal{M}_{t_1} \subseteq \mathcal{M}_{t_2}$, when $t_1 \le t_2$. For fixed $s \ge 0$ and $\Gamma \in \mathcal{B}_S$ let us consider the set $A_{t_1} = \{\widetilde{w}(s) \in \Gamma\} \bigcap \{\eta_{t_1} \ge s\}$. If $A_{t_1} = \emptyset$ then $A_{t_1} \in \mathcal{M}_{t_2}$.

Let $A_{t_1} \ne \emptyset$. Then $A_{t_1} \bigcap \{\eta_{t_2} \ge s\} \ne \emptyset$, because $\{\eta_t\}$ is the increasing process. Also $A_{t_1} = A_{t_2} \bigcap \{\eta_{t_1} \ge s\}$. Since $A_{t_2} \in \widetilde{\mathcal{F}}_{t_2}$ and $\{\eta_{t_1} \ge s\} \in \mathcal{F}_{t_1}^w \subseteq \mathcal{F}_{t_2}^w$, $A_{t_1} \in \mathcal{M}_{t_2}$.

The lemma is proved.

Lemma 2. *The process $\{\zeta(t)\}$ is a square integrable martingale w.r.t. $\{\mathcal{M}_t\}$, and its characteristic is equal to $\{\eta_t\}$.*

Proof. Firstly, we show that the process $\zeta(t)$ is $\mathcal{M}_t$-measurable for all $t \ge 0$. We can approximate $\zeta(t)$ by step functions in the following way

$$\zeta(t) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \widetilde{w}\left(\frac{k-1}{n}\right) 1\!\!\mathrm{I}_{\left\{\frac{k-1}{n} \le \eta_t < \frac{k}{n}\right\}}.$$

We can write

$$\widetilde{w}\left(\frac{k-1}{n}\right) 1\!\!\mathrm{I}_{\left\{\frac{k-1}{n}\le\eta_t<\frac{k}{n}\right\}} = \left[\widetilde{w}\left(\frac{k-1}{n}\right) 1\!\!\mathrm{I}_{\left\{\frac{k-1}{n}\le\eta_t\right\}}\right]\left[1\!\!\mathrm{I}_{\left\{\eta_t<\frac{k}{n}\right\}}\right],$$

where the process in the first brackets is $\widetilde{\mathcal{F}}_t$-measurable, and the process in the second brackets is $\mathcal{F}^w_t$-measurable. Then $\zeta(t)$ is $\mathcal{M}_t$-measurable as the limit of measurable functions.

The second moment of the process $\zeta(t)$ is finite for all $t \ge 0$ because

$$\mathbb{E}\widetilde{w}(\eta_t)^2 = \mathbb{E}\mathbb{E}\left(\widetilde{w}(\eta_t)^2/\mathcal{F}^w_\infty\right) = \mathbb{E}\left(\mathbb{E}\widetilde{w}(\widehat{t})^2\right)_{\widehat{t}=\eta_t} = \mathbb{E}\left(\widehat{t}\right)_{\widehat{t}=\eta_t} = \mathbb{E}\eta_t < \infty.$$

In the second equality, we used Lemma 1 in [6, p.67].

Let us prove that, for an arbitrary bounded $\mathcal{M}_s$-measurable random value ξ, the relation $\mathbb{E}\left(\zeta(t)-\zeta(s)\right)\xi = 0$ holds for all $t \ge s$. It is enough to check this relation for the indicators of sets generating the σ-algebra $\mathcal{M}_s$. For all $k \ge 1, u_i \in [0,+\infty), \Gamma_i \in \mathcal{B}_S, 1 \le i \le k, \Gamma \in \mathcal{B}_{C[0,+\infty)}$ (the Borel σ-algebra on $C([0,+\infty))$)

$$\mathbb{E}\left[\widetilde{w}(\eta_t)-\widetilde{w}(\eta_s)\right] 1\!\!\mathrm{I}_{\left\{w(\cdot)|_0^s\in\Gamma\right\}} \prod_{i=0}^k 1\!\!\mathrm{I}_{\{\widetilde{w}(u_i)\in\Gamma_i\}} 1\!\!\mathrm{I}_{\{u_i\le\eta_s\}} =$$

$$= \mathbb{E}\left[1\!\!\mathrm{I}_{\left\{w(\cdot)|_0^s\in\Gamma\right\}}\mathbb{E}\left(\left[\widetilde{w}(\eta_t)-\widetilde{w}(\eta_s)\right]\prod_{i=0}^k 1\!\!\mathrm{I}_{\{\widetilde{w}(u_i)\in\Gamma_i\}} 1\!\!\mathrm{I}_{\{u_i\le\eta_s\}}/\mathcal{F}^w_\infty\right)\right] =$$

$$= \mathbb{E}\left[1\!\!\mathrm{I}_{\left\{w(\cdot)|_0^s\in\Gamma\right\}}\mathbb{E}\left(\left[\widetilde{w}(T_2)-\widetilde{w}(T_1)\right]\prod_{i=0}^k 1\!\!\mathrm{I}_{\{\widetilde{w}(u_i)\in\Gamma_i\}} 1\!\!\mathrm{I}_{\{u_i\le T_1\}}\right)\right]\Bigg|_{T_2=\eta_t, T_1=\eta_s} = 0,$$

here we denote the trajectory of the Wiener process $\{w(t)\}$ on $[0,s]$ by $w(\cdot)|_0^s$. Here we use the independence of the processes $\{\widetilde{w}(t)\}, \{\eta_t\}$ and Lemma 1 in [6, p.67] in the second equality.

In the same way, one can observe that the process $\{\widetilde{w}(\eta_t)^2 - \eta_t\}$ is martingale w.r.t. $\{\mathcal{M}_t\}$. This means that the characteristics of the martingale $\{\widetilde{w}(\eta_t)\}$ is equal to $\{\eta_t\}$.

Lemma is proved.

Lemma 3. *$\{w(t)\}$ is a Wiener process w.r.t. $\{\mathcal{M}_t\}$.*

Remark 1. This result is not obvious because the σ-algebra $\{\mathcal{M}_t\}$ is larger than $\{\mathcal{F}^w_t\}$ and σ-algebras $\{\mathcal{F}^w_t\}$ and $\{\widetilde{\mathcal{F}}_t\}$ are not independent.

Proof. It is enough to prove that the process $\{w(t)\}$ is a martingale with characteristic t w.r.t. $\{\mathcal{M}_t\}$. Firstly we show that, for an arbitrary bounded $\mathcal{M}_s$-measurable random value ξ, the relation $\mathbb{E}\left(w(t)-w(s)\right)\xi = 0$ holds for all $t \ge s$. Again we check this relation for the indicators of sets generating the σ-algebra $\mathcal{M}_s$. For all $k \ge 1, u_i \in [0,+\infty), \Gamma_i \in \mathcal{B}_S, 1 \le i \le k, \Gamma \in \mathcal{B}_{C[0,+\infty)}$

$$\mathbb{E}\left[w(t)-w(s)\right] 1\!\!\mathrm{I}_{\left\{w(\cdot)|_0^s\in\Gamma\right\}} \prod_{i=0}^k 1\!\!\mathrm{I}_{\{\widetilde{w}(u_i)\in\Gamma_i\}} 1\!\!\mathrm{I}_{\{u_i\le\eta_s\}} =$$

$$= \mathbb{E}\left[1\!\!\mathrm{I}_{\left\{w(\cdot)|_0^s\in\Gamma\right\}}\prod_{i=0}^k 1\!\!\mathrm{I}_{\{\widetilde{w}(u_i)\in\Gamma_i\}} 1\!\!\mathrm{I}_{\{u_i\le\eta_s\}}\mathbb{E}\left(w(t)-w(s)/\mathcal{F}^w_s \vee \mathcal{F}^{\widetilde{w}}_\infty\right)\right] = 0.$$

In the same way one can show that the process $\{w(t)^2 - t\}$ is martingale w.r.t. $\{\mathcal{M}_t\}$. This means that the characteristic of the martingale $\{w(t)\}$ is t.

Lemma is proved.

For a given $x_0 \in \Re^d$,-measurable function $\alpha : S \to S$, and operator $\beta : S \to \mathcal{L}_+(S)$ ($\mathcal{L}_+(S)$ denotes the space of all linear symmetric nonnegative operators on S), we consider the stochastic equation

$$x(t) = x_0 + \int_0^t \left(q\nu + \alpha(x^S(\tau))\right) d\eta_\tau + \int_0^t \widetilde{\beta}(x^S(\tau)) d\widetilde{w}(\eta_\tau) + w(t) \tag{1}$$

in $\Re^d$, where $\widetilde{\beta}(\cdot) = \beta^{1/2}(\cdot)$. We call the strong solution to Eq. (1) to be the $\{\mathcal{M}_t\}$-adapted process $\{x(t)\}$ which satisfies equality (1).

Theorem 1. *Assume that there exists $K > 0$ such that*

1. $\sup_{x\in S} \left(\left|\alpha(x)\right| + \left\|\widetilde{\beta}(x)\right\|\right) \le K$,
2. $\left|\alpha(x) - \alpha(y)\right|^2 + \left\|\widetilde{\beta}(x) - \widetilde{\beta}(y)\right\|^2 \le K \left|x - y\right|^2$, *for all $x, y \in S$.*

Then, for all $T > 0$, the solution to Eq. (1) exists and is unique when $t \in [0, T]$.

It is enough to prove the existence and uniqueness for the process $\{x^S(t)\}$. This, in turn, is a consequence of the existence and uniqueness theorems for the stochastic equations with arbitrary martingales and stochastic measures (see [7], p. 278-296).

Remark 2. It was proved in [8] that the solution of Eq. (1) has a measurable modification as the function of the starting point.

3. Markov property of the constructed process.

By $\{x(t, x)\}$, we denote the solution to Eq. (1) started from $x \in \Re^d$.

Theorem 2. $\{x(t, x)\}$ *has Markov property.*

Proof. We prove the theorem if we show that the relation

$$\mathbb{E} 1\!\!\mathrm{I}_{\{x(t,x)\in\Gamma\}} \xi = \mathbb{E} \Phi_{t-s}(x(s, x), \Gamma)\xi, \tag{2}$$

holds for an arbitrary bounded $\mathcal{M}_s$-measurable random value ξ, where $\Phi_{t-s}(\cdot, \Gamma) : \Re^d \to \Re$ is a measurable function for all $0 \le s \le t, \Gamma \in \mathcal{B}_{\Re^d}$.

Firstly, we note that the process $\{x(t, x)\}$ has the property

$$\theta_s x(t - s, z)|_{z=x(s,x)} = x(t, x), \quad 0 \le s \le t, \tag{3}$$

where θ_s is the shift operator (see, [10, p.121]). We can deal with the process

$$\{x(t, x(s, x))\}$$

because the process $\{x(t, x)\}$ is measurable as a function of the initial point. Equality (3) holds true, because each of the processes on both sides of (3) satisfies the equation

$$x(t, x) = x(s, x) + \int_s^t \left(q\nu + \alpha(x^S(u, x))\right) d\eta_u + \int_s^t \widetilde{\beta}(x^S(u, x)) d\widetilde{w}(\eta_u) + w(t) - w(s), \tag{4}$$

that has the unique solution.

Let us denote $(\gamma(\cdot) - \gamma(s))|_s^t = (\gamma(\cdot) - \gamma(s)) 1\!\!\mathrm{I}_{[s,t]}(\cdot)$ for an arbitrary process $\{\gamma(t)\}, s \le t$. Equalities (3) and (4) mean that we can express the process $\{x(t, x)\}$ in the form

$$x(t, x) = F(x(s, x), (w(\cdot) - w(s))|_s^t, (\eta_\cdot - \eta_s)|_s^t, (\widetilde{w}(\eta_\cdot) - \widetilde{w}(\eta_s))|_s^t), \quad 0 \le s \le t, \tag{5}$$

where $F : \Re^d \times C[0, +\infty) \times C[0, +\infty) \to \Re^d$ is a measurable functional. Therefore, we will prove (2), if we show that, for all $0 \le s \le t, \Gamma \in \mathcal{B}_{(C[0,+\infty))^3}$, the relation

$$\mathbb{E} 1\!\!\mathrm{I}_{\{((w(\cdot)-w(s))|_s^t, (\eta_\cdot - \eta_s)|_s^t, (\widetilde{w}(\eta_\cdot) - \widetilde{w}(\eta_s))|_s^t) \in \Gamma\}} \xi = \mathbb{E} \Phi^1_{t-s}(x(s, x), \Gamma)\xi, \tag{6}$$

holds true, where $\Phi^1_{t-s}(\cdot, \Gamma) : \Re^d \to \Re$ is a measurable function for all t, s, Γ.

We prove (6) in the next way. We construct $\Phi^1_{t-s}(\cdot,\cdot)$ for the indicators of sets generating the σ-algebra $\mathcal{M}_s$. We show that, for all $k\ge 1, u_i\in[0,+\infty), \widetilde{\Gamma}_i\in\mathcal{B}_S, 1\le i\le k, \Gamma_0,\Gamma_1,\Gamma_2,\Gamma_3\in\mathcal{B}_{C[0,+\infty)}$, the relation

$$\mathbb{E}1\!\!1_{\{(w(\cdot)-w(s))|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(\eta_s))|_s^t\in\Gamma_3\}}1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\prod_{i=1}^k\Big(1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}\times$$

$$\times 1\!\!1_{\{u_i\le\eta_s\}}\Big)=\mathbb{E}\Phi^2_{t-s}(x(s,x),\Gamma_1,\Gamma_2,\Gamma_3)1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le\eta_s\}}\tag{7}$$

holds, where $\Phi^2_{t-s}(\cdot,\Gamma_1,\Gamma_2,\Gamma_3):\Re^d\to\Re$ is a measurable function for all $t,s,\Gamma_1,\Gamma_2,\Gamma_3$.

Let us take the conditional expectation w.r.t. $\mathcal{F}^w_s$ on the left-hand side of relation (7). Then we obtain

$$\mathbb{E}1\!\!1_{\{(w(\cdot)-w(s))|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(\eta_s))|_s^t\in\Gamma_3\}}1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\times$$

$$\times\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le\eta_s\}}=\mathbb{E}\Big(1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\mathbb{E}\Big[1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\times$$

$$\times\ 1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}/\mathcal{F}^w_s\Big]\Big|_{z=w(s),r=\eta_s}\Big).\tag{8}$$

Let us show that the sets $\{(w(\cdot)-z)|_s^t\in\Gamma_1\}\bigcap\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}\bigcap\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}$ and $\bigcap_{i=1}^k\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}\bigcap\{u_i\le r\}$ are independent when the σ-algebra $\mathcal{F}^w_s$ is fixed. Really, for an arbitrary bounded $\mathcal{F}^w_s$-measurable random value ζ, we have

$$\mathbb{E}\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}=$$

$$=\mathbb{E}\Big(\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\mathbb{E}\Big[1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\times$$

$$\times\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}/\mathcal{F}^w_\infty\Big]\Big)=\mathbb{E}\Big(\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\times$$

$$\times\mathbb{E}\Big[1\!\!1_{\{(\widetilde{w}(r(\cdot))-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}\Big]\Big|_{r(\cdot)=\eta_\cdot}\Big)=\mathbb{E}\Big(\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}\times$$

$$\times\ 1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\mathbb{E}\Big[1\!\!1_{\{(\widetilde{w}(r(\cdot))-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\Big|_{r(\cdot)=\eta_\cdot}\Big]\mathbb{E}\Big[\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}\Big]\Big)=$$

$$=\mathbb{E}\Big(\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\mathbb{E}\Big[1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}}/\mathcal{F}^w_\infty\Big]\Big)\times$$

$$\times\mathbb{E}\left(\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}\right)=\mathbb{E}\Big(\zeta 1\!\!1_{\{(w(\cdot)-z)|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-r)|_s^t\in\Gamma_2\}}\times$$

$$\times\ 1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(r))|_s^t\in\Gamma_3\}}\Big)\mathbb{E}\left(\prod_{i=1}^k 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le r\}}\right).$$

In the third equality, we take into account the independence of the sets $\{(\widetilde{w}(r(\cdot))-\widetilde{w}(r))|_s^t\in\Gamma_3\}$ and $\bigcap_{i=1}^k\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}\bigcap\{u_i\le r\}$.

Therefore, we obtain that the expression on the right-hand side of (8) equals

$$\mathbb{E}\Big(1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\mathbb{E}\Big[1\!\!1_{\{(w(\cdot)-w(s))|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(\eta_s))|_s^t\in\Gamma_3\}}/\mathcal{F}^w_s\Big]\times$$

$$\times \mathbb{E}\left[\prod_{i=1}^{k} 1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}} 1\!\!1_{\{u_i\le\eta_s\}}/\mathcal{F}_s^w\right]\Bigg). \tag{9}$$

Let us denote the conditional joint distribution of the processes $\{(w(\cdot)-w(s))|_s^t\}$ and $\{(\eta_\cdot-\eta_s)|_s^t\}$ in the following way:

$$\mathbb{P}\left\{(w(\cdot)-w(s))|_s^t\in\Gamma_1,(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2/\mathcal{F}_s^w\right\}=\mu_{t-s}(x^\nu(s),\Gamma_1,\Gamma_2).$$

We recall that the process $\{\eta_t\}$ is an additive functional of $\{x^\nu(t)\}$, and thus the distribution of $\{(\eta_\cdot-\eta_s)|_s^t\}$ depends only on the distribution of $x^\nu(s)$. Then we have

$$\mathbb{E}\left(1\!\!1_{\{(w(\cdot)-w(s))|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(\eta_s))|_s^t\in\Gamma_3\}}/\mathcal{F}_s^w\right)=$$

$$=\int\mathbb{P}\left\{y(\cdot)\in\Gamma_1,\theta(\cdot)\in\Gamma_2,\widehat{w}(\theta(\cdot))\in\Gamma_3\right\}\mu_{t-s}(x^\nu(s),dy(\cdot),d\theta(\cdot))=$$

$$=\Phi_{t-s}^2(x^\nu(s),\Gamma_1,\Gamma_2,\Gamma_3),$$

where $\widehat{w}(\cdot)$ is a Wiener process in S.

Therefore, (9) equals

$$\mathbb{E}\left(1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\mathbb{E}\left[1\!\!1_{\{(w(\cdot)-w(s))|_s^t\in\Gamma_1\}}1\!\!1_{\{(\eta_\cdot-\eta_s)|_s^t\in\Gamma_2\}}1\!\!1_{\{(\widetilde{w}(\eta_\cdot)-\widetilde{w}(\eta_s))|_s^t\in\Gamma_3\}}/\mathcal{F}_s^w\right]\times\right.$$

$$\left.\times\mathbb{E}\left[\prod_{i=1}^{k}1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le\eta_s\}}/\mathcal{F}_s^w\right]\right)=\mathbb{E}\left(1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\Phi_{t-s}^2(x^\nu(s),\Gamma_1,\Gamma_2,\Gamma_3)\times\right.$$

$$\left.\times\mathbb{E}\left[\prod_{i=1}^{k}1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le\eta_s\}}/\mathcal{F}_s^w\right]\right)=\mathbb{E}\left(1\!\!1_{\{w(\cdot)|_0^s\in\Gamma_0\}}\Phi_{t-s}^2(x^\nu(s),\Gamma_1,\Gamma_2,\Gamma_3)\times\right.$$

$$\left.\times\prod_{i=1}^{k}1\!\!1_{\{\widetilde{w}(u_i)\in\widetilde{\Gamma}_i\}}1\!\!1_{\{u_i\le\eta_s\}}\right).$$

This proves equality (7) and then, after the standard limiting procedure, equality (6). The theorem is proved.

In the next theorem, we characterize the process constructed in Theorem 1 as a generalized diffusion process in the sense of Portenko (see [1]).

Theorem 3. *For all $\theta\in\Re^d,\varphi\in\mathcal{C}_0(\Re^d)$ (the space of all real-valued continuous functions in $\Re^d$ with compact support), the relations*

$$\lim_{t\downarrow0}\frac{1}{t}\int_{\Re^d}\varphi(z)\mathbb{E}|x(t,z)-x(0,z)|^4dz=0,$$

$$\lim_{t\downarrow0}\frac{1}{t}\int_{\Re^d}\varphi(z)\mathbb{E}(x(t,z)-x(0,z),\theta)dz=\int_S\varphi(z)(q\nu+\alpha(z),\theta)d\sigma_z$$

$$\lim_{t\downarrow0}\frac{1}{t}\int_{\Re^d}\varphi(z)\mathbb{E}(x(t,z)-x(0,z),\theta)^2dz=(\theta,\theta)\int_{\Re^d}\varphi(z)dz+\int_S(\beta(z)\theta,\theta)\varphi(z)d\sigma_z$$

hold, where $d\sigma$ is the Lebesgue measure on S.

This result means that the process $\{x(t,x)\}$ is a generalized diffusion process with the generalized drift vector being equal to $(q\nu+\alpha(x^S))\delta_S(x)$ and the generalized diffusion matrix being equal to $I+\beta(x^S)\delta_S(x)$, here I is the identity matrix in $\Re^d$, and $\delta_S(\cdot)$ is the delta-function concentrated on S.

The proof of this theorem is similar to that of the corresponding theorem in [11].

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