Theory of Stochastic Processes
Vol. 11 (27), no. 3–4, 2005, pp. 71–81

UDC 519.21

D. V. GUSAK AND E. V. KARNAUKH

ON THE EXIT FROM A FINITE INTERVAL FOR THE RISK PROCESSES WITH STOCHASTIC PREMIUMS

We consider the almost semicontinuous step-process $\xi(t)$. The conditional characteristic functions of the jumps of $\xi(t)$ have the form $\mathrm{E}\left[e^{i\alpha\xi_k}/\xi_k > 0\right] = c(c - i\alpha)^{-1}$. For such processes, the boundary functionals related to the exit from a finite interval are investigated.

The problems on the exit from a finite interval for the process $\xi(t)$ $(t \geq 0, \xi(0) = 0)$ with stationary independent increments were considered by many authors (see, for example [1, ch. IV, § 2]). In [1], the joint distributions of extrema and the distributions of values of the process up to the exit from the interval were expressed in terms of rather complicate series of the "convolutions" of

$$\Gamma^{\pm}(s, x, y) = \mathrm{E}\left[e^{-s\tau^{\pm}(\pm x)}, \gamma^{\pm}(\pm x) \leq y\right],$$

where

$$\tau^{\pm}(\pm x) = \inf\{t > 0 : \pm\xi(t) > x\}, \gamma^{\pm}(\pm x) = \pm\xi(\tau^{\pm}(\pm x)) \mp x, \ x > 0.$$

Simpler relations for the Wiener processes are established in [1, p. 463] and in [2, § 27]. In [3] - [6], the mentioned problems were investigated for semicontinuous processes $\xi(t)$ ($\xi(t)$ have jumps of one sign). For these processes, the distribution density of $\xi(t)$ up to the exit from the interval was represented [7], [8] in terms of the resolvent functions $R_s(x)$ (introduced by V.S. Korolyuk in [3]).

We consider the compound Poisson process

$$\xi(t) = \sum_{k \leq \nu(t)} \xi_k,$$

where $\nu(t)$ is the Poisson process with rate $\lambda > 0$. The distributions of ξ_k satisfy the next condition ($F(x)$ is a cumulative distribution function)

$$\text{(1)} \qquad \mathrm{P}\{\xi_k < x\} = qF(x)I\{x \leq 0\} + (1 - pe^{-cx})I\{x > 0\}, \ c > 0, \ p + q = 1.$$

The process $\xi(t)$ is the almost upper semicontinuous piecewise constant process. We can represent $\xi(t)$ as the claim surplus process $\xi(t) = C(t) - S(t)$ with the stochastic premium function

$$C(t) = \sum_{k \leq \nu_1(t)} \eta_k, \ \eta_k > 0, \ \mathrm{E}\, e^{i\alpha\eta_k} = \frac{c}{c - i\alpha}, \ c > 0,$$

and with the process of claims $S(t) = \sum_{k \leq \nu_2(t)} \xi'_k$, $\xi'_k > 0$. Here, $\nu_1(t)$, $\nu_2(t)$ are the independent Poisson processes with rates λ_1, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = \lambda$ (for details, see [8]).

2000 *AMS Mathematics Subject Classification.* Primary 60G50; Secondary 60K10.
Key words and phrases. Almost semicontinuous processes, risk process with stochastic premiums, functionals connected with the exit from an interval.

Note that $C(t) \to 0$ and $\xi(t) \to -S(t)$ as $c \to \infty$, where $-S(t)$ is a non-increasing process.

Let $C_c(t)$ be the process with the cumulant

$$\psi_c(\alpha) = \lambda_c \left(\frac{c}{c - i\alpha} - 1 \right), \quad \lambda_c = ac, \ a > 0,$$

then $\psi_c(\alpha) \underset{c\to\infty}{\longrightarrow} i\alpha a$, consequently $C_c(t) \underset{c\to\infty}{\longrightarrow} at$, and $\xi_c(t) = C_c(t) - S(t) \to \xi^0(t) = at - S(t)$, where the limit process $\xi^0(t)$ is the classical upper semicontinuous risk process with the non-stochastic premium function $C(t) = at$.

Let θ_s be the exponentially distributed random variable ($\mathrm{P}\{\theta_s > t\} = e^{-st}; \ s, t > 0$). Then the randomly stopped process $\xi(\theta_s)$ has the characteristic function (ch.f.)

$$\varphi(s, \alpha) = \mathrm{E} e^{i\alpha\xi(\theta_s)} = \frac{s}{s - \psi(\alpha)},$$

where

$$\psi(\alpha) = \lambda p(c(c - i\alpha)^{-1} - 1) + \lambda q(\varphi(\alpha) - 1), \ \varphi(\alpha) = \int_{-\infty}^{0} e^{i\alpha x} dF(x). \tag{2}$$

Let us denote the first exit time from the interval $(x - T, x)$, $0 < x < T$, $T > 0$:

$$\tau(x, T) = \inf \{ t > 0 : \xi(t) \notin (x - T, x) \},$$

and the events

$$A_+(x) = \{\omega : \xi(\tau(x, T)) \geq x\}, \ A_-(x) = \{\omega : \xi(\tau(x, T)) \leq x - T\}.$$

Then

$$\tau(x, T) \doteq \begin{cases} \tau^+(x, T) = \tau^+(x), \ \omega \in A_+(x); \\ \tau^-(x, T) = \tau^-(x - T), \ \omega \in A_-(x). \end{cases}$$

Overshoots at the moments of the exit from the interval are denoted by the following relations:

$$\gamma_T^-(x) = x - T - \xi(\tau^-(x, T)), \ \gamma_T^+(x) = \xi(\tau^+(x, T)) - x.$$

The main task of our paper is the finding of the following moment generating functions (m.g.f.) of the functionals connected with the exit from the interval:

$$\begin{aligned}
Q(T, s, x) &= \mathrm{E}\, e^{-s\tau(x,T)}, \\
Q^T(s, x) &= \mathrm{E} \left[e^{-s\tau^+(x,T)}, \ A_+(x) \right], \\
Q_T(s, x) &= \mathrm{E} \left[e^{-s\tau^-(x,T)}, \ A_-(x) \right], \\
V^\pm(s, \alpha, x, T) &= \mathrm{E} \left[e^{i\alpha\gamma_T^\pm(x) - s\tau^\pm(x,T)}, \ A_\pm(x) \right], \\
V_\pm(s, \alpha, x, T) &= \mathrm{E} \left[e^{i\alpha\xi(\tau^\pm(x,T)) - s\tau^\pm(x,T)}, \ A_\pm(x) \right], \\
V(s, \alpha, x, T) &= \mathrm{E} \left[e^{i\alpha\xi(\theta_s)}, \ \tau(x, T) > \theta_s \right],
\end{aligned}$$

Let us denote the extrema $\xi^\pm(t) = \sup\limits_{0\leq s\leq t} (\inf) \xi(s)$, $\xi^\pm = \sup\limits_{0\leq s<\infty} (\inf) \xi(s)$, the joint distribution of $\{\xi(\theta_s), \xi^+(\theta_s), \xi^-(\theta_s)\}$:

$$\begin{aligned}
H_s(T, x, y) &= \mathrm{P} \left\{ \xi(\theta_s) < y, \xi^+(\theta_s) < x, \xi^-(\theta_s) > x - T \right\} \\
&= \mathrm{P} \left\{ \xi(\theta_s) < y, \tau(x, T) > \theta_s \right\},
\end{aligned}$$

and

$$P_{\pm}(s,x) = \mathrm{P}\left\{\xi^{\pm}(\theta_s) < x\right\},\ x \gtrless 0,\ p_{\pm}(s) = \mathrm{P}\left\{\xi^{\pm}(\theta_s) = 0\right\},\ q_{\pm}(s) = 1 - p_{\pm}(s);$$

$$\varphi_{\pm}(s,\alpha) = \pm \int_0^{\pm\infty} e^{i\alpha x} dP_{\pm}(s,x),$$

$$T^{\pm}(s,x) = \mathrm{E}\left[e^{-s\tau^{\pm}(x)}, \tau^{\pm}(x) < \infty\right],\ x \gtrless 0.$$

Lemma 1. *For the process $\xi(t)$ with cumulant (2), the main factorization identity is represented by the relations*

$$\varphi(s,\alpha) = \varphi_+(s,\alpha)\varphi_-(s,\alpha),\ \Im\alpha = 0; \tag{3}$$

$$\varphi_+(s,\alpha) = \frac{p_+(s)(c - i\alpha)}{\rho_+(s) - i\alpha}, \tag{4}$$

where $\rho_+(s) = cp_+(s)$ is the positive root of Lundberg's equation $\psi(-ir) = s$, $s > 0$.

$$\mathrm{P}\left\{\xi^+(\theta_s) > x\right\} = T^+(s,x) = q_+(s)e^{-c\rho_+(s)x}, x > 0. \tag{5}$$

If $m > 0$:

$$\lim_{s\to 0} \rho_+(s)s^{-1} = \rho_+'(0) = m^{-1},\ \lim_{s\to 0} P_-(s,x) = \mathrm{P}\left\{\xi^- < x\right\},\ x < 0. \tag{6}$$

If $m < 0$:

$$\lim_{s\to 0} \rho_+(s) = \rho_+ > 0;\ \lim_{s\to 0} s^{-1}\mathrm{P}\left\{\xi^-(\theta_s) > x\right\} = \mathrm{E}\,\tau^-(x),\ x < 0. \tag{7}$$

If $\sigma_1^2 = D\xi(1) < \infty$ and $m = \lambda\left(pc^{-1} - q\widetilde{F}(0)\right) = 0$ $\left(\widetilde{F}(0) = \int_{-\infty}^0 F(x)dx\right)$, then

$$\begin{aligned} &\lim_{s\to 0} \rho_+(s)s^{-1/2} = \frac{\sqrt{2}}{\sigma_1};\ \lim_{s\to 0} s^{-1/2}P_-'(s,x) = f_0(x),\ x < 0, \\ &f_0(x) = k_0\frac{\partial}{\partial x}\left(\int_0^\infty \mathrm{P}\left\{\widetilde{\xi}_0(t) < x\right\}dt\right) = -k_0\frac{\partial}{\partial x}\mathrm{E}\,\tau_0(x), x < 0; \end{aligned} \tag{8}$$

where $k_0 = c\sigma_1\left(\sqrt{2}\right)^{-1}$, $\tau_0(x) = \inf\left\{t > 0 : \widetilde{\xi}_0(t) < x\right\}$, $x < 0$; $\widetilde{\xi}_0(t)$ is the decreasing process with the spectral measure

$$\Pi_0(dx) = \lambda q\left(cF(x)dx + dF(x)\right),\ x < 0.$$

Proof. Relations (3) - (7) were proved in [7] - [8]. If $m = 0$ $\left(p = cq\widetilde{F}(0)\right)$, then

$$\varphi(s,\alpha) = \frac{s(c - i\alpha)}{s(c - i\alpha) - i\alpha\lambda(p - q\widetilde{F}(\alpha)(c - i\alpha))},\ \widetilde{F}(\alpha) = \int_{-\infty}^0 e^{i\alpha x}F(x)dx.$$

On the basis of the factorization identity (3) as $s \to 0$, we get

$$\frac{1}{\sqrt{s}}\varphi_-(s,\alpha) = \frac{\sqrt{s}}{p_+(s)}\frac{\rho_+(s) - i\alpha}{s(c - i\alpha) - i\alpha\lambda\left(p - q\widetilde{F}(\alpha)(c - i\alpha)\right)} \to \widetilde{f}_0(\alpha),$$

$$\widetilde{f}_0(\alpha) = \frac{c\sigma_1}{\sqrt{2}}\frac{1}{-\lambda q\left[\left(\widetilde{F}(\alpha) - \widetilde{F}(0)\right)c + \varphi(\alpha) - 1\right]} = \frac{c\sigma_1}{\sqrt{2}}\frac{1}{-\widetilde{\psi}_0(\alpha)},$$

$$\widetilde{\psi}_0(\alpha) = \int_{-\infty}^0 \left(e^{i\alpha x} - 1\right)\Pi_0(dx),\ \Pi_0(dx) = \lambda q\left(cF(x)dx + dF(x)\right),\ x < 0.$$

Let's denote

$$\varphi_0(s,\alpha) = \mathrm{E}\, e^{i\alpha\widetilde{\xi}_0(\theta_s)} = \frac{s}{s-\widetilde{\psi}_0(\alpha)},$$

where $\widetilde{\xi}_0(t)$ is the decreasing process with the cumulant $\widetilde{\psi}_0(\alpha)$. Since

$$\frac{c\sigma_1}{\sqrt{2}}\varphi_0(s,\alpha)s^{-1} \to \widetilde{f}_0(\alpha) = \int_{-\infty}^{0} e^{i\alpha x} f_0(x)dx,\ s\to 0,$$

we get that

$$f_0(x) = k_0 \frac{\partial}{\partial x}\left(\int_0^\infty \mathrm{P}\left\{\widetilde{\xi}_0(t) < x\right\} dt\right),$$

or

$$-f_0(x) = k_0 \frac{\partial}{\partial x}\int_0^\infty \mathrm{P}\left\{\tau_0(x) > t\right\} dt = k_0 \frac{\partial}{\partial x}\mathrm{E}\,\tau_0(x),\ x<0.$$

Let's introduce the set of boundary functions on the interval $I \subset (-\infty,\infty)$

$$\mathfrak{L}(I) = \left\{G(x) : \int_I |G(x)|dx < \infty\right\}$$

and the set of integral transforms

$$\mathfrak{R}^0(I) = \left\{g^0(\alpha) : g^0(\alpha) = C + \int_I e^{i\alpha x} G(x)dx\right\}.$$

Let's denote the projection operations on $\mathfrak{R}^0((-\infty,\infty))$ by the following relations:

$$\begin{gathered}\left[g^0(\alpha)\right]_I = \int_I e^{i\alpha x}G(x)dx,\ \ \left[g^0(\alpha)\right]_I^0 = C + \int_I e^{i\alpha x}G(x)dx,\\ \left[g^0(\alpha)\right]_- = \left[g^0(\alpha)\right]_{(-\infty,0)},\ \left[g^0(\alpha)\right]_+ = \left[g^0(\alpha)\right]_{(0,\infty)}.\end{gathered}$$

The main results of our paper are included in the following two assertions.

Theorem 1. *For the process $\xi(t)$ with cumulant* (2), *$Q^T(s,x)$ has the form* $(0<x<T)$

$$\begin{aligned}(9)\quad Q^T(s,x) = q_+(s)e^{-\rho_+(s)x}\int_{x-T}^{0} e^{\rho_+(s)y}dP_-(s,y)\times\\ \times\left[e^{-\rho_+(s)T}\int_{-\infty}^{-T} e^{c(T+y)}dP_-(s,y) + \int_{-T}^{0} e^{\rho_+(s)y}dP_-(s,y)\right]^{-1}.\end{aligned}$$

Theorem 2. *For the process $\xi(t)$ with cumulant* (2), *the joint distributions of $\left\{\tau^+(x,T),\gamma_T^+(x)\right\}$ and $\{\tau^+(x,T),\xi(\tau^+(x,T))\}$ are determined by the relations*

$$(10)\qquad \begin{cases} V^+(s,\alpha,x,T) = \dfrac{c}{c-i\alpha}Q^T(s,x),\ 0<x<T,\\ V_+(s,\alpha,x,T) = e^{i\alpha x}V^+(s,\alpha,x,T) = \dfrac{c\,e^{i\alpha x}}{c-i\alpha}Q^T(s,x).\end{cases}$$

The ch.f. of $\xi(\theta_s)$ before the exit time from the interval has the form

$$\begin{aligned}(11)\quad V(s,\alpha,x,T) &= \varphi_+(s,\alpha)\left[\varphi_-(s,\alpha)\left(1-V_+(s,\alpha,x,T)\right)\right]_{[x-T,\infty)}\\ &= \varphi_+(s,\alpha)\left[\varphi_-(s,\alpha)\left(1-c\,e^{i\alpha x}(c-i\alpha)^{-1}Q^T(s,x)\right)\right]_{[x-T,\infty)},\end{aligned}$$

the corresponding distribution has the density $(x-T<z<x,\ z\neq 0)$

$$(12)\quad h_s(T,x,z)=\frac{\partial}{\partial z}H_s(T,x,z)=$$
$$=\left(p_+(s)P'_-(s,z)-q_+(s)\rho_+(s)\int_z^0 e^{\rho_+(s)(y-z)}dP_-(s,y)\right)I\{z<0\}+$$
$$+\rho_+(s)Q^T(s,x)\int_{z-x}^0 e^{\rho_+(s)(y-(z-x))}dP_-(s,y),$$

and the following atomic probability

$$\mathrm{P}\{\xi(\theta_s)=0,\tau(x,T)>\theta_s\}=\mathrm{P}\{\xi(\theta_s)=0\}=p_-(s)p_+(s)=\frac{s}{s+\lambda}.$$

Proof of Theorem 1. From the stochastic relations for $\tau^+(x,T)$, $\gamma_T^+(x)$ ($\xi=\xi_1$ has the cumulative distribution function $F_1(x)$, ζ is the moment of the first jump of $\xi(t)$),

$$\tau^+(x,T)\doteq\begin{cases}\zeta,\ \xi>x,\\ \zeta+\tau^+(x-\xi,T),\ x-T<\xi<x,\end{cases}$$
$$\gamma_T^+(x)\doteq\begin{cases}\xi-x,\ \xi>x,\\ \gamma_T^+(x-\xi),\ x-T<\xi<x,\end{cases}$$

we have the following equation for $V^+(s,\alpha,x)=V^+(s,\alpha,x,T)$:

$$(13)\qquad (s+\lambda)V^+(s,\alpha,x)=\frac{\lambda pc}{c-i\alpha}e^{-cx}+\lambda\int_{x-T}^{x}V^+(s,\alpha,x-z)dF_1(z),\ 0<x<T.$$

If $\alpha=0$, then, from (13), we obtain the equation for $Q^T(s,x)$

$$(14)\qquad (s+\lambda)Q^T(s,x)=\lambda pe^{-cx}+\lambda\int_{x-T}^{x}Q^T(s,x-z)dF_1(z),\ 0<x<T.$$

Since $\mathrm{P}(A_+(x))=1$ for $x<0$, we have the boundary conditions

$$Q^T(s,x)=\begin{cases}0,\ x>T,\\ 1,\ x<0.\end{cases}$$

After the replacement
$$\overline{Q}^{\,T}(s,x)=1-Q^T(s,x)$$
, relation (14) yields the equation for $\overline{Q}^{\,T}(s,x)$ $(0<x<T)$

$$(s+\lambda)\overline{Q}^{\,T}(s,x)=s+\lambda F(x-T)+\lambda\int_0^T\overline{Q}^{\,T}(s,z)F_1'(x-z)dz,$$

which, after prolonging for $x>0$, has the form

$$(15)\qquad (s+\lambda)\overline{Q}^{\,T}(s,x)=sC(x)+\lambda\int_{-\infty}^{\infty}\overline{Q}^{\,T}(s,z)F_1'(x-z)dz+C_T^{>}(s,x),$$
$$C(x)=I\{x>0\},\ C_T^{>}(s,x)=\overline{C}_T(s)e^{-cx},\ x>0,$$

$$(16)\qquad \overline{C}_T(s)=\lambda p\left[e^{cT}-c\overline{Q}_s^{\,*}(T)\right],\ \overline{Q}_s^{\,*}(T)=\int_0^T e^{cx}\overline{Q}^{\,T}(s,x)dx.$$

Let's introduce the function $C_\epsilon(x)=e^{-\epsilon x}C(x)$, $x>0$, and consider, instead of (15), the equation for $Y_\epsilon(T,s,x)$ $(\epsilon>0)$:

$$(17)\qquad (s+\lambda)Y_\epsilon(T,s,x)=sC_\epsilon(x)+\lambda\int_{-\infty}^{\infty}Y_\epsilon(T,s,x-z)dF_1(z)+C_T^{>}(s,x),\ x>0.$$

Denote

$$y_\epsilon(T,s,\alpha)=\int_0^\infty e^{i\alpha x}Y_\epsilon(T,s,x)dx,\ \widetilde{C}_\epsilon(\alpha)=\int_0^\infty e^{i\alpha x}C_\epsilon(x)dx,$$
$$\widetilde{C}_T(s,\alpha)=\int_0^\infty e^{i\alpha x}C_T^{>}(s,x)dx.$$

By performing the integral transformation of (17), we obtain the equation

$$(s-\psi(\alpha))y_\epsilon(T,s,\alpha)=s\widetilde{C}_\epsilon(\alpha)+\widetilde{C}_T(s,\alpha)-[y_\epsilon(\alpha)\varphi(\alpha)]_-$$

or

$$sy_\epsilon(T,s,\alpha)\varphi^{-1}(s,\alpha)=s\widetilde{C}_\epsilon(\alpha)+\widetilde{C}_T(s,\alpha)-[y_\epsilon(\alpha)\varphi(\alpha)]_-\,. \tag{18}$$

After using the factorization decomposition (3) and the projection operation $[\,]_+$, relation (18) yields

$$sy_\epsilon(T,s,\alpha)\varphi_+^{-1}(s,\alpha)=\Big[\varphi_-(s,\alpha)\Big(s\widetilde{C}_\epsilon(\alpha)+\widetilde{C}_T(s,\alpha)\Big)\Big]_+$$

or

$$sy_\epsilon(T,s,\alpha)=\varphi_+(s,\alpha)\Big[\varphi_-(s,\alpha)\Big(s\widetilde{C}_\epsilon(\alpha)+\widetilde{C}_T(s,\alpha)\Big)\Big]_+\,. \tag{19}$$

By inverting relation (19), we obtain

$$sY_\epsilon(T,s,x)=s\int_0^x B_\epsilon(x-y)dP_+(s,y)+\int_0^x B(s,x-y,T)dP_+(s,y), \tag{20}$$
$$B_\epsilon(x)=\int_{-\infty}^x e^{-\epsilon(x-y)}dP_-(s,y)=\int_{-\infty}^0 e^{-\epsilon(x-y)}dP_-(s,y)=$$
$$=e^{-\epsilon x}\mathrm{E}\,e^{\epsilon\xi^-(\theta_s)},$$
$$B(s,x,T)=\overline{C}_T(s)\int_{-\infty}^{x-T}e^{-c(x-y)}dP_-(s,y),\ x>0.$$

Taking into account that $C_\epsilon(x)\to I\{x>0\}$ as $\epsilon\to 0$, $Y_\epsilon(T,s,x)\to\overline{Q}^{\,T}(s,x)$ as $\epsilon\to 0$, $0<x<T$. So Eq. (20) yields

$$s\overline{Q}^{\,T}(s,x)=sP_+(s,x)+p_+(s)B(s,x,T)+\int_{+0}^x B(s,x-z,T)P_+'(s,z)dz.$$

Taking into account that

$$\begin{aligned}
q_+(s)\rho_+(s)&\int_0^x\int_{-\infty}^{z-T}e^{-c(z-y)}dP_-(s,y)e^{-\rho_+(s)(x-z)}dz=\\
=&q_+(s)\rho_+(s)\int_{-\infty}^{x-T}e^{-\rho_+(s)x+cy}dP_-(s,y)\int_{\max(0,y+T)}^x e^{-cq_+(s)z}dz\\
=&p_+(s)\Big[\int_{-\infty}^{-T}e^{cy-\rho_+(s)x}dP_-(s,y)+\\
&+\int_{-T}^{x-T}e^{\rho_+(s)(y+T-x)-cT}dP_-(s,y)-\int_{-\infty}^{x-T}e^{-c(x-y)}dP_-(s,y)\Big],
\end{aligned}$$

we have

$$\begin{aligned}
s\overline{Q}^{\,T}(s,x)=sP_+(s,x)+p_+(s)\overline{C}_T(s)e^{-\rho_+(s)x}\times\\
\times\Big[\int_{-\infty}^{-T}e^{cy}dP_-(s,y)+\int_{-T}^{x-T}e^{-cT+\rho_+(s)(y+T)}dP_-(s,y)\Big].
\end{aligned}$$

From the last equation, we can find $\overline{C}_T(s)$ and $\overline{Q}\,^*_s(T)$ and then get (9).

Let's note that $Q^T(s,x) \to \overline{P}_+(s,x)$ as $T \to \infty$ and $Q^T(s,x) \to 0$ as $c \to \infty$. If we consider, instead of $\xi(t)$, the process $\xi_c(t) = C_c(t) - S(t)$, then relation (9) yields

$$Q_c^T(s,x) = q_+^c(s)\mathrm{E}\left[e^{\rho_+^c(s)(\xi_c^-(\theta_s)+T-x)}, \xi_c^-(\theta_s)+T-x>0\right] \times$$
$$\times \left(\mathrm{E}\left[e^{c(\xi_c^-(\theta_s)+T)}, \xi_c^-(\theta_s)+T<0\right] + \mathrm{E}\left[e^{\rho_+^c(s)(\xi_c^-(\theta_s)+T)}, \xi_c^-(\theta_s)+T>0\right]\right)^{-1}.$$

Taking into account that, for $x>0$, $\mathrm{P}\{\xi_c^+(\theta_s)>x\} = q_+^c(s)e^{-\rho_+^c(s)x} \underset{c\to\infty}{\longrightarrow} e^{-\rho_0^+(s)x}$, where $\rho_0^+(s)$ is the positive solution of the equation

$$\psi^0(-ir) := ar - \lambda_2\left(\int_{-\infty}^0 e^{rx}dF(x) - 1\right) = 0,$$

we get $Q_c^T(s,x) \to Q_\infty^T(s,x)$ as $c\to\infty$. If we denote

$$\xi_\pm^0(t) = \sup_{0\le u\le t}(\inf)\xi^0(u),$$

then

$$\begin{aligned}
Q_\infty^T(s,x) &= \mathrm{E}\left[e^{\rho_0^+(s)(\xi_-^0(\theta_s)+T-x)}, \xi_-^0(\theta_s)+T-x>0\right] \times \\
&\times \left(\mathrm{E}\left[e^{\rho_0^+(s)(\xi_-^0(\theta_s)+T)}, \xi_-^0(\theta_s)+T>0\right]\right)^{-1} \\
&= \int_0^{T-x} e^{\rho_0^+(s)(T-x-y)}d\mathrm{P}\left\{-\xi_-^0(\theta_s)<y\right\} \times \\
&\left(\int_0^T e^{\rho_0^+(s)(T-y)}d\mathrm{P}\left\{-\xi_-^0(\theta_s)<y\right\}\right)^{-1} \\
&= R_s(T-x)R_s^{-1}(T),
\end{aligned}$$

where the last relation is the well-known formula (see [3]) for the upper semicontinuous processes.

Proof of Theorem 2. The first relation in (10) follows from Eqs. (13) and (14). The second relation follows from the first one. The first equality in (11) was proved in [9]. After inverting (11), we get

(21)
$$\begin{aligned}
h_s(T,x,z) =& p_+(s)\frac{\partial}{\partial z}P_-(s,z)I\{z<0\} + q_+(s)\rho_+(s)\int_{x-T}^{\min\{z,0\}} e^{-\rho_+(s)(z-y)}dP_-(s,y) - \\
&- Q^T(s,x)\Big[p_+(s)\frac{\partial}{\partial z}\mathrm{P}\left\{\xi^-(\theta_s)+\theta_c'+x\le z\right\} + \\
&+ q_+(s)\rho_+(s)\int_{x-T}^z e^{-\rho_+(s)(z-y)}dP\left\{\xi^-(\theta_s)+\theta_c'+x<z\right\}\Big].
\end{aligned}$$

Using the integral transformation of (21) with respect to the distribution of θ_c', we get formula (12).

Corollary 1. *For the joint distribution* $\{\tau^-(x,T), \xi(\tau^-(x,T))\}$, *we have*

(22) $$s\mathrm{E}\left[e^{-s\tau^-(x,T)}, \xi(\tau^-(x,T))<z,\ A_-(x)\right] = \int_{x-T}^x \Pi_-(z-y)dH_s(T,x,y),\ z\le x-T,$$

where $H_s(T,x,y)$ *is determined by its density* (12) *and* $\Pi_-(x) = \int_{-\infty}^x \Pi(dy),\ x<0.$

The probability of the lack of exit (non-exit) from the interval $(x-T,x)$ *has the form*

$$\text{(23)}\quad \mathrm{P}\left\{\tau(x,T)>\theta_s\right\}=\mathrm{P}\left\{\xi^-(\theta_s)>x-T\right\}-\\ -Q^T(s,x)\left[\int_{-\infty}^{-T}e^{c(z+T)}dP_-(s,z)+\mathrm{P}\left\{\xi^-(\theta_s)>-T\right\}\right].$$

The m.g.f. for $\tau(x,T)$ *and* $\tau^-(x,T)$ *are determined in the following way:*

$$\text{(24)}\qquad \begin{cases} Q(T,s,x)=1-\mathrm{P}\left\{\tau(x,T)>\theta_s\right\},\ 0<x<T,\\ Q_T(s,x)=Q(T,s,x)-Q^T(s,x),\ 0<x<T.\end{cases}$$

Proof. Formula (22) follows from [6, Theorem 7.3]. By substituting (12) in (22), we obtain the relation in terms of $Q^T(s,x)$ and the truncated distribution of $\xi^-(\theta_s)+\theta'_c$. Taking into account that

$$\mathrm{P}\left\{\tau(x,T)>\theta_s\right\}=\int_{x-T}^{x}dH_s(T,x,z)=\\ =\mathrm{P}\left\{\xi^-(\theta_s)>x-T\right\}-q_+(s)\int_{x-T}^{0}e^{\rho_+(s)(y-(x-T))}dP_-(s,z)+\\ +Q^T(s,x)\left[\int_{-T}^{0}e^{\rho_+(s)(z+T)}dP_-(s,z)-\mathrm{P}\left\{\xi^-(\theta_s)>-T\right\}\right],$$

and using formula (9), we obtain (23) after some simple transformations. Substituting (23) into the first relation of (24), we find the m.g.f. of $\tau(x,T)$, and then we can get the m.g.f. of $\tau^-(x,T)$ (see the second relation in (24)).

On the basis of formulas (6) - (8), we can get the following statement about the limit behavior of $Q^T(s,x)$ and $h_s(T,x,z)$ as $s\to 0$.

Corollary 2. *The function* $h'_0(T,x,z)=\lim_{s\to 0}s^{-1}h_s(T,x,z)$ $(x-T<z<x,\ z\neq 0,\ 0<x<T)$ *according to the sign of* m *has the following forms:*
if $m>0$

$$\text{(25)}\quad h'_0(T,x,z)=\frac{1}{m}\left(c^{-1}\frac{\partial}{\partial z}\mathrm{P}\left\{\xi^-<z\right\}-\mathrm{P}\left\{\xi^->z\right\}\right)I\left\{z<0\right\}+\\ +\frac{1}{m}Q^T(x)\mathrm{P}\left\{\xi^->z-x\right\};$$

if $m<0$

$$\text{(26)}\quad h'_0(T,x,z)=\left(-p_+\frac{\partial}{\partial z}\mathrm{E}\,\tau^-(z)+q_+\rho_+\int_z^0 e^{\rho_+(y-z)}d\mathrm{E}\,\tau^-(y)\right)I\left\{z<0\right\}-\\ -Q^T(x)\rho_+\int_{z-x}^{0}e^{\rho_+(y-(z-x))}d\mathrm{E}\,\tau^-(y);$$

if $m=0$

$$\text{(27)}\quad h'_0(T,x,z)=\left(-\frac{\partial}{\partial z}\mathrm{E}\,\tau_0(z)-c\lambda^{-1}+c\int_z^0\frac{\partial}{\partial y}\mathrm{E}\,\tau_0(y)dy\right)I\left\{z<0\right\}+\\ +cQ^T(x)\left(\lambda^{-1}-\int_{z-x}^{0}\frac{\partial}{\partial y}\mathrm{E}\,\tau_0(y)dy\right).$$

The ruin probability

$$Q^T(x)=\lim_{s\to 0}Q^T(s,x)$$

(*according to the sign of* m) *is determined from* (9) *in the following way:*

(28)

$$Q^T(x)=\begin{cases}\displaystyle\int_{x-T}^{0} d\mathrm{P}\left\{\xi^- < y\right\}\times \\ \displaystyle\qquad\times\left[\int_{-\infty}^{-T} e^{c(T+y)} d\mathrm{P}\left\{\xi^- < y\right\} + \int_{-T}^{0} d\mathrm{P}\left\{\xi^- < y\right\}\right]^{-1}, \ m>0, \\ \displaystyle q_+ e^{-\rho_+ x}\left(\frac{1}{\lambda p_+} - \int_{x-T}^{0} e^{\rho_+ y}\frac{\partial}{\partial y}\mathrm{E}\,\tau^-(y)dy\right)\times \\ \displaystyle\times\left[\frac{1}{\lambda p_+} - e^{-\rho_+ T}\int_{-\infty}^{-T} e^{c(T+y)}\frac{\partial}{\partial y}\mathrm{E}\,\tau^-(y)dy - \right. \\ \displaystyle\qquad\qquad \left. - \int_{-T}^{0} e^{\rho_+ y}\frac{\partial}{\partial y}\mathrm{E}\,\tau^-(y)dy\right]^{-1}, \ m<0, \\ \displaystyle\left(\lambda^{-1} - \int_{x-T}^{0}\frac{\partial}{\partial y}\mathrm{E}\,\tau_0(y)dy\right)\times \\ \displaystyle\quad\times\left[\lambda^{-1} - \int_{-\infty}^{-T} e^{c(T+y)}\frac{\partial}{\partial y}\mathrm{E}\,\tau_0(y)dy - \int_{-T}^{0}\frac{\partial}{\partial y}\mathrm{E}\,\tau_0(y)dy\right]^{-1}, \ m=0.\end{cases}$$

The distribution of $\xi(\tau^-(x,T))$ *has the form*

$$\begin{aligned}\mathrm{P}\left\{\xi(\tau^-(x,T)) < z, \ A_-(x)\right\} =& \frac{1}{\lambda}\Pi_-(z) + \int_{x-T}^{0-}\Pi_-(z-y)h_0'(T,x,y)dy + \\ &+ \int_{0+}^{x}\Pi_-(z-y)h_0'(T,x,y)dy, \ z<x-T.\end{aligned} \tag{29}$$

Corollary 3. *For the process* $\xi(t)$ *with the cumulant function*

$$\psi(\alpha)=\lambda p(c(c-i\alpha)^{-1}-1)+\lambda q(b(b+i\alpha)^{-1}-1), \tag{30}$$

$Q^T(x)$ *is represented in the following way* ($0<x<T$)*:*

(31)

$$Q^T(x)=\begin{cases}\left(1-q_- e^{\rho_-(x-T)}\right)\left(1-q_- c\left(c+\rho_-\right)^{-1}e^{-\rho_- T}\right)^{-1}, \ m>0, \\ q_+ e^{-\rho_+ x}\left(1-b(\rho_+ +b)^{-1}e^{\rho_+(x-T)}\right)\left(1-b(\rho_+ +b)^{-1}q_+ e^{-\rho_+ T}\right)^{-1}, \ m<0, \\ \dfrac{c(1+b(T-x))}{b+c+bcT}, \ m=0.\end{cases}$$

If $\xi(t)$ *is a symmetric process* ($p=q=1/2$, $b=c$)*, then*

$$Q^T(x)=\frac{1+c(T-x)}{2+cT}, \ Q_T(x)=\frac{1+cx}{2+cT}, \quad (0<x<T).$$

Proof. Let's note that the process with cumulant (30) is the almost upper and lower semicontinuous process. Then, in addition to relations (4) - (5), we have

$$\varphi_-(s,\alpha)=\frac{p_-(s)(b+i\alpha)}{\rho_-(s)+i\alpha}, \tag{32}$$

where $-\rho_-(s)=-bp_-(s)$ is the negative root of the equation $\psi(-ir)=s$, $s>0$,

$$\mathrm{P}\left\{\xi^-(\theta_s)<x\right\}=T^-(s,x)=q_-(s)e^{\rho_-(s)x}, x<0. \tag{33}$$

If $m > 0$, then

$$\mathrm{P}\left\{\xi^-(\theta_s) < x\right\} \underset{s\to 0}{\longrightarrow} \mathrm{P}\left\{\xi^- < x\right\} = q_- e^{bp_- x},\ x < 0, \quad p_-(s) \underset{s\to 0}{\longrightarrow} p_- > 0. \tag{34}$$

Taking into account that $p_+(s)p_-(s) = s\,(s+\lambda)^{-1}$, we have, for $m < 0$, $q_-'(s) = -p_-'(s) \to -(\lambda p_+)^{-1}$ as $s \to 0$. Hence,

$$\mathrm{E}\,\tau^-(x) = -\frac{\partial}{\partial s} T^-(s,x)|_{s=0} = \frac{1-bx}{\lambda p_+},\ x < 0. \tag{35}$$

If $m = 0$, then we have, for $\widetilde{\xi}_0(t)$, $\Pi_0(dx) = \lambda_0 b e^{bx} dx$, $x < 0$, $\lambda_0 = \lambda q(c+b)b^{-1}$. Moreover,

$$\widetilde{\xi}_0^-(t) = \widetilde{\xi}_0(t),\ p_-^0(s) = \mathrm{P}\left\{\widetilde{\xi}_0(\theta_s) = 0\right\} = \frac{s}{s+\lambda_0}.$$

Hence, the m.g.f. of $\tau_0(x)$ has the form

$$T_0^-(s,x) = \mathrm{E}\,e^{-s\tau_0(x)} = q_-^0(s) e^{bp_-^0(s)x},\ x < 0.$$

Since $(p_-^0)'(s) = -(q_-^0)'(s) \to \lambda_0^{-1}$ as $s \to 0$, we get

$$\mathrm{E}\,\tau_0(x) = -\frac{\partial}{\partial s}\, T_0^-(s,x)\big|_{s=0} = \frac{1-bx}{\lambda_0},\ x < 0. \tag{36}$$

Substituting formulas (34) - (36) into the corresponding relations of (28), we get (31).

Remark. We should note that it is easy to get the representation of the m.g.f. of the functionals related to the exit from the interval for the almost lower semicontinuous process $\eta(t)$ (with the parameter $b > 0$, by considering that $\xi(t) = -\eta(t)$). Particularly,

$$Q_T(s,x) = q_-(s)\int_0^x e^{\rho_-(s)(x-y)} dP_+(s,y) \times \\ \times \left[\int_T^\infty e^{b(T-y)} dP_+(s,y) + \int_0^T e^{\rho_-(s)(T-y)} dP_+(s,y)\right]^{-1}.$$

Let $\xi(t)$ be the almost upper semicontinuous piecewise constant process. Then $\xi_1(t) = at + \xi(t)$, $a < 0$, is the almost upper semicontinuous piecewise linear process. For the process $\xi_1(t)$ on the basis of the stochastic relations for $\tau^+(x,T)$,

$$\tau^+(x,T) \doteq \begin{cases} \zeta,\ \xi + a\zeta > x, \\ \zeta + \tau^+(x - \xi - a\zeta),\ x - T < \xi + a\zeta < x, \end{cases}$$

we have the integro-differential equation for $Q^T(s,x)$

$$a\frac{\partial}{\partial x} Q^T(s,x) = \lambda \int_{x-T}^x Q^T(s,x-z)\,dF_1(z) - (s+\lambda)Q^T(s,x) + \lambda p e^{-cx},\ 0 < x < T.$$

Introducing the function $\overline{Q}^{\,T}(s,x) = 1 - Q^T(s,x)$ and following the reasoning analogous to that for the piecewise constant process $\xi(t)$, we can get the representation of the functionals related to the exit from the interval $(x-T,\ x)$ for the piecewise linear processes.

Two boundary-value problems for the integer-valued random walks are considered in [10] and, for the process with stationary independent increments, are treated in [11].

Bibliography

1. I.I. Gikhman, A.V. Skorokhod, *Theory of Stochastic Processes*, vol. 2, Nauka, Moscow, 1973, pp. 620. (in Russian)
2. A.V. Skorokhod, *Stochastic Processes with Independent Increments*, Nauka, Moscow, 1964, pp. 280. (in Russian)
3. V.S. Korolyuk, *Boundary Problems for Compound Poisson Processes*, Naukova Dumka, Kyiv, 1975, pp. 140. (in Russian)
4. V.M. Shurenkov, *Limit distribution of the exit time from an expanding interval and of the position at exit time of a process with independent increments and one-signed jumps*, Prob. Theory Appl. **23** (1978), no. 2, 402-407.
5. V.S. Korolyuk, N.S. Bratiichuk, B. Pirdjanov, *Boundary-Value Problems for Random Walks*, Ilim, Ashkhabad, 1987, pp. 250. (in Russian)
6. N.S. Bratiichuk, D.V. Gusak, *Boundary Problems for Processes with Independent Increments*, Naukova Dumka, Kyiv, 1990, pp. 264. (in Russian)
7. D.V. Gusak, *Compound Poisson processes with two-sided reflection*, Ukr. Math. J. **53** (2002), no. 12, 1616-1625.
8. D.V. Gusak, *Risk process with stochastic premiums and distributions of their functionals*, Theory of Stoch. Process. **11(27)** (2005), no. 1-2, 29-39.
9. E.A. Pecherskiy, *Some identities related to the exit of a random walk out of a segment and a semiinterval*, Probab. Theory Appl. **19** (1974), no. 1, 104-119. (in Russian)
10. T.V. Kadankova, *Two-sided boundary problems for the random walks with geometrically distributed negative jumps*, Teor. Imov. and Mat. Stat. **68** (2003), 49-60. (in Ukrainian)
11. V.F. Kadankov, T.V. Kadankova, *On the distribution of the moment of the first exit from an interval and the value of overshoot through the boundary for the processes with independent increments and random walks*, Ukr. Math. J. **57** (2005), no. 10, 1359-1385.

Institute of Mathematics, Nat. Acad. of Sci. of Ukraine, 3, Tereshchenkivs'ka Str., Kyiv 10601, Ukraine
E-mail: random@imath.kiev.ua

Taras Shevchenko Kyiv National University, 64, Volodymyrs'ka Str., Kyiv 10601, Ukraine
E-mail: kveugene@univ.kiev.ua