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SPECTRAL PROPERTIES OF NON-HOMOGENEOUS TIMOSHENKO BEAM AND ITS CONTROLLABILITY

Controllability of slowly rotating non-homogeneous beam clamped to a disc is considered. It is assumed that at the beginning the beam remains at the position of rest and it is supposed to rotate by the given angle and achieve desired position. The rotor of propelling engine is in the middle of the disk. The movement is governed by the system of two differential equations with non-constant coefficients: linear mass density, flexural rigidity, rotational inertia and shear stiffness. To solve the problem of controllability, the spectrum of the operator generating the dynamics of the model is studied. Then the problem of controllability is reduced to the moment problem that is, in turn, solved with the use of the asymptotics of the spectrum and Ullrich Theorem.

Introduction. The considered model of Timoshenko beam clamped to a disc of driving motor was first described by X.J. Xiong [1]. The derivation of that model was described also in monograph [2]. The controllability and stabilization of homogeneous Timoshenko beam was elaborated by W. Krabs and G.M. Sklyar first in [3] and then in a series of papers that was concluded with the monograph [2]. The first author to consider controllability of non-homogeneous Timoshenko beam was M. Shubov [4–6]. She considers the different model of the beam, but we used her ideas while deriving equations of current model.

In this paper we present our results published in [7] and [8]. Described are theorems and some ideas of proofs. For detailed proofs we send reader to mentioned articles.

1. Model equation and spectral operators connected to it. We consider the motion of a beam in a horizontal plane. The left end of the beam is clamped to the disk of a driving motor. We denote by r the radius of that disk and let $\theta = \theta(t)$ be the rotation angle considered as a function of time ($t \geq 0$). Further on, we assign to a (uniform) cross section at x, with $0 \leq x \leq 1$ the following: $E(x)$ which is the flexural rigidity, $K(x)$ – shear stiffness, $\varrho(x)$ – mass of the cross section and $R(x)$ – rotary inertia. All of the above functions are assumed to be real and bounded by two positive numbers. We also assume that their first and second derivatives are bounded. The length of the beam is assumed to be 1. We denote by $w(x;t)$ the deflection of the center line of the beam and by $\xi(x,t)$ the rotation angle of the cross section area at the location x and at time t. Then w and ξ are governed by the following system of two hyperbolic differential equations:

$$\begin{aligned} &\varrho(x)\ddot{w}(x,t) - \big(K(x)(w'(x,t) + \xi(x,t))\big)' = -\ddot{\theta}(t)\varrho(x)(x+r), \\ &R(x)\ddot{\xi}(x,t) - (E(x)\xi'(x,t))' + K(x)(w'(x,t) + \xi(x,t)) = \ddot{\theta}(t)R(x). \end{aligned} \tag{1}$$

In addition to (1) we impose the following boundary conditions:

$$w(0,t)=\xi(0,t)=0,$$
$$w'(1,t)+\xi(1,t)=0, \quad \xi'(1,t)=0$$

for $t\geq 0$.

We consider $L^2([0,1],\mathbb{C}^2)$ as an underlying set with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \int_0^1 \varrho(x)y_1(x)\overline{y_2(x)}dx + \int_0^1 R(x)z_1(x)\overline{z_2(x)}dx. \tag{2}$$

Due to hypotheses imposed on ϱ and R, the norm generated by (2) is equivalent to the standard L^2 norm.

Let $H=H_0^2([0,1],\mathbb{C}^2)$ be the set of all functions from $L^2([0,1],\mathbb{C}^2)$ (with the inner product (2)) that are twice differentiable and whose value at 0 is $(0,0)^T$. We define the linear operator $A: D(A)\to H$ by the formula

$$A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\varrho}(K(y'+z))' \\ -\frac{1}{R}\big((Ez')'-K(y'+z)\big) \end{pmatrix}, \tag{3}$$

where K,E,ϱ,R,y and z are functions of variable $x\in[0,1]$ and

$$D(A)=\left\{\begin{pmatrix} y \\ z \end{pmatrix} : y(0)=z(0)=0,\ y'(1)+z(1)=z'(1)=0\right\}\subset H.$$

It is easy to see that $D(A)$ is dense in H. Using the defined operator A and putting

$$f_1(x,t)=-\ddot{\theta}(t)(r+x), \qquad f_2(x,t)=\ddot{\theta}(t), \tag{4}$$

we can rewrite the equations (1) in the vector form

$$\begin{pmatrix} \ddot{w}(x,t) \\ \ddot{\xi}(x,t) \end{pmatrix} + A\begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}. \tag{5}$$

It follows readily that the operator $A: D(A)\to H$ is positive, symmetric, invertible and self-adjoint [7]. Therefore there exists the unique weak solution to (1) given by

$$\begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \sum_{j=1}^{\infty}\frac{1}{\sqrt{\lambda_j}}\int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \sin\sqrt{\lambda_j}(t-s)ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \tag{6}$$

The inner product we use here is defined in (2), the functions f_1 and f_2 are defined in (4) and $\begin{pmatrix} y_j \\ z_j \end{pmatrix}$ for $j\in\mathbb{N}$ are the eigenvectors of the operator A that correspond to eigenvalues λ_j. Also we notice, that the first (time) derivative of the above solution is

$$\begin{pmatrix} \dot{w}(x,t) \\ \dot{\xi}(x,t) \end{pmatrix} = \sum_{j=1}^{\infty} \int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \cos\sqrt{\lambda_j}(t-s)ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \qquad (7)$$

To use the solution (6) for the controllability purposes, we need to know at least approximate location of eigenvalues of the operator A.

2. Asymptotic behaviour of eigenvalues. In order to study the location of eigenvalues of the operator A defined by (3), we need to consider the following system of spectral equations

$$\begin{aligned} -\big(K(x)(y'(x)+z(x))\big)' &= \lambda\varrho(x)y(x), \\ -\big((E(x)z'(x)\big)' + K(x)(y'(x)+z(x)) &= \lambda R(x)z(x) \end{aligned}$$

with the boundary conditions

The asymptotic location of eigenvalues are given by the following theorem [7].

Theorem 1. *The set of eigenvalues of the operator A is given by formulas*

$$\lambda_n^{(0)} = \left(\int_0^1 \sqrt{\frac{\varrho(t)}{K(t)}}\,dt\right)^{-2} \left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(0)}\right)^2, \qquad (8)$$

$$\lambda_n^{(1)} = \left(\int_0^1 \sqrt{\frac{R(t)}{E(t)}}\,dt\right)^{-2} \left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(1)}\right)^2 \qquad (9)$$

with $\varepsilon_n^{(0)}, \varepsilon_n^{(1)} \to 0$ as $n \to \infty$. Thus the spectrum of the operator A asymptotically splits into two sets – $\Lambda^{(0)}$, whose elements are described by (8) and $\Lambda^{(1)}$ containing all elements of the form (9).

The proof of the above theorem is solving the spectral equation. It is brought to the system of two integral equations that is, in turn, solved with the use of Neumann series. It is shown in [7] that the eigenvalues are at most double. However, we suspect that the eigenvalues are asymptotically single, but we do not have a rigorous proof of that fact.

3. Rest to rest controllability. Given the beam, whose movement is described by (1), we want to rotate it from the state of rest at time $t = 0$ to the state of rest at the time $t = T > 0$. Thus we have the following boundary conditions:

$$\begin{array}{ccccccccl} w(x,0) & = & \dot{w}(x,0) & = & \xi(x,0) & = & \dot{\xi}(x,0) & = 0, \\ w(x,T) & = & \dot{w}(x,T) & = & \xi(x,T) & = & \dot{\xi}(x,T) & = 0 \end{array} \qquad (10)$$

for $x \in [0,1]$. The beam is controlled by motor that rotates it from angle 0 to θ_T . The control is given by angular acceleration $\ddot{\theta}(t)$ and our goal is to find that function. The beginning position of rest means that the motor does not work, i.e. the function θ is a member of $H_0^2(0,T)$, where

$$H_0^2(0,T) = \left\{ f \in H^2(0,T) : f(0) = \dot{f}(0) = 0 \right\}.$$

At the end of the movement the beam is at the position $\theta(T) = \theta_T$ and the motor does not move, i.e. $\dot{\theta}(T) = 0$.

Thus, to solve the problem of controllability from rest to rest, we need for given time $T > 0$ and angle $\theta_T \in \mathbb{R}$, $\theta_T \neq 0$ to find a function $\theta \in H_0^2(0,T)$ with

$$\theta(T) = \theta_T, \qquad \dot{\theta}(T) = 0. \tag{11}$$

Using weak solution (6) and its derivative (7) we arrive at the conclusion that conditions (10) are equivalent to

$$\frac{1}{\sqrt{\lambda_n}} \int_0^T \left\langle \begin{pmatrix} f_1(\cdot,t) \\ f_2(\cdot,t) \end{pmatrix}, \begin{pmatrix} y_n \\ z_n \end{pmatrix} \right\rangle \sin(T-t)\sqrt{\lambda_n} dt = 0$$

and

$$\int_0^T \left\langle \begin{pmatrix} f_1(\cdot,t) \\ f_2(\cdot,t) \end{pmatrix}, \begin{pmatrix} y_n \\ z_n \end{pmatrix} \right\rangle \cos(T-t)\sqrt{\lambda_n} dt = 0$$

for all $n \in \mathbb{N}$. Here the eigenvalues λ_n are not yet distinguish on those that belong to $\Lambda^{(0)}$ and $\Lambda^{(1)}$. Next, upon putting

$$a_n = \int_0^1 R(x)\overline{z_n(x)}dx - \int_0^1 \varrho(x)(r+x)\overline{y_n(x)}dx \tag{12}$$

we obtain

$$\left\langle \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}, \begin{pmatrix} y_n(x) \\ z_n(x) \end{pmatrix} \right\rangle = a_n \ddot{\theta}(t)$$

for all positive integer n. We recall that the above inner product is defined by the formula (2).

We remark here that for the controllability from rest to rest, the condition $a_n \neq 0$ for all positive integer n (the formulas for a_n an are given by (12)) is not necessary. It becomes crucial while considering controllability from rest to arbitrary condition. The values of those parameters depend on the radius r of the disc (in general, on the ratio radius to the length of the beam). However, it was proved in [8] that there are only countable many values of r for which some of a_n's are zeroes.

Employing (12) and well-known trigonometric formulas we obtain

$$\begin{aligned} a_n \int_0^T \ddot{\theta}(t)\sin(t\sqrt{\lambda_n})dt = 0, \\ a_n \int_0^T \ddot{\theta}(t)\cos t\sqrt{\lambda_n}dt = 0. \end{aligned} \tag{13}$$

Now we notice that if $\theta \in H_0^2(0,T)$, then the conditions (11) are equivalent to

$$\begin{aligned} &\int_0^T \ddot{\theta}(t)dt = 0, \\ &\int_0^T t\ddot{\theta}(t)dt = -\theta_T. \end{aligned} \tag{14}$$

Gathering (13) and (14) we obtain that the problem of controllability from rest to rest is equivalent to the following moment problem.

Moment Problem. Find $u \in L^2(0,T)$ such that for all $n \in \mathbb{N}$ the conditions

$$\begin{aligned} \int_0^T u(t)\cos t\sqrt{\lambda_n}dt &= 0, \\ \int_0^T u(t)\sin t\sqrt{\lambda_n}dt &= 0; \\ \int_0^T u(t)dt &= 0, \\ \int_0^T tu(t)dt &= -\theta_T \end{aligned}$$

are satisfied.

To find the solution to the stated Moment Problem we consider the system

$$\left\{t, 1, \cos t\sqrt{\lambda_n^{(i)}}, \sin t\sqrt{\lambda_n^{(i)}} : n \in \mathbb{N}, i \in \{0,1\}\right\}. \tag{15}$$

Here the $\lambda_n^{(i)}$'s are the eigenvalues described in Theorem 1. Just for convenience, we rewrite the system (15) in the form

$$V \cup \{t\}, \quad \text{where} \quad V = \left\{1, \cos t\sqrt{\lambda_n^{(i)}}, \sin t\sqrt{\lambda_n^{(i)}} : n \in \mathbb{N}, i \in \{0,1\}\right\}.$$

Let W be the closure of the linear span over V. Using the Theorem of Russel [9], one can prove [7] that the above system is minimal for $T > 2(J^{(0)} + J^{(1)})$, where

$$J^{(0)} = \int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx \quad \text{and} \quad J^{(1)} = \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx.$$

Let $f(t) = t$. Considering T given above, we have the existence of exactly one $h_0 \in W$such that

$$\|h_0 - f\|_2 \leq \|h - f\|_2 \quad \text{for all } h \in W,$$

where $\|\cdot\|_2$ denotes the L^2-norm in $L^2(0,T)$. For that h_0 we have

$$\int_0^T (f(t)-h_0(t))h(t)dt = 0 \quad \text{for all } h \in W.$$

In particular, the above implies

$$\begin{gathered}\int_0^T (f(t)-h_0(t))dt = 0,\\ \int_0^T (f(t)-h_0(t))\cos t\sqrt{\lambda_n^{(i)}}dt = 0, \quad n\in\mathbb{N}, i\in\{0,1\},\\ \int_0^T (f(t)-h_0(t))\sin t\sqrt{\lambda_n^{(i)}}dt = 0, \quad n\in\mathbb{N}, i\in\{0,1\}\end{gathered}$$

and

$$\int_0^T (f(t)-h_0(t))f(t)dt = \|h_0-f\|_2^2 > 0.$$

Upon defining

$$u(t) = -\frac{\theta_T}{\|h_0-f\|_2^2}(f(t)-h_0(t))$$

for $t\in[0,T]$ we receive $u\in L^2(0,T)$ that solves the Moment Problem.

Thus we have the following theorem:

Theorem 2. *The problem of controllability from rest to rest is solvable if*

$$T > 2\left(\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx + \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx\right).$$

4. Controllability from rest to arbitrary position. We assume that at time $t=0$ the beam remains at the position of rest, i.e.

$$w(x,0) = \dot{w}(x,0) = \xi(x,0) = \dot{\xi}(x,0) = \theta(0) = \dot{\theta}(0) = 0$$

for $x\in[0,1]$. At a given time T, we need to achieve the following position

$$\begin{aligned} w(x,T) &= w_T(x), & \dot{w}(x,T) &= \dot{w}_T(x),\\ \xi(x,T) &= \xi_T(x), & \dot{\xi}(x,T) &= \dot{\xi}_T(x), \end{aligned} \tag{16}$$

where functions w_T, $\dot{w}_T$, ξ_T, $\dot{\xi}_T$ defined on $[0,1]$ are given. The problem of controllability from rest to arbitrary position is:

Problem of controllability. *Given time $T>0$, numbers θ_T, $\dot{\theta}_T\in\mathbb{R}$ and position (16), find a function $\theta\in H_0^2(0,T)$ satisfying*

$$\theta(T) = \theta_T, \qquad \dot{\theta}(T) = \dot{\theta}_T \tag{17}$$

and such that the weak solution (6) of (1) satisfies (16).

Consideration similar to the above one leads to following moment problem.

Moment Problem. Find $u \in L^2(0,T)$ such that for all $n \in N$ and $k \in \{0,1\}$ the conditions

$$\int_0^T u(t)\cos t\sqrt{\lambda_n^{(k)}}dt = \dot{c}_n^{(k)}, \qquad \int_0^T u(t)\sin t\sqrt{\lambda_n^{(k)}}dt = c_n^{(k)},$$

$$\int_0^T u(t)dt = \dot{\theta}_T, \qquad \int_0^T tu(t)dt = \theta_T$$

are satisfied.

Here $(c_n^{(k)})_n$ and $(\dot{c}_n^{(k)})_n$ are sequences whose values depend on end conditions (16) and on eigenvalues.

The stated problem is divided onto three cases.

Case 1: $\boldsymbol{J^{(0)} = J^{(1)} = J}$.

In the solution the Ullrich Theorem [10] is used and the final result is given by theorem [8]:

Theorem 3. *Provided*

$$\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx = \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx \quad \text{and} \quad T \geq 4\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}} > dx,$$

the problem of controllability from the state of rest to arbitrary position is solvable if and only if the following condition is satisfied

$$\sum_{n=1}^{\infty}\left(|c_{n0}|^2 + \left|\frac{c_{n0}-c_{n1}}{\sqrt{\lambda_n^{(0)}}-\sqrt{\lambda_n^{(1)}}}\right|^2\right) < \infty$$

with $c_{nk} = (\pi/J)(\dot{c}_n^{(k)} + ic_n^{(k)})$.

Case 2: $\boldsymbol{J^{(1)}/J^{(0)} = \dfrac{p}{q}}$ is rational number and p and q are relatively prime positive odd integers.

Without loss of generality, we may assume that $J^{(1)}/J^{(0)} = \gamma > 1$. Here the Ullrich-type Theorem [11] in place of Ullrich Theorem is used . The final result is the following theorem [7]

Theorem 4. *If* $\displaystyle\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx \Big/ \int_0^1 \sqrt{\frac{R(x)}{E(x)}}dx = \frac{p}{q}$ *with* p, q *relatively prime odd positive integers and*

$$T \geq 2\left(\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx + \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx\right),$$

the problem of controllability from the state of rest to arbitrary position is solvable if and only if the condition

$$\sum_{n=1}^{\infty}\left(|c_{n0}|^2+|c_{n1}|^2+\left|\frac{c_{((1-q)/2)+qn,0}-c_{((1-p)/2)+pn,1}}{\sqrt{\lambda^{(0)}_{((1-q)/2)+qn}}-\sqrt{\lambda^{(1)}_{((1-p)/2)+pn}}}\right|^2\right)<\infty$$

is satisfied.

Case 3: $\boldsymbol{J^{(1)}/J^{(0)}=\frac{p}{q}}$ is rational number, p and q are relatively prime positive integers and exactly one of them is even.

We proceed in the same way like in Case 2 and finally get the following theorem

Theorem. *If* $\int_0^1\sqrt{\frac{\varrho(x)}{K(x)}}\,dx\Big/\int_0^1\sqrt{\frac{R(x)}{E(x)}}\,dx=\frac{p}{q}$ *with* p, q *relatively prime positive integers, from which exactly one is even and*

$$T\geq 2\left(\int_0^1\sqrt{\frac{\varrho(x)}{K(x)}}\,dx+\int_0^1\sqrt{\frac{R(x)}{E(x)}}\,dx\right),$$

the problem of controllability from the state of rest to arbitrary position is solvable if and only if the condition

$$\sum_{n=-\infty}^{\infty}\left(|c_{n0}|^2+|c_{n1}|^2\right)<\infty.$$

is satisfied.

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Institute of Mathematics, Szczecin University, Poland
`sklar@sus.univ.szczecin.pl, szkibiel@poczta.onet.pl`

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