НОВІ МЕТОДИ
В СИСТЕМНОМУ АНАЛІЗІ, ІНФОРМАТИЦІ ТА
ТЕОРІЇ ПРИЙНЯТТЯ РІШЕНЬ

УДК 517.9

GLOBAL ATTRACTOR FOR NON-AUTONOMOUS WAVE EQUATION WITHOUT UNIQUENESS OF SOLUTION

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In the paper the non-autonomous wave equation with non-smooth right-hand side is considered. It is proved that all its weak solutions generate multi-valued non autonomous dynamical system, which has invariant global attractor in the phase space.

Introduction. One of the main directions to investigate the asymptotic behaviour of solutions of non-linear problems is given by the mathematical physics through the theory of minimal attracting sets (global attractors). The topic methods of this theory and a great number of applications are described in [1–3]. This theory presents some generalizations in the cases of non-uniqueness of solutions [4–7] and also non-autonomous problems [8–11].

From this point of view, non-linear wave equation is difficult for studying because under conditions of global resolvebility it does not generate compact semigroup (even with smooth non-linearity). Different variants of additional conditions on non-linear term, which provide the existence of global attractor in spite of non-compactness of semigroup are discussed in [1, 2].

In [7] it is suggested a new idea of verifying Ladyzheuskaya's condition (or asymptotic semi-compactness condition) in order to prove the existence of global attractor for wave equation without the restrictive conditions imposed in the non-linearity for uniqueness of solution. In this paper we use a similar approach in situations of non-autonomous problem.

Setting of the problem We consider the problem

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + f(t,u) = 0, \\ u|_{\partial\Omega} = 0, & (1) \\ u|_{t=\tau} = u_\tau(x), \quad u_t|_{t=\tau} = v_\tau(x), & (2) \end{cases}$$

where $\gamma > 0$ is constant, $\Omega \subset \mathbb{R}^n$ is bounded domain with smooth boundary, $n \geq 3$, $\tau \in \mathbb{R}$ and non-linear term f satisfies the following condition

$$f, f_t' \in C(\mathbb{R}^2), \ \liminf_{|u|\to\infty} \inf_{t\in\mathbb{R}} \frac{f(t,u)}{u} > -\lambda_1,$$

$$|f(t,u)| \leq C\left(1 + |u|^{\frac{n}{n-2}}\right), \quad |f_t'(t,u)| \leq \alpha(t) + \beta(t)|u|, \qquad (3)$$

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where $C>0$ is constant, $\lambda_1>0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, $\alpha(\cdot)\ge 0$, $\beta(\cdot)\ge 0$ are given continuous functions from $L^1(\mathbb{R})$.

We denote by $|\cdot|$, $(\cdot,\cdot)$ and $\|\cdot\|, ((\cdot,\cdot))$ the norm and scalar product in $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively.

Our aim is to study the asymptotic behaviour of $\varphi(t)=\begin{pmatrix} u(t) \\ u_t(t)\end{pmatrix}$ in the phase space $E=H_0^1(\Omega)\times L^2(\Omega)$ on $t\to\infty$ by the methods of the theory of global attractors of multivalued non-autonomous dynamical systems.

Definition 1. Function $\varphi(\cdot)=\begin{pmatrix} u(\cdot) \\ u_t(\cdot)\end{pmatrix}$ is called solution of (1) on (τ,T), if $u(\cdot)\in L^\infty(\tau,T;H_0^1(\Omega))$, $u_t(\cdot)\in L^\infty(\tau,T;L^2(\Omega))$ and $\forall\psi\in H_0^1(\Omega)$ $\forall\eta\in$ $\in\mathbb{C}_0^\infty(\tau,T)$

$$-\int_\tau^T (u_t,\psi)\eta_t+\int_\tau^T\big(\gamma(u_t,\psi)+((u,\psi))+(f(t,u),\psi)\big)\eta=0\,, \tag{4}$$

where u_t denotes the distributional derivative with respect to t of u.

Note, that since $H_0^1(\Omega)$ is continuously embedded in $L^{\frac{2n}{n-2}}(\Omega)$, by (3) for every $u\in L^\infty(\tau,T;H_0^1(\Omega))$ we have $f(t,u)\in L^2(\tau,T;L^2(\Omega))$. Then for each solution $\varphi(\cdot)=\begin{pmatrix} u(\cdot) \\ u_t(\cdot)\end{pmatrix}$ of (1) from [2] we have $u(\cdot)\in\mathbb{C}([\tau,T];H_0^1(\Omega))$, $u_t(\cdot)\in$ $\in\mathbb{C}([\tau,T];L^2(\Omega))$, $\forall\psi\in H_0^1(\Omega)$ $(u_t(\cdot),\psi)\in\mathbb{C}^1(\tau,T)$ and $\forall t\in(\tau,T)$

$$\frac{d}{dt}(u_t,\psi)+\gamma(u_t,\psi)+((u,\psi))+(f(t,u),\psi))=0\,. \tag{5}$$

Firstly we prove that under conditions (3) the problem (1), (2) $\forall T>\tau$ $\forall\varphi_\tau=\begin{pmatrix} u_\tau \\ v_\tau\end{pmatrix}\in E$ has at least one solution on $[\tau,T]$, and each solution of (1), (2) (independently from the method of finding) satisfies certain energy equality (Lemma 5).

Note that there is no Lipschitz's condition on f with respect to variable u, so the problem (1), (2) is not necessary uniquely resolved.

Since f depends on t, solutions of (1), (2) do not generate semigroup, but under additional condition on f as a function of t we can construct non-autonomous analogue of semigroup.

For this purpose, following by [9], we consider the space $\mathbb{M}=\mathbb{C}(\mathbb{R};\mathbb{R}^2)$ of continuous vector-functions $p(\cdot)=\begin{pmatrix} p_1(\cdot) \\ p_2(\cdot)\end{pmatrix}$ and equip it with a uniform convergence topology on each segment $[\nu_1,\nu_2]\subset\mathbb{R}$, that is

$$p_n \to p \text{ in } \mathbb{M} \Leftrightarrow \forall [\nu_1, \nu_2] \subset \mathbb{R} \sup_{\nu \in [\nu_1, \nu_2]} \| p_n(\nu) - p(\nu) \|_{\mathbf{R}^2} \to 0 .$$

It is known that with such topology $\mathbb{M}$ is a complete metric space.

Further we consider the space $\mathbb{C}(\mathbb{R};\mathbb{M})$ of continuous functions $g(t)$, $t \in \mathbb{R}$ with values in $\mathbb{M}$. It is also equipped with a uniform convergence topology on each segment $[t_1, t_2] \subset \mathbb{R}$ that is

$$g_n \to g \text{ in } \mathbb{C}(\mathbb{R};\mathbb{M}) \Leftrightarrow \forall [t_1, t_2] \subset \mathbb{R} \sup_{t \in [t_1, t_2]} \rho_{\mathbb{M}}(g_n(t), g(t)) \to 0 .$$

It is known that with such topology $\mathbb{C}(\mathbb{R};\mathbb{M})$ is a complete metric space. For every $g \in \mathbb{C}(\mathbb{R};\mathbb{M})$ we put

$$H(g) = \mathrm{cl}_{\mathbb{C}(\mathbb{R};\mathbb{M})} \{ g(t+h) \mid h \in \mathbb{R} \} .$$

The function $g \in \mathbb{C}(\mathbb{R};\mathbb{M})$ is called translation-compact (tr.-c.) in $\mathbb{C}(\mathbb{R};\mathbb{M})$ if the set $H(g)$ is compact in $\mathbb{C}(\mathbb{R};\mathbb{M})$.

Our additional condition on function f , which we use to construct the non-autonomous dynamical system is the following:

$$\begin{pmatrix} f \\ f_t' \end{pmatrix} \text{ is tr.-c. in } \mathbb{C}(\mathbb{R};\mathbb{M}) . \tag{6}$$

As an example of the function f which satisfies (3), (6), we can consider $f(t,u) = e^{-t^2} u + h(u)$, where $h \in \mathbb{C}(\mathbb{R})$ (but not smooth),

$$\liminf_{|u| \to \infty} \frac{h(u)}{u} > -\lambda_1 \text{ and } |h(u)| \leqslant C \left(1 + |u|^{\frac{n}{n-2}} \right) .$$

Then $|f(t,u)| \leqslant \widetilde{C} \left(1 + |u|^{\frac{n}{n-2}} \right)$ $\liminf\limits_{|u| \to \infty} \inf\limits_{t \in \mathbb{R}} \left(e^{-t^2} + \dfrac{h(u)}{u} \right) = \liminf\limits_{|u| \to \infty} \dfrac{h(u)}{u} > -\lambda_1$, $|f_t'(t,u)| = \left| -2te^{-t^2} u \right| \leqslant 2|t| e^{-t^2} |u|$ and $\begin{pmatrix} f \\ f_t' \end{pmatrix}$ is obviously tr. -c. in $\mathbb{C}(\mathbb{R};\mathbb{M})$.

We note, that in this example f and f_t' are not almost-periodic in Bohr sense. We denote

$$\Sigma = H \begin{pmatrix} f \\ f_t' \end{pmatrix} . \tag{7}$$

From [9] we have that continuous shift group $\{ T(h) : \Sigma \to \Sigma \}_{h \in \mathbb{R}}$, $T(h)\sigma(t) = \sigma(t+h)$ acts on Σ.

Now we need the following Lemma.

Lemma 1. Each function $\sigma \in \Sigma$ has the form $\sigma = \begin{pmatrix} g \\ g_t' \end{pmatrix}$, and functions g, g_t' satisfy the following conditions:

$$\liminf_{|u| \to \infty} \inf_{t \in \mathbb{R}} \frac{g(t,u)}{u} > -\lambda_1, \; |g(t,u)| \leq C \left(1 + |u|^{\frac{n}{n-2}} \right), \; |g_t'(t,u)| \leq \alpha_\sigma(t) + \beta_\sigma(t) |u| ,$$

where $\int_{-\infty}^{+\infty} \alpha_\sigma(t)\,dt \le \int_{-\infty}^{+\infty} \alpha(t)dt$, $\int_{-\infty}^{+\infty} \beta_\sigma(t)dt \le \int_{-\infty}^{+\infty} \beta(t)dt$.

Proof . For each $\sigma = \begin{pmatrix} g \\ l \end{pmatrix} \in \Sigma$ according to (6) there exists sequence $\{h_n\}$ such that $\forall [t_1, t_2] \subset \mathbb{R}$ $\forall [v_1, v_2] \subset \mathbb{R}$

$$\sup_{t\in[t_1,t_2]} \sup_{v\in[v_1,v_2]} \left(\left| f(t+h_n, v) - g(t,v) \right| + \left| f'_t(t+h_n, v) - l(t,v) \right| \right) \to 0,\ n \to \infty .$$

From this we can easy obtain $l(t,v) = g'_t(t,v)$. Since $f(t+h_n, v) \le C\left(1 + |v|^{\frac{n}{n-2}}\right)$, we have $|g(t,v)| \le C\left(1 + |v|^{\frac{n}{n-2}}\right)$. Choosing $\varepsilon > 0$ such that $\liminf_{|v|\to\infty} \inf_{t\in\mathbb{R}} \frac{f(t,v)}{v} > -\lambda_1 + \varepsilon$, we have

$$\exists R > 0\ \ \forall |v| \ge R\ \ \forall t \in \mathbb{R}\ \ \forall n \ge 1\ \ \frac{f(t+h_n, v)}{v} > -\lambda_1 + \varepsilon .$$

So $\frac{g(t,v)}{v} \ge -\lambda_1 + \varepsilon$ and we obtain $\liminf_{|v|\to\infty} \inf_{t\in\mathbb{R}} \frac{g(t,v)}{v} \ge -\lambda_1 + \varepsilon > -\lambda_1$.

Since $|f'_t(t+h_n, v)| \le \alpha(t+h_n) + \beta(t+h_n)|v|$, we have for $h_n \to \infty$ $g'_t(t,v) = 0$ and for $h_n \to h_0$ $|g'_t(t,v)| \le \alpha(t+h_0) + \beta(t+h_0)|v|$, where $\int_{-\infty}^{+\infty} \alpha(t+h_0)\,dt = \int_{-\infty}^{+\infty} \alpha(t)\,dt$, $\int_{-\infty}^{+\infty} \beta(t+h_0)dt = \int_{-\infty}^{+\infty} \beta(t)dt$. Lemma is proved.

Now we dip the problem (1), (2) into the family of similar problems:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + g(t,u) = 0, & \\ u|_{\partial\Omega} = 0, & (1)_\sigma \\ u|_{t=\tau} = u_\tau(x), \quad u_t|_{t=\tau} = v_\tau(x), & (2)_\sigma \end{cases}$$

where $\sigma = \begin{pmatrix} g \\ g'_t \end{pmatrix} \in \Sigma.$

As functions g, g'_t satisfy the conditions (3), for each $\sigma \in \Sigma$ the problem $(1)_\sigma$, $(2)_\sigma$ is globally resolved for all $\varphi_\tau = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} \in E$. The main object which we consider in this paper is a family of multivalued maps $\{U_\sigma : \mathbb{R}_d \times E \to 2^E\}_{\sigma\in\Sigma}$, $\mathbb{R}_d = \{(t,\tau) \in \mathbb{R}^2 \mid t \ge \tau\}$

$$U_\sigma(t,\tau,\varphi_\tau) = \left\{ \varphi(t) \mid \varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} \text{ is solution of } (1)_\sigma,\ \varphi(\tau) = \varphi_\tau \right\}. \quad (8)$$

For the family (8) our goal is to prove the existence in phase space E of minimal invariant uniformly attracting set — global attractor.

Elements of abstract theory of global attractors for multivalued non-autonomous dynamical systems. Let (X,ρ) be a complete metric space. We denote by $P(X)(\beta(X))$ the set of all non-empty (non-empty bounded) subsets of X, $\forall A,B\subset X$ $\operatorname{dist}(A,B)=\sup\limits_{x\in A}\inf\limits_{y\in B}\rho(x,y)$, $O_\delta(A)=\{x\in X\,|\operatorname{dist}(x,A)<\delta\}$, $B_r=\{x\in X\,|\,\rho(x,0)\le r\}$. Let Σ be some complete metric space, $\{T(h):\Sigma\to\to\Sigma\}_{h\in\mathbb{R}}$ be some continuous group acting on Σ.

Definition 2. The family of multivalued maps $\{U_\sigma:\mathbb{R}_d\times X\to P(X)\}_{\sigma\in\Sigma}$ is called family of multivalued processes (MP) or non-autonomous multivalued dynamical system, if $\forall\sigma\in\Sigma$, $\forall x\in X$:

1) $U_\sigma(\tau,\tau,x)=x$ $\forall\tau\in\mathbb{R}$;

2) $U_\sigma(t,\tau,x)\subset U_\sigma(t,s,U_\sigma(s,\tau,x))$ $\forall t\ge s\ge\tau$;

3) $U_\sigma(t+h,\tau+h,x)\subset U_{T(h)\sigma}(t,\tau,x)$ $\forall t\ge\tau$, $\forall h\in\mathbb{R}$.

The family of MP is called strict, if in conditions 2), 3) equality takes place.

We denote $U_\Sigma(t,\tau,x)=\bigcup\limits_{\sigma\in\Sigma}U_\sigma(t,\tau,x)$.

Definition 3. The set $\Theta_\Sigma\subset X$ is called global attractor of the family of $MP\{U_\sigma\}_{\sigma\in\Sigma}$, if $\Theta_\Sigma\ne X$ and

1) Θ_Σ is uniformly attracting set, that is

$\forall B\in\beta(X)$ $\forall\tau\in\mathbb{R}$ $\operatorname{dist}(U_\Sigma(t,\tau,B),\Theta_\Sigma)\to 0$, $t\to\infty$;

2) Θ_Σ is minimal uniformly attracting set, that is for arbitrary uniformly attracting set Y we have $\Theta_\Sigma\subset\operatorname{cl}_X Y$.

Global attractor Θ_Σ is called semi-invariant (invariant) if $\forall(t,\tau)\in\mathbb{R}_d$ $\Theta_\Sigma\subset U_\Sigma(t,\tau,\Theta_\Sigma)$, $(\Theta_\Sigma=U_\Sigma(t,\tau,\Theta_\Sigma))$.

Lemma 2. 1) If the family of $MP\{U_\sigma\}_{\sigma\in\Sigma}$ satisfies the following conditions:

$$\forall B\in\beta(X)\ \exists T=T(B)\bigcup_{t\ge T}U_\Sigma(t,0,B)\in\beta(X), \tag{9}$$

$$\forall B\in\beta(X)\ \forall\{t_n\,|\,t_n\to\infty\}\ \forall\{\xi_n\,|\,\xi_n\in U_\Sigma(t_n,0,B)\}\ \text{the sequence } \{\xi_n\} \text{ is precompact in } X, \tag{10}$$

then there exists global attractor Θ_Σ,

$$\Theta_\Sigma=\bigcup_\tau\Theta_\Sigma(\tau)=\Theta_\Sigma(0), \tag{11}$$

where $\Theta_\Sigma(\tau)=\bigcup\limits_{B\in\beta(X)}\omega_\Sigma(\tau,B)$, $\omega_\Sigma(\tau,B)=\bigcap\limits_{s\ge\tau}\overline{\bigcup\limits_{t\ge s}U_\Sigma(t,\tau,B)}$ is compact in X;

2) if, additionally, $\forall t\ge 0$ the map

$$X\times\Sigma\ni(x,\sigma)\to U_\sigma(t,0,x) \tag{12}$$

has closed graph, then Θ_Σ is semi-invariant;

3) if, additionally, the family of $MP\{U_\sigma\}_{\sigma\in\Sigma}$ is strict, then Θ_Σ is invariant.

Proof. The properties 1), 2) directly derived from the result of [11]. Now we prove 3). From [11] we have the embedding $\omega_\Sigma(0,B)\subset \subset U_\Sigma(t,0,\omega_\Sigma(0,B))$ $\forall B\in\beta(X)$ $\forall t\ge 0$. So $\forall p\ge 0$ $U_\Sigma(t+p,t,\omega_\Sigma(0,B))\subset \subset U_\Sigma(t+p,t,U_\Sigma(t,0,\omega_\Sigma(0,B)))=U_\Sigma(t+p,0,\omega_\Sigma(0,B))$. Then $U_\Sigma(p,0,\omega_\Sigma(0,B))= =U_{T(t)\Sigma}(p,0,\omega_\Sigma(0,B))=U_\Sigma(t+p,t,\omega_\Sigma(0,B))\subset U_\Sigma(t+p,0,\omega_\Sigma(0,B))$. . From this for all $p\ge 0$, for all $\tau\ge p$

$$U_\Sigma(p,0,\omega_\Sigma(0,B))\subset\bigcup_{k\ge\tau}U_\Sigma(k,0,\omega_\Sigma(0,B))\subset\overline{\bigcup_{k\ge\tau}U_\Sigma(k,0,\omega_\Sigma(0,B))}.$$

So,

$$U_\Sigma(p,0,\omega_\Sigma(0,B))\subset\bigcap_{\tau\ge p}\overline{\bigcup_{k\ge t}U_\Sigma(k,0,\omega_\Sigma(0,B))}=\omega_\Sigma(0,\omega_\Sigma(0,B))\subset\Theta_\Sigma .$$

Therefore, $\forall\ p\ge 0$ $U_\Sigma(p,0,\Theta_\Sigma)\subset\Theta_\Sigma$.

Then $\forall\tau\in\mathbb{R}$ $U_\Sigma(p+\tau,\tau,\Theta_\Sigma)=U_{T(\tau)\Sigma}(p,0,\Theta_\Sigma)=U_\Sigma(p,0,\Theta_\Sigma)\subset\Theta_\Sigma$ and Lemma is proved.

Properties of solutions of the problem (1), (2). We put $F(t,u)= =\int_0^u f(t,s)ds$, $F_t'(t,u)=\int_0^u f_t'(t,s)ds$. Then F, $F_t'\in\mathbb{C}(\mathbb{R}^2)$ and according to (3) there exist constants $\lambda<\lambda_1$, $C_1>0$, $C_2\in\mathbb{R}$ which only depend on $C>0$, $n\ge 3$ and $\lambda_1>0$ such that $\forall(t,u)\in\mathbb{R}^2$

$$|F(t,u)|\le C_1\left(1+|u|^{\frac{2n-2}{n-2}}\right),\quad F(t,u)\ge-\frac{\lambda}{2}u^2+C_2,$$
$$|F_t'(t,u)|\le\alpha(t)|u|+\frac{\beta(t)}{2}|u|^2. \tag{13}$$

In view of (13) for every function $\varphi(\cdot)=\begin{pmatrix}u(\cdot)\\ u_t(\cdot)\end{pmatrix}\in\mathbb{C}([\tau,T];E)$ we can correctly define the following functionals:

$$V(t,\varphi(t))=\frac{1}{2}|u_t(t)|^2+\frac{1}{2}\|u(t)\|^2+(F(t,u(t)),1),$$
$$I(t,\varphi(t))=V(t,\varphi(t))+\frac{\gamma}{2}(u_t(t),u(t)),$$
$$H(t,\varphi(t))=\left(F_t'(t,u(t)),1\right)+\gamma(F(t,u(t)),1)-\frac{\gamma}{2}(f(t,u(t)),u(t)).$$

Lemma 3. The following properties take place:

1) functions $(F(\cdot,u(\cdot)),1)$, $\left(F_t'(\cdot,u(\cdot)),1\right)$, $(f(\cdot,u(\cdot)),u(\cdot))$, $(f(\cdot,u(\cdot)),u_t(\cdot))\in \in\mathbb{C}([\tau,T])$;

2) if $\{\rho_n(\cdot)\}\subset\mathbb{C}([\tau,T];H_0^1(\Omega))$ and $\forall t\in[\tau,T]$ $\rho_n(t)\to u(t)$ in $H_0^1(\Omega)$, then $\forall t\in[\tau,T]$

$$(F(t,\rho_n(t)),1)\to(F(t,u(t)),1),\quad\left(F_t'(t,\rho_n(t)),1\right)\to\left(F_t'(t,u(t)),1\right),$$

$$(f(t,\rho_n(t)),\rho_n(t))\to(f(t,u(t)),u(t)).$$

If, additionally, $\{\rho_n(\cdot)\}\subset\mathbb{C}^1([\tau,T];H_0^1(\Omega))$ and $\forall t\in[\tau,T]$ $\rho_n'(t)\to u_t(t)$ in $L^2(\Omega)$, then $(f(t,\rho_n(t)),\rho_n'(t))\to(f(t,u(t)),u_t(t))$.

Proof. In the proof of this Lemma and in all results, given below, we use the following version of the dominated convergence Lebesgue's Theorem: if for measurable functions $\{\xi_n\}_{n\ge1}$, ξ we have $\xi_n\to\xi$ a.e., $|\xi_n|<\eta_n$ a.e. and $\eta_n\to\eta$ in L^1, then $\xi_n\to\xi$ in L^1.

We consider the function $(f(\cdot,u(\cdot)),u_t(\cdot))$ (for others one can apply the same arguments). Let $t_n\to t_0$. Then $u(t_n)\to u(t_0)$ in $H_0^1(\Omega)$, $u_t(t_n)\to u_t(t_0)$ in $L^2(\Omega)$, so $u(t_n,x)\to u(t_0,x)$ a.e., $u_t(t_n,x)\to u_t(t_0,x)$ a.e. Since $f\in\mathbb{C}(\mathbb{R}^2)$, we obtain $f(t_n,u(t_n,x))u_t(t_n,x)\to f(t_0,u(t_0,x))u_t(t_0,x)$ a.e. Moreover, in view of (3) $|f(t_n,u(t_n,x))u_t(t_n,x)|\le C|u_t(t_n,x)|+C|u(t_n,x)|^{\frac{n}{n-2}}|u_t(t_n,x)|$. As $H_0^1(\Omega)\subset\subset L^{\frac{2n}{n-2}}(\Omega)$, we have $u(t_n,x)\to u(t_0,x)$ in $L^{\frac{2n}{n-2}}$. Since $u_t(t_n,x)\to u_t(t_0,x)$ in $L^2(\Omega)$, we easy obtain $|u(t_n,x)|^{\frac{n}{n-2}}|u_t(t_n,x)|\to|u(t_0,x)|^{\frac{n}{n-2}}|u_t(t_0,x)|$ in $L^1(\Omega)$.

Applying Lebesgue's theorem, we have $f(t_n,u(t_n,x))u_t(t_n,x)\to\to f(t_0,u(t_0,x))u_t(t_0,x)$ in $L^1(\Omega)$ and thus $f(\cdot,u(\cdot))u_t(\cdot)\in\mathbb{C}([\tau,T])$. Statement 2 can be proved in the same way. Lemma is proved.

As a consequence of Lemma 3 we immediately obtain that $\forall\varphi(\cdot)=\begin{pmatrix}u(\cdot)\\u_t(\cdot)\end{pmatrix}\in\mathbb{C}([\tau,T];E)$ functions $V(\cdot,\varphi(\cdot)),I(\cdot,\varphi(\cdot)),H(\cdot,\varphi(\cdot))$ belong to $\mathbb{C}([\tau,T])$.

Lemma 4. For every $u(\cdot)\in\mathbb{C}([\tau,T];H_0^1(\Omega))$, $u_t(\cdot)\in\mathbb{C}([\tau,T];L^2(\Omega))$ function $(F(\cdot,u(\cdot)),1)$ belongs to $\mathbb{C}^1(\tau,T)$ and $\forall t\in(\tau,T)$

$$\frac{d}{dt}(F(t,u(t)),1)=(F_t'(t,u(t)),1)+(f(t,u(t)),u_t(t)).\tag{14}$$

Proof. From Lemma 3 it suffices to show that $\forall[t_0,t_1]\subset(\tau,T)$ $\forall\eta\in\mathbb{C}_0^\infty(t_0,t_1)$

$$-\int_{t_0}^{t_1}(F(t,u(t)),1)\eta_t=\int_{t_0}^{t_1}((F_t'(t,u(t)),1)+(f(t,u(t)),u_t(t)))\eta.\tag{15}$$

We can mollify u with respect to t to obtain a sequence $\{\rho_n(\cdot)\}\subset\mathbb{C}^1([t_0,t_1];H_0^1(\Omega))$ with $\rho_n\to u$ in $\mathbb{C}([t_0,t_1];H_0^1(\Omega))$, $\rho_n'\to u_t$ in $\mathbb{C}([t_0,t_1];L^2(\Omega))$. Equality (15) obviously holds for $\rho_n(\cdot)$. Using Lemma 3 and boundness of $\begin{pmatrix}\rho_n\\\rho_n'\end{pmatrix}$ in $\mathbb{C}([t_0,t_1];E)$ we can apply Lebesgue's theorem and obtain (15) by passing to the limit in the same identify for ρ_n. Lemma is proved.

Lemma 5. Under conditions (3) $\forall \tau \in \mathbb{R}$ $\forall T > \tau$ $\forall \varphi_\tau = \begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} \in E$ problem (1), (2) has at least one solution on (τ, T). Moreover, for each solution $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}$ of problem (1) on (τ, T) the functions $(u_t(\cdot), u(\cdot))$, $V(\cdot, \varphi(\cdot))$, $I(\cdot, \varphi(\cdot))$ belong to $\mathbb{C}^1(\tau, T)$ and $\forall t \in (\tau, T)$ we have

$$\frac{d}{dt} V(t, \varphi(t)) = -\gamma |u_t(t)|^2 + \left(F_t'(t, \varphi(t)), 1\right), \tag{16}$$

$$\frac{d}{dt}(u_t(t), u(t)) = |u_t(t)|^2 - \gamma (u_t(t), u(t)) - \|u(t)\|^2 - (f(t, u(t)), u(t)), \tag{17}$$

$$\frac{d}{dt} I(t, \varphi(t)) = -\gamma I(t, \varphi(t)) + H(t, \varphi(t)). \tag{18}$$

Proof. We construct solution of (1),(2) using the Faedo-Galerkin method. Let $\{\omega_j\}_{j=1}^{\infty}$ be a complete system of functions in $H_0^1(\Omega)$ and $u_m(t) = \sum_{i=1}^{m} g_i^{(m)}(t)\omega_i$ be the Galerkin approximation, satisfying the following ordinary differential system

$$\frac{d^2}{dt^2}(u_m, \omega_j) + \gamma \frac{d}{dt}(u_m, \omega_j) + ((u_m, \omega_j)) + (f(t, u_m), \omega_j) = 0, \; j = 1, \dots, m \tag{19}$$

with the initial conditions

$$u_m(\tau) = u_\tau^m, \quad u_m'(\tau) = v_\tau^m,$$

where $u_\tau^m \to u_\tau$, $m \to \infty$ in $H_0^1(\Omega)$, $v_\tau^m \to v_\tau$, $m \to \infty$ in $L^2(\Omega)$. Local existence of $u_m(\cdot)$ is obvious. Existence on $[\tau, T]$ will be guaranteed by following a priori estimates:

$$(u_m'', u_m') + \gamma |u_m'|^2 + ((u_m, u_m')) + (f(t, u_m), u_m') = 0,$$

$$\frac{d}{dt}\left\{ |u_m'|^2 + \|u_m\|^2 + 2(F(t, u_m), 1) \right\} + 2\gamma |u_m'|^2 - 2(F_m'(t, u_m), 1) = 0.$$

From this equality and (13) we deduce that $\forall t \ge \tau$

$$|u_m'(t)|^2 + \|u_m(t)\|^2 \le C_3 \Bigg(|u_m'(\tau)|^2 + \|u_m(\tau)\|^2 + \|u_m(\tau)\|^{\frac{2n-2}{n-2}} + 1 +$$

$$+ \int_\tau^t (\alpha(s) + \beta(s)) \left(|u_m'(s)|^2 + \|u_m(s)\|^2 \right) ds \Bigg), \tag{20}$$

where constant $C_3 > 0$ depends only on $\lambda_1 > 0$, $C > 0$, $n \ge 3$. Using Gronwall inequality, we obtain:

$$|u_m'(t)|^2 + \|u_m(t)\|^2 \le C_3 \Big(|u_m'(\tau)|^2 + \|u_m(\tau)\|^2 +$$

$$+\|u_m(\tau)\|^{\frac{2n-2}{n-2}}+1\Bigg)e^{\int_\tau^t(\alpha(s)+\beta(s))ds}. \tag{21}$$

From (21) we deduce that $\begin{pmatrix} u_m \\ u'_m \end{pmatrix}$ is bounded in $L^\infty(\tau,T;E)$.

So we can extract a subsequence, still denoted m, such that

$$u_m \to u \ \text{ in } \ L^\infty\left(\tau,T;H_0^1(\Omega)\right) \ \text{weak}-\text{star},$$

$$u'_m \to u_t \ \text{ in } \ L^\infty\left(\tau,T;L^2(\Omega)\right) \ \text{weak}-\text{star}.$$

Thanks to a classical compactness theorem

$$u_m \to u \ \text{ in } \ L^2\left(\tau,T;L^2(\Omega)\right) \ \text{strongly}.$$

Hence on some subsequence $u_m(t,x)\to u(t,x)$ a.e. and so $f(t,u_m(t,x))\to$ $\to f(t,u(t,x))$ a.e. From (21) $\{u_m(t)\}$ is bounded in $L^\infty\left(\tau,T;H_0^1(\Omega)\right)$, so $\{f(t,u_m(t))\}$ is bounded in $L^2\left(\tau,T;L^2(\Omega)\right)$. Then in a standard way we obtain $f(t,u_m(t))\to f(t,u(t))$ in $L^2\left(\tau,T;L^2(\Omega)\right)$ weakly. It allows us to pass to the limit in (19) and find that $\varphi(\cdot)=\begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}\in L^\infty(\tau,T;E)$ and satisfies (4). Thus $\varphi(\cdot)$ is a solution of (1), $\varphi(\cdot)\in \mathbb{C}([\tau,T];E)$. Moreover, as $\{u''_m\}$ is bounded in $L^2\left(\tau,T;H^{-1}(\Omega)\right)$, from compactness theorem we have

$$\forall\, t\in[\tau,T] \ u_m(t)\to u(t) \ \text{ weakly in } \ L^2(\Omega),$$

$$\forall\, t\in[\tau,T] \ u'_m(t)\to u_t(t) \ \text{ weakly in } \ H^{-1}(\Omega)$$

and, again applying (21), $\varphi_m(t)=\begin{pmatrix} u_m(t) \\ u'_m(t) \end{pmatrix}\to\varphi(t)$ weakly in E. In particular, $\varphi_m(\tau)=\begin{pmatrix} u_\tau^m \\ \upsilon_\tau^m \end{pmatrix}\to\varphi(\tau)=\begin{pmatrix} u_\tau \\ \upsilon_\tau \end{pmatrix}$ in E and existence is proved.

Now let $\varphi(\cdot)=\begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}$ is an arbitrary solution of (1), (2) on (τ,T).

Since $f(t,u(t))\in L^2\left(\tau,T;L^2(\Omega)\right)$, from [2] we deduce that in the sense of scalar distributions on (τ,T)

$$\frac{1}{2}\frac{d}{dt}\left(|u_t|^2+\|u\|^2\right)=\left(-\gamma u_t(t)-f(t,u(t)),u_t(t)\right). \tag{22}$$

Similarly to the proof of Lemma 4 we can obtain in the sense of distributions

$$\langle u_{tt},u\rangle=\frac{d}{dt}(u_t,u)-|u_t|^2, \tag{23}$$

where $\langle \cdot,\cdot \rangle$ is the scalar product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. From (23) and (1) we have equality (17) in the sense of distributions on (τ, T).

According to $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} \in \mathbb{C}([\tau, T]; E)$ we deduce that functions $(u_t(\cdot), u(\cdot))$, $|u_t(\cdot)|^2 + \|u(\cdot)\|^2$ belong to $\mathbb{C}^1(\tau, T)$ and so identities (17), (22) take place in classical sense $\forall t \in (\tau, T)$. Then using the result of Lemma 4 and (17), (22) we can easily obtain (16)-(18). Lemma is proved.

Remark 1. As $T > \tau$ is arbitrary, we can state a global resolvebility of (1), (2), that is we say that $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix} \in \mathbb{C}([\tau, +\infty]; E)$ is a solution of (1), (2), if $\varphi(\tau) = \varphi_\tau$ and $\varphi(\cdot)$ satisfies (4) $\forall T > \tau$.

Remark 2. It is easy to see that if (16)-(18) hold, then for each solution $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}$ of (1) we can repeat arguments, using in proof of Lemma 5 and obtain (21). Hence, for arbitrary solution $\varphi(\cdot) = \begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}$ of (1), for which $\|u(\tau)\|^2 + |u_t(\tau)|^2 \le R$, we have

$$\forall t \ge \tau \qquad \|u(t)\|^2 + |u_t(t)|^2 \le K(R), \tag{24}$$

where constant $K(R) > 0$ depends only on constants $R > 0,\ \lambda_1 > 0,\ C > 0,\ n \ge 3$ and values of $\int_{-\infty}^{+\infty} \alpha(t)dt$, $\int_{-\infty}^{+\infty} \beta(t)dt$.

Main results. For every $\sigma = \begin{pmatrix} g \\ g_t' \end{pmatrix} \in \Sigma$ we consider the problem $(1)_\sigma$, $(2)_\sigma$. In view of Lemmas 1, 5 for every $\tau \in \mathbb{R}$, $\varphi_\tau = \begin{pmatrix} u_\tau \\ \upsilon_\tau \end{pmatrix} \in E$ the problem $(1)_\sigma$, $(2)_\sigma$ has at least one solution on $(\tau, +\infty)$ and for all solutions of $(1)_\sigma$, $(2)_\sigma$ the equalities $(16)-(18)$ take place, if we change V, I, H on V_σ, I_σ, H_σ respectively.

Lemma 6. Let $\varphi_n(\cdot)$ be a solution of $(1)_{\sigma_n}$, where $\sigma_n = \begin{pmatrix} g_n \\ g_n' \end{pmatrix} \to \sigma = \begin{pmatrix} g \\ g' \end{pmatrix}$ in Σ and $\varphi_n(\tau) \to \varphi_\tau$ weakly in E.

Then $\forall T > \tau\ \ \forall t \in [\tau, T]\ \ \varphi_n(t) \to \varphi(t)$ weakly in E, where $\varphi(\cdot)$ is solution of $(1)_\sigma$, $\varphi(\tau) = \varphi_\tau$ and $(F_{\sigma_n}(t, u_n(t)), 1) \to (F_\sigma(t, u(t)), 1)$, $(F'_{\sigma_n}(t, u_n(t)), 1) \to$ $\to (F'_\sigma(t, u(t)), 1)$ $(f_{\sigma_n}(t, u_n(t)), u_n(t)) \to (f_\sigma(t, u(t)), u(t))$ where $f_{\sigma_n} := g_n$; $f_\sigma := g$.

Proof. Thanks to Lemma 1, (16)-(18) and boundness of $\{\varphi_n(\tau)\}$ in E we can in the same way as in Lemma 5 obtain for $\varphi_n(\cdot)=\begin{pmatrix} u_n(\cdot) \\ u_n'(\cdot) \end{pmatrix}$:

$$\forall\, t \ge \tau \ \ \|u_n(t)\|^2 + |u_n'(t)|^2 \le C_3\Big(|u_n'(\tau)|^2 + \|u_n(\tau)\|^2 + \\ + \|u_n(\tau)\|^{\frac{2n-2}{n-2}} + 1\Big) e^{\int_{-\infty}^{+\infty}(\alpha(t)+\beta(t))dt}. \qquad (25)$$

So using the compactness theorem we can extract a subsequence such, that

$$\varphi_n \to \varphi = \begin{pmatrix} u \\ u_t \end{pmatrix} \ \text{ in } \ L^\infty(\tau,T;E) \ \text{ weak} - \text{star},$$

$$\varphi_n(t) \to \varphi(t) \ \text{ in } E \ \text{ weakly } \ \forall t \in [\tau,T], \qquad (26)$$

$$u_n \to u \ \text{ in } L^2(\tau,T;L^2(\Omega)) \ \text{ strongly}$$

$$u_n(t,x) \to u(t,x) \ \text{ a.e.}$$

From Lemma 1 and (25) $\{g_n(t,u_n)\}$ is bounded in $L^2(\tau,T;L^2(\Omega))$. According to convergence $\sigma_n \to \sigma$ in Σ we have $\forall R>0$

$$\sup_{t\in[\tau,T]} \sup_{|v|\le R} |g_n(t,v) - g(t,v)| \to 0, \ \ n\to\infty.$$

Hence $g_n(t,u_n(t,x)) \to g(t,u(t,x))$ a.e. and from Lions Lemma we obtain $g_n(t,u_n) \to g(t,u)$ in $L^2(\tau,T;L^2(\Omega))$ weakly. It allows us to pass to the limit in (4), wrote for $\varphi_n(\cdot)$, and we deduce that $\varphi(\cdot)=\begin{pmatrix} u(\cdot) \\ u_t(\cdot) \end{pmatrix}$ is solution of $(1)_\sigma$, $\varphi(\tau)=\varphi_\tau$.

Now we prove that $\forall t\in[\tau,T]$ $(F_{\sigma_n}(t,u_n(t)),1) \to (F_\sigma(t,u(t)),1)$ (other statements can be proved by similar arguments). Firstly $F_{\sigma_n}(t,u_n(t,x)) \to$ $\to F_\sigma(t,u(t,x))$ for a.a. $x\in\Omega$ and from Lemma 1 and (13) $|F_{\sigma_n}(t,u_n(t,x))| \le$ $\le C_1\left(1+|u_n(t,x)|^{\frac{2n-2}{n-2}}\right)$. As $\forall\, t\in[\tau,T]$

$$\int_\Omega |u_n(t,x)-u(t,x)|^{\frac{2n-2}{n-2}} dx = \int_\Omega |u_n(t,x)-u(t,x)| \cdot |u_n(t,x)-u(t,x)|^{\frac{n}{n-2}} dx \le \\ \le |u_n(t)-u(t)| \cdot \|u_n(t)-u(t)\|^{\frac{n}{n-2}},$$

and $u_n(t)\to u(t)$ in $L_2(\Omega)$ strongly, from (25) we deduce that $|u_n(t,x)|^{\frac{2n-2}{n-2}} \to$ $\to |u(t,x)|^{\frac{2n-2}{n-2}}$ in $L^1(\Omega)$. So we can apply Lebesgue theorem and obtain that $\forall t\in[\tau,T]$ $F_{\sigma_n}(t,u_n(t,x)) \to F_\sigma(t,u(t,x))$ in $L^1(\Omega)$. Lemma is proved.

Remark 3. From Lemma 6 we have that $\forall t \in [\tau, T]$ $H_{\sigma_n}(t, \varphi_n(t)) \to \to H_\sigma(t, \varphi(t))$ and the following estimate holds:

$$\sup_{t \in [t,T]} \left| H_{\sigma_n}(t, \varphi_n(t)) \right| \le C_5, \tag{27}$$

where constant $C_5 > 0$ dependes only on C_4 from $\| \varphi_n(\tau) \| \le C_4$.

Theorem. Under conditions (3), (6) the family of maps, constructed in (8), is a strict family of $MP\{U_\sigma : \mathbb{R}_d \times E \to P(E)\}_{\sigma \in \Sigma}$, for which there exists an invariant global attractor in the phase space E.

Proof. Let us prove that the family (8) satisfies Definition 2 with equalities in 2), 3). Condition 1) is obvious. Let $\xi \in U_\sigma(t, \tau, \varphi_\tau)$. Then $\xi = \varphi(t), \varphi(\cdot)$ is solution of $(1)_\sigma$ on $(\tau, +\infty)$, $\varphi(\tau) = \varphi_\tau$. Then $\forall s \in [\tau, T]$ $\varphi(s) \in U_\sigma(s, \tau, \varphi_\tau)$. We put $\psi(p) = \varphi(p),\ p \ge s$. Then $\psi(\cdot)$ is solution of $(1)_\sigma$ on $(s, +\infty)$, $\psi(s) = \varphi(s)$. So $\xi = \psi(t) \in U_\sigma(t, s, \varphi(s)) \subset U_\sigma(t, s, U_\sigma(s, \tau, \varphi_\tau))$.

Let $\xi \in U_\sigma(t, s, U_\sigma(s, \tau, \varphi_\tau))$. Then $\xi \in U_\sigma(t, s, \eta)$, $\eta \in U_\sigma(s, \tau, \varphi_\tau)$. Hence $\xi = \varphi(t)$, $\varphi(\cdot)$ is solution of $(1)_\sigma$ on $(s, +\infty)$, $\varphi(s) = \eta$, $\eta = \psi(s)$, $\psi(\cdot)$ is solution of $(1)_\sigma$ on $(\tau, +\infty)$, $\psi(\tau) = \varphi_\tau$. We put $\theta(p) = \begin{cases} \psi(p), & p \in [\tau, s] \\ \varphi(p), & p > s \end{cases}$. Then $\xi = \varphi(t) = \theta(t)$, $\theta(\cdot)$ is solution of $(1)_\sigma$ on $(\tau, +\infty)$, $\theta(\tau) = \psi(\tau) = \varphi_\tau$. Thus $\xi \in U_\sigma(t, \tau, \varphi_\tau)$

Let $\xi \in U_\sigma(t + h, \tau + h, \varphi_\tau)$. Then $\xi = \varphi(t + h)$, $\varphi(\cdot)$ is solution of $(1)_\sigma$ on $(\tau + h, +\infty)$, $\varphi(\tau + h) = \varphi_\tau$. We put $v(p) = \varphi(p + h)$, $p \ge \tau$. Then $v(\cdot)$ is solution of $(1)_{T(h)\sigma}$ on $(\tau, +\infty)$, $v(\tau) = \varphi_\tau$, so $\xi = v(t) \in U_{T(h)\sigma}(t, \tau, \varphi_\tau)$

Let $\xi \in U_{T(h)\sigma}(t, \tau, \varphi_\tau)$. Then $\xi = \varphi(t)$, $\varphi(\cdot)$ is solution of $(1)_{T(h)\sigma}$ on $(\tau, +\infty)$, $\varphi(\tau) = \varphi_\tau$. We put $v(p) = \varphi(p - h)$, $p \ge \tau + h$. Then $v(\tau + h) = \varphi_\tau$, $v(\cdot)$ is solution of $(1)_\sigma$ on $(\tau + h; +\infty)$, that is $\xi = v(t + h) \in U_\sigma(t + h, \tau + h, \varphi_\tau)$. So, $\{U_\sigma\}_{\sigma \in \Sigma}$ is a strict family of MP.

Now we verify conditions 1)−3) of Lemma 2. From estimate (24) with $\tau = 0$ we immediately obtain property (9).

Let $\xi_n \in U_{\sigma_n}(t, 0, \eta_n)$, $\xi_n \to \xi$, $\eta_n \to \eta$ in E. Since Σ is compact, we can claim $\sigma_n \to \sigma$ in Σ. Then $\xi_n = \varphi_n(t)$, $\varphi_n(\cdot)$ is solution of $(1)_{\sigma_n}$, $\varphi_n(0) = = \eta_n \to \eta$. From Lemma 6 we deduce that $\forall s \ge 0$ $\varphi_n(s) \to \varphi(s)$ weakly in E, where $\varphi(s) \in U_\sigma(s, 0, \eta)$. Thus $\xi_n = \varphi_n(t) \to \varphi(t) = \xi \in U_\sigma(t, 0, \eta)$ and property 2) is proved.

To finish the proof we should check the property (10). Let $\xi_n \in \in U_{\sigma_n}(t_n, 0, \eta_n)$, $\eta_n \in B \in \beta(E)$, $t_n \to \infty$, $\sigma_n \to \sigma$. Then $\xi_n = \varphi_n(t_n)$, $\varphi_n(\cdot)$ is solution of $(1)_{\sigma_n}$, $\varphi_n(0) = \eta_n$. Using (24) we have that $\{\varphi_n(t_n)\}$ is bounded in E. Hence there exists $\theta \in E$ such that on some subsequence $\xi_n = \varphi_n(t_n) \to \theta$ weakly in E. In the same way $\forall M \ge 0$ $\varphi_n(t_n - M) \to \theta_M$ weakly in E.

Moreover $\forall t \geq 0 \quad \varphi_n(t_n - M + t) \in U_{\sigma_n}(t_n - M + t, t_n - M, \varphi_n(t_n - M)) = = U_{T(t_n - M)\sigma_n}(t, 0, \varphi_n(t_n - M))$. It follows that $\varphi_n(t_n - M + t) = v_n(t)$, $v_n(\cdot)$ is a solution of $(1)_{T(t_n - M)\sigma_n}$, $v_n(0) = \varphi_n(t_n - M)$. Since $\tilde{\sigma}_n := T(t_n - M)\sigma_n \to \to \tilde{\sigma}$ in Σ, from Lemma 6 we obtain that $\forall t \geq 0 \quad v_n(t) \to v(t)$ weakly in E, where $v(t) \in U_{\tilde{\sigma}}(t, 0, \theta_M)$. In particular, $v_n(M) = \xi_n \to v(M) = \theta \in U_{\tilde{\sigma}}(M, 0, \theta_M)$ weakly in E.

From equality (18) writed for $v_n(\cdot)$ we have $\forall t \geq 0$

$$I_{\tilde{\sigma}_n}(t, v_n(t)) = I_{\tilde{\sigma}_n}(0, v_n(0)) e^{-\gamma t} + \int_0^t e^{\gamma(p-t)} H_{\tilde{\sigma}_n}(p, v_n(p)) dp$$

and with $t = M$

$$I_{\tilde{\sigma}_n}(M, \xi_n) = I_{\tilde{\sigma}_n}(0, v_n(0)) e^{-\gamma M} + \int_0^M e^{\gamma(p-M)} H_{\tilde{\sigma}_n}(p, v_n(p)) dp .$$

Hence

$$\liminf_{n\to\infty} I_{\tilde{\sigma}_n}(M, \xi_n) \leq \limsup_{n\to\infty} I_{\tilde{\sigma}_n}(0, v_n(0)) e^{-\gamma M} + + \limsup_{n\to\infty} \int_0^M e^{\gamma(p-M)} H_{\tilde{\sigma}_n}(p, v_n(p)) dp . \tag{28}$$

Thanks to (24) $\limsup_{n\to\infty} I_{\tilde{\sigma}_n}(0, v_n(0)) \leq C_6$,where constant $C_6 > 0$ does not depend on n and M . Moreover, from Remark 3 we conclude that

$$\limsup_{n\to\infty} \int_0^M e^{\gamma(p-M)} H_{\tilde{\sigma}_n}(p, v_n(p)) dp = \int_0^M e^{\gamma(p-M)} H_{\tilde{\sigma}}(p, v(p)) dp . \tag{29}$$

If we write equality (18) for function $v(\cdot)$ and for $t = M$, then

$$I_{\tilde{\sigma}}(M, v(M)) = I_{\tilde{\sigma}}(0, v(0)) e^{-\gamma M} + \int_0^M e^{\gamma(p-M)} H_{\tilde{\sigma}}(p, v(p)) dp . \tag{30}$$

Moreover, if we denote $v(\cdot) = \begin{pmatrix} \omega(\cdot) \\ \omega_t(\cdot) \end{pmatrix}$, then

$$\liminf_{n\to\infty} I_{\tilde{\sigma}_n}(M, \xi_n) \geqslant \geqslant \frac{1}{2} \liminf_{n\to\infty} \|\xi_n\|_E^2 + \frac{\gamma}{2}(\omega_t(M), \omega(M)) + (F_{\tilde{\sigma}}(M, \omega(M)), 1). \tag{31}$$

From (28)–(31) we obtain

$$\frac{1}{2} \liminf_{n\to\infty} \|\xi_n\|_E^2 + \frac{\gamma}{2}(\omega_t(M), \omega(M)) + (F_{\tilde{\sigma}}(M, \omega(M)), 1) \leqslant$$

$$\leqslant C_6 e^{-\gamma M} - I_{\tilde{\sigma}_n}(M, v(M)) - I_{\tilde{\sigma}_n}(0, v(0)) e^{-\gamma M} =$$

$$= C_6 e^{-\gamma M} - I_{\widetilde{\sigma}}(0, v(0)) e^{-\gamma M} + \frac{1}{2}\|\theta\|_E^2 + \frac{\gamma}{2}(\omega_t(M), \omega(M)) + (F_{\widetilde{\sigma}}(M, \omega(M)), 1).$$

So

$$\frac{1}{2}\liminf_{n\to\infty}\|\xi_n\|_E^2 \leqslant C_6 e^{-\gamma M} - I_{\widetilde{\sigma}}(0, v(0)) e^{-\gamma M} + \frac{1}{2}\|\theta\|_E^2. \qquad (32)$$

From (24) $\|\varphi_n(t_n - M)\|_E^2 \leqslant K(B)$, where constant $K(B) > 0$ does not depend on n, M. As $\varphi_n(t_n - M) \to \theta_M$ weakly in E, we have $\|\theta_M\|_E^2 \leqslant \leqslant \liminf_{n\to\infty}\|\varphi_n(t_n - M)\|_E^2 \leqslant K(B)$. Since $\theta_M = v(0)$, then we can pass to limit in (32) for $M \to \infty$ and obtain

$$\frac{1}{2}\liminf_{n\to\infty}\|\xi_n\|_E^2 \leqslant \frac{1}{2}\|\theta\|_E^2.$$

In view of weak convergence ξ_n to θ in E we have inverse inequality, so $\xi_n \to \theta$ strongly in E. Theorem is proved.

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Received 05.10.2005

From the Editorial Board: The article corresponds completely to submitted manuscript.