Theory of Stochastic Processes
Vol. 11 (27), no. 3–4, 2005, pp. 58–62

UDC 519.21

A. C. CHANI

# ON THE STABILITY WITH PROBABILITY ONE FOR A CLASS OF STOCHASTIC SEMIGROUPS

The formulated theorem specifies the result of A.V. Skorokhod concerning the stability with probability one for the solution of a linear stochastic system built on a continuous homogeneous semimartingale with independent increments.

We specify the result by A.V. Skorokhod published in monograph [1] concerning the conditions of the asymptotic stability with probability one of a homogeneous stochastic semigroup $U_t^0$ with values in the space $L(R^d)$. The semigroup $U_t^0$ is given by the solution of the linear stochastic differential equation

$$dU_t^0 = AU_t^0 dt + \sum_{k=1}^{n} B_k U_t^0 dw_k(t), \quad 0 \le t < \infty, \tag{1}$$

where $A, B_1, B_2, ..., B_n \in L(R^d)$, $w_1(t), w_2(t), ..., w_n(t)$ are independent Wiener processes with values in $R^1$, and the stability is understood as follows: for each $x \in R^d$,

$$P\left(\lim_{t\to\infty} U_t^0 x = 0\right) = 1. \tag{2}$$

The mentioned result ([1], p. 244, Theorem 18) means that it is sufficient for the stability of the semigroup $U_t^0$ for all $x \in R^d$ lying on a unit sphere with the center at zero that the following inequality be fulfilled:

$$D(x) = (Ax, x) + \frac{1}{2}\sum_{k=1}^{n} |B_k x|^2 - \sum_{k=1}^{n} (B_k x, x)^2 < 0, \tag{3}$$

where $|\cdot|$ is the norm in $R^d$ generated by a scalar product $(\cdot,\cdot)$. In the particular case with $B_1 = B_2 = ... = B_n = 0$, hypothesis (3) appears as $(Ax, x) < 0$.

In other words, for the stability of the nonrandom semigroup $U_t^0$ which is a solution of the equation

$$dU_t^0 = AU_t^0 dt, \tag{4}$$

it is required that the matrix $A^* + A$ be negatively defined

$$((A^* + A)x, x) < 0, \quad |x| = 1, \tag{5}$$

where $A^*$ is the matrix symmetric to $A$.

Condition (5) for systems of the form (4) is overstated because the classical result of Lyapunov only requires in this case that the matrix $A$ be stable. This means that we can found a symmetric positive definite matrix $H$ such that

$$((A^*H + HA)x, x) < 0, \quad |x| = 1.$$

---

2000 *AMS Mathematics Subject Classification.* Primary 60F15, 60H25.
*Key words and phrases.* Stability, stochastic system.

Thus, it is natural to deduce the conditions for system (1) to be stable, from which the result of Lyapunov will follow in the absence of stochastic additives. The present work is devoted to the solution of the above-mentioned problem.

We now pass to exact statements.

All processes are defined on a space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $t \geq 0$, which satisfies the usual conditions. We consider the stochastic process

$$Y_t = At + \sum_{k=1}^{n} B_k w_k(t)$$

receiving the values in the space $(L(R^d), \mathcal{M})$, $d \geq 1$, where $L(R^d)$ is the set of real $d \times d$-matrices alloted with a Borel $\sigma$-algebra $\mathcal{M}$ with respect to the Euclidean norm generated by a scalar product $(A, B) = Sp(B^* A)$. We define $U_t$ as a solution of the equation

$$U_t = I + \int_0^t dY_s U_s, \tag{6}$$

where $I$ is the identity matrix from $L(R^d)$. By $L_s^+(R^d)$, we denote the set of symmetric positive definite matrices from $L(R^d)$ and, with the help of $V \in L_s^+(R^d)$, apply it to the similarity transformation

$$U_t(V) = V U_t V^{-1}.$$

Consider the matrix $V$ as a parameter and define the family of processes

$$\vartheta = \{U_t(V);\ V \in L_s^+(R^d)\}.$$

Let $V$ be an arbitrary matrix from $L_s^+(R^d)$, and let

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d > 0$$

be its eigenvalues. Then the well-known inequalities

$$\lambda_d |U_t y| \leq |U_t(V)x| \leq \lambda_1 |U_t y|,$$

$$y = V^{-1}x, \quad x \in R^d,$$

are valid. The mappings $x \to y = V^{-1}x$ and $y \to x = Vy$ are one-to-one mutually inverse. Therefore, the stability in the sense of definition (2) of the process $U_t$ yields the stability of $U_t(V)$, and conversely. Thus, one can judge the stability of the $\vartheta$ family of processes by the stability of any of its representatives.

On the other hand, each process $U_t(V)$ from $\vartheta$ is the solution of an equation of the form (6):

$$U_t(V) = I + \int_0^t dY_s(V) U_s(V),$$

where

$$Y_t(V) = A(V)t + \sum_{k=1}^{n} B_k(V) w_k(t),$$

$$A(V) = VAV^{-1}, \quad B_k(V) = VB_kV^{-1}, \quad k = 1, 2, ..., n.$$

Therefore, the result of A.V. Skorokhod is applicable to the process $U_t(V)$, i.e. $U_t(V)$ will be stable with probability one, if the following inequality is fulfilled:

$$D_V(x) = (VAV^{-1}x, x) + \frac{1}{2}\sum_{k=1}^{n} |VB_kV^{-1}x|^2 - \sum_{k=1}^{n} (VB_kV^{-1}x, x)^2 < 0, \quad |x| = 1. \tag{7}$$

The mentioned arguments allow us to formulate the following theorem.

**Theorem.** *The process $U_t$ defined as a solution of Eq. (6) will be stable with probability one, if there exists a symmetric positive definite matrix $V$, for which inequality (7) is fulfilled.*

The formulated theorem makes sense, if there exist the stable processes $U_t$ with probability one, for which condition (3) is not fulfilled. We provide two simple examples which show that such processes exist. In addition, it is useful to simultaneously clarify the issue about the stability of the corresponding processes in the square mean sense. The point is that neglecting the third term on the left-hand side of inequality (7) turns it into the necessary and sufficient condition of stability in the quadratic mean sense:

$$(VAV^{-1}x, x) + \frac{1}{2}\sum_{k=1}^{n} |VB_kVx^{-1}|^2 < 0, \quad |x| = 1. \tag{8}$$

In fact, if we introduce the designation $V^2 = C$, then inequality (8) is equivalent to

$$((A^*C + CA)y, y) + \sum_{k=1}^{n} (B_k^* C B_k y, y) < 0, \quad |y| = 1, \tag{9}$$

where $y = |V^{-1}x|^{-1}V^{-1}x$. The fact of the existence of a symmetric positive definite matrix $C$, for which inequality (9) is fulfilled, is the necessary and sufficient condition of stability in the quadratic mean sense for the solution of Eq. (1) ([1], p. 227, Theorem 12 and p. 221, formula (47)).

*Example 1.* Let $U_t$ be a solution of Eq. (6) in the phase space $L(R^2)$. We choose the stable matrices

$$A = \begin{pmatrix} -\lambda & a \\ 0 & -\lambda \end{pmatrix}, \quad \lambda > 0,$$

$$B_1 = \begin{pmatrix} \mu & b \\ 0 & \mu \end{pmatrix}, \quad B_2 = B_3 = ... = B_n = 0.$$

Concerning the parameters $a, \mu, b$, we assume that they are connected by the equation

$$a + \mu b = 0. \tag{10}$$

Let us deal with the stability of the process $U_t$ in the square mean sense. To this end, we rewrite condition (8) in the matrix form:

$$D = VAV^{-1} + V^{-1}A^*V + V^{-1}B_1^*V^2B_1V^{-1} < 0.$$

The left-hand side of this inequality is a homogeneous function of the zero order in $V$. Therefore, it is sufficient to search for a matrix $V$ in the form

$$V = \begin{pmatrix} 1 & \beta \\ \beta & \alpha \end{pmatrix}, \quad \Delta = \alpha - \beta^2 > 0.$$

We have

$$VAV^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} -\lambda & a \\ 0 & -\lambda \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ -\beta & 1 \end{pmatrix} =$$

$$= \frac{1}{\Delta}\begin{pmatrix} -\lambda\Delta - a\beta & a \\ -a\beta^2 & -\lambda\Delta + a\beta \end{pmatrix},$$

$$V^{-1}A^*V = (VAV^{-1})^* = \frac{1}{\Delta}\begin{pmatrix} -\lambda\Delta - a\beta & -a\beta^2 \\ a & -\lambda\Delta + a\beta \end{pmatrix},$$

$$VAV^{-1} + V^{-1}A^*V = \frac{1}{\Delta}\begin{pmatrix} -2(\lambda\Delta + a\beta) & a(1-\beta^2) \\ a(1-\beta^2) & -2(\lambda\Delta - a\beta) \end{pmatrix}.$$

Further,

$$VB_1V^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \mu & b \\ 0 & \mu \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ -\beta & 1 \end{pmatrix} = \frac{1}{\Delta}\begin{pmatrix} \mu\Delta - b\beta & b \\ -b\beta^2 & \mu\Delta + b\beta \end{pmatrix},$$

$$V^{-1}B_1^*V = (VB_1V^{-1})^* = \frac{1}{\Delta}\begin{pmatrix} \mu\Delta - b\beta & -b\beta^2 \\ b & \mu\Delta + b\beta \end{pmatrix},$$

$$V^{-1}B_1^*V^2B_1V^{-1} = \frac{1}{\Delta^2}\begin{pmatrix} (\mu\Delta - b\beta)^2 + b^2\beta^4 & \mu b(1-\beta^2)\Delta - b^2\beta(1+\beta^2) \\ \mu b(1-\beta^2)\Delta - b^2\beta(1+\beta^2) & (\mu\Delta + b\beta)^2 + b^2 \end{pmatrix}.$$

After simple transformations with regard for equality (10), we get finally

$$D = \frac{1}{\Delta^2}\begin{pmatrix} (\mu^2 - 2\lambda)\Delta^2 + b^2\beta(1+\beta^2) & -b^2\beta(1+\beta^2) \\ -b^2\beta(1+\beta^2) & (\mu^2 - 2\lambda)\Delta^2 + b^2(1+\beta^2) \end{pmatrix}.$$

For the matrix $D$ to be negative definite, it is necessary and sufficient that the system of inequalities

$$\begin{cases} (\mu^2 - 2\lambda)\Delta^2 + b^2\beta^2(1+\beta^2) < 0, \\ det\ D = (\mu^2 - 2\lambda)\left[(\mu^2 - 2\lambda)\Delta^2 + b^2(1+\beta^2)^2\right] > 0 \end{cases}$$

be satisfied. Treating $\Delta \in (0,\infty)$ and $\beta \in (-\infty,\infty)$ as independent variables, we draw conclusion that this system of inequalities has a solution if and only if

$$\mu^2 < 2\lambda.$$

For arbitrary $\beta$, it is sufficient to take

$$\Delta > \frac{b^2(1+\beta^2)^2}{2\lambda - \mu^2}.$$

Thus, the relevant solution $U_t$ will be stable in the square mean in the region of parameters

$$Q_2 = \{(\lambda, a, \mu, b) : \lambda > 0,\ a + \mu b = 0,\ \mu^2 < 2\lambda\}.$$

We now examine the question of the stability of $U_t$ with probability one.

We have

$$\begin{aligned} 2D_V(x) &= \big((VAV^{-1} + V^{-1}A^*V + V^{-1}B_1^*V^2B_1V^{-1})x, x\big) - 2(VB_1V^{-1}x, x)^2 = \\ &= \frac{1}{\Delta^2}\Big\{\left[(\mu^2 - 2\lambda)\Delta^2 + b^2\beta^2(1+\beta^2)\right]x_1^2 - 2b^2\beta(1+\beta^2)x_1x_2 + \big[(\mu^2 - 2\lambda)\Delta^2 + \\ &\quad + b^2(1+\beta^2)\big]x_2^2 - 2\left[(\mu\Delta - b\beta)x_1^2 + b(1-\beta^2)x_1x_2 + (\mu\Delta + b\beta)x_2^2\right]^2\Big\} = \\ &= \frac{1}{\Delta^2}\Big\{\left[(\mu^2 - 2\lambda)\Delta^2 + b^2\beta^2(1+\beta^2)\right]x_1^2 - 2b^2\beta(1+\beta^2)x_1x_2 + \big[(\mu^2 - 2\lambda)\Delta^2 + \\ &\quad + b^2(1+\beta^2)\big](1 - x_1^2) - 2\left[\mu\Delta + b\beta - 2b\beta x_1^2 + b(1-\beta^2)x_1x_2\right]^2\Big\} = \\ &= \frac{1}{\Delta^2}\big\{(\mu^2 - 2\lambda)\Delta^2 + b^2(1+\beta^2) - 2(\mu\Delta + b\beta)^2 + b\left[b(\beta^4 - 1) + 8\beta(\mu\Delta + b\beta)\right]x_1^2 - \\ &\quad - 2b\left[b\beta(1+\beta^2) + 2(1-\beta^2)(\mu\Delta + b\beta)\right]x_1x_2 - 2b^2\left[(1-\beta^2)x_1x_2 - 2\beta x_1^2\right]^2. \end{aligned}$$

For $\beta = 0$,

$$2D_V(x) = -(\mu^2 + 2\lambda) + \frac{1}{\Delta^2}(b^2x_2^2 - 4b\mu\Delta x_1x_2 - 2b^2x_1^2x_2^2), \quad x_1^2 + x_2^2 = 1.$$

Having selected a rather big value $\Delta = \alpha$, we define the symmetric positive definite matrix

$$V = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \Delta = \alpha > 0,$$

for which condition (7) of Theorem will be fulfilled:

$$D_V(x) < 0 \quad \text{with} \quad |x| = 1.$$

This condition quarantees the stability of the process $U_t$ on the set

$$Q_0 = \{(\lambda, a, \mu, b) : \lambda > 0,\ a + \mu b = 0\}.$$

Concerning condition (3) with $\Delta = 1$, we get

$$2D(x) = 2D_I(x) = -(\mu^2 + 2\lambda) + b^2 x_2^2 - 4b\mu x_1 x_2 - 2b^2 x_1^2 x_2^2, \quad x_1^2 + x_2^2 = 1.$$

Calculating the functional $2D(x)$ at the points

$$x_1 = (0;1), \quad x_2 = (1;0), \quad x_3 = \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right), \quad x_4 = \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}},\right)$$

we obtain

$$-(\mu^2 + 2\lambda) + b^2, \quad -(\mu^2 + 2\lambda), \quad -(\mu^2 + 2\lambda) - 4b\mu, \quad -(\mu^2 + 2\lambda) + 4b\mu,$$

respectively. Therefore, for the fulfillment of condition (3), it is necessary that $b^2 \vee 4|b\mu| < \mu^2 + 2\lambda$. This means that the set of stability $Q_1$ which is defined by inequality (3) lies in the region

$$Q = \{(\lambda, a, \mu, b) : \lambda > 0,\ a + \mu b = 0,\ b^2 \vee 4|\mu b| < \mu^2 + 2\lambda\}.$$

In this case, $Q_1 \subset Q \subset Q_0$ and $Q \neq Q_0$. The exact description of the set $Q_1$ is not the purpose of the present work and demands a separate study. As a result, we obtain the following picture:

$$Q_2 \subset Q_0 \quad \text{and} \quad Q_2 \neq Q_0,$$
$$Q_1 \subset Q_0 \quad \text{and} \quad Q_1 \neq Q_0.$$

*Example 2.* In Example 1, we change only the matrix $A$ and choose it to be unstable:

$$A = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad \lambda > 0,$$

$$B_1 = \begin{pmatrix} \mu & b \\ 0 & \mu \end{pmatrix}, \quad B_2 = B_3 = \dots = B_n = 0, \quad a + \mu b = 0.$$

Then

$$Q_2 = \varnothing, \quad Q_0 = \{(\lambda, a, \mu, b) : \lambda > 0,\ a + \mu b = 0,\ 2\lambda < \mu^2\},$$
$$Q_1 \subset Q = \{(\lambda, a, \mu, b) : \lambda > 0,\ a + \mu b = 0,\ 2\lambda + (b^2 \vee 4|b\mu|) < \mu^2\}.$$

## Bibliography

1. A. V. Skorokhod, *Asymptotic methods of theory of stochastic differential equations*, Naukova Dumka, Kiev, 1987.

74, R.Luxemburg Str., Donetsk 83114, Ukraine
*E-mail*: chani@iamm.ac.donetsk.ua