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Theory of integral acoustoelasticity for 3-D stress-strained state

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Starting with the model of small elastic disturbance in a non-uniformly strained body and taking into account the weakness of the body's acoustical inhomogeneity and anisotropy induced by strain, a theory for integral acoustoelasticity has been developed in the paper. The theory establishes mathematical models for interaction of narrow longitudinally and transversally polarized ultrasonic beams with 3-D strain field in the body. Ray integrals of acoustoelasticity have been established with the use of the model. These relationships connect measured phase parameters of longitudinally and transversally polarized ultrasonic beams, crossing the body in any direction, with integrals of initial strain distribution along this direction. They can be used to formulate problems for tomography of the body's stress-strained state.

Key words: strain field, acoustoelasticity, acoustical tensor field tomography.

Introduction. Acoustoelasticity is a feature of solids to change their acoustical properties under strain. Physical nature of this effect consists in the dependence of the mass density and elasticity moduli on strain and in non-additivity of strains of initial state and a disturbance [1, 2]. In the case of homogeneous initial strained state acoustoelasticity relationships were obtained [1-4]. They connect phase velocities of plane waves with components of initial strain tensor and elasticity moduli of the body.

If the body is non-uniformly strained it becomes acoustically anisotropic and inhomogeneous. Propagation of small elastic disturbances in such object is described by a system of hyperbolic type differential equations with variable coefficients [5]. Thereupon problems of wave field analysis in such an objects become much more complicated. But acoustical anisotropy and inhomogeneity induced by elastic strain are weak. This makes possible to simplify the mathematical model for interaction of acoustical waves with non-uniformly strained solids. For instance, in papers [6, 7] the weakness of acoustical inhomogeneity was used to build an iteration process for a problem of small pulsed disturbance propagation in non-uniformly strained solid continuum. This approach enabled us to establish the integral acoustoelasticity relationships. They express time periods for elastic pulses travelling along a given segment in strained continuum via integrals of initial strain distribution on the segment.

1. Small elastic disturbance in a non-uniformly strained solid

Propagation of small elastic disturbance in non-uniformly strained elastic body $\mathcal{B}$ is described in geometrically linear approach by hyperbolic system of equations [5]

$$\rho \frac{\partial^2 w_i}{\partial t^2} = \frac{\partial}{\partial x_j}\left(C^{\varepsilon}_{ijkl} \frac{\partial w_k}{\partial x_l} \right), \tag{1}$$

where $\rho, t, w_i \left(i = \overline{1,3} \right)$ and x_i, stand for mass density, time variable, components of the disturbance displacement vector **w** and Cartesian coordinates; $C^{\varepsilon}_{ijkl} \left(i, j, k, l = \overline{1,3} \right)$ are dependent on initial strain moduli of elasticity for small elastic disturbance

$$C^{\varepsilon}_{ijkl} = C_{ijkl} + \Gamma_{ijklmn} \varepsilon_{mn} . \tag{2}$$

In this formula $\varepsilon_{mn} \left(m, n = \overline{1,3} \right)$ stands for Cartesian components of initial strain tensor, C_{ijkl} and Γ_{ijklmn} are of order two and three elasticity moduli of the body.

The formula (2) is valid for small elastic strains ε_{mn} of an infinitesimal order α_{ε}. Elastic disturbance is small as against initial strain field. This means that displacement gradients $\partial w_l / \partial x_k$ are quantities of higher order of smallness in comparison with strains ε_{mn}. We will consider the components $\partial w_l / \partial x_k$ as quantities of the infinitesimal order $\alpha_e = \alpha_{\varepsilon}^2$.

For isotropic bodies, the components C_{ijkl} and Γ_{ijklmn} represent isotropic tensors of rank four and six respectively

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj} \right), \qquad \Gamma_{ijklmn} = \Gamma'_{((ij)(kl)(mn))},$$

$$\Gamma'_{ijklmn} = \frac{1}{3}(l + 2m) \delta_{ij} \delta_{kl} \delta_{mn} - m \delta_{ij} \left(\delta_{kl} \delta_{mn} - \delta_{kn} \delta_{lm} \right) + \frac{n}{6} \in_{ikm} \in_{jln} .$$

Here δ_{ij} and $\in_{jln}$ stand for Kronecker's delta and Levi-Civita symbols; λ, μ and l, m, n denote Lamé and Murnagan constants. Parentheses in the denotation $\Gamma'_{((ij)(kl)(mn))}$ mean symmetrization with respect to the enclosed indices.

For many engineering materials the moduli λ, μ and *l*, *m*, *n* are quantities of the same order of magnitude. Hence the second term in the formula (2) is quantities of the order α_{ε} as compared to the first one. So, acoustic anisotropy induced by strain is weak.

Let $\mathcal{L}_{\mathbf{n}}$ be a straight line crossing the area $\mathcal{V}$ in the direction of unit vector $\mathbf{n} = \left(n_1, n_2, n_3 \right)$ and $\left\| \partial \varepsilon^e_{ij} / \partial x_k \right\|_{L_0} \equiv \left(l^{\varepsilon \mathbf{n}} \right)^{-1}$ be a norm of strain tensor gradient on the segment $L_{\mathbf{n}} = \mathcal{L}_{\mathbf{n}} \cap \mathcal{V}$. The value $l^{\varepsilon \mathbf{n}}$ is characteristic of optical inhomogeneity of the body $\mathcal{B}$ — the greater is $l^{\varepsilon \mathbf{n}}$, the weaker is optical inhomogeneity along the direction **n**.

2. Directional sounding of strained body

External narrow ultrasonic beam (pulsed or continuous) can be used for elastic waves in the body excitation. A schematic model for such sounding implementation is shown in fig. 1. It includes an ultrasound vibration generator 1, for instance, a piezoelectric transducer and an acoustic waveguide 2 with bevel face 3. The waveguide has been fabricated from the same material as the body. Owing to this differences in acoustical properties of the waveguide and the body $\mathcal{B}$ are small quantities of infinitesimal order α_ε. The plate of the transducer 1 is rigidly connected to the bevel face 3 of the waveguide 2. Depending on polarization, it produces normal or tangential displacement on some area of the bevel face 3. In-plane dimensions of the transducer plate 1 are much bigger than the wavelength. Practically parallel and homogeneous in its cross-section ultrasonic beam 4 is formed in the waveguide owing to this. The beam propagates in direction **n** normal to the bevel face. The waveguide is applied to the body surface with some small pressure, necessary to produce cohesion in tangential direction. The area of contact of the waveguide and the body is wetted by immersion liquid. Another waveguide 5, identical to the first one, is applied to the opposite surface of the body. It serves to transfer the beam from the body to sensing devices without distorting the wave. Such sounding technique minimizes reflection and dispersion of the incident wave on the «waveguide–body» and «body–waveguide» boundaries.

Uniform in its cross-section sounding beam crosses the interface «waveguide–body» and penetrates into the body's volume $\mathcal{V}$. Here it interacts with acoustically inhomogeneous medium and gains some gradients in normal to **n** directions. However, as the medium inhomogeneity is weak and the beam's diameter is small enough, acquired nonuniformity of sounding wave field will be also small. We will use this to simplify the mathematical model (1). To do this we rewrite the system (1) in a Cartesian system $\{y_1, y_2, y_3\}$, whose y_3 axis is directed along **n**

$$(1+\varepsilon)\frac{\partial^2 w_i}{\partial t^2} = a_{ij}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_j}{\partial y_3{}^2} + b_{ij}^{\varepsilon\mathbf{n}}\frac{\partial w_j}{\partial y_3} + a_{oij}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_j}{\partial y_o \partial y_3} + a_{opij}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_j}{\partial y_o \partial y_p} + b_{oij}^{\varepsilon\mathbf{n}}\frac{\partial w_j}{\partial y_o}, \tag{3}$$

where $o, p = 1,2$, $\varepsilon = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$ is the first invariant of the initial strain tensor,

$$a_{il}^{\varepsilon\mathbf{n}} = \rho_0{}^{-1} C_{ijkl}^{\varepsilon} n_j n_k, \quad b_{ij}^{\varepsilon\mathbf{n}} = \rho_0{}^{-1}\left(\frac{\partial C_{ijkl}^{\varepsilon}}{\partial y_3} n_j + \frac{\partial C_{ijkl}^{\varepsilon}}{\partial y_o} n_{oj}\right) n_k,$$

$$a_{oij}^{\varepsilon\mathbf{n}} = \frac{1}{\rho_0} C_{ijkl}^{\varepsilon}\left(n_{oj} n_k + n_j n_{ok}\right), \quad a_{opij}^{\varepsilon\mathbf{n}} = \frac{1}{\rho_0} C_{ijkl}^{\varepsilon} n_{oj} n_{pk}, \quad b_{oij}^{\varepsilon\mathbf{n}} = \frac{1}{\rho_0}\left(\frac{\partial C_{ijkl}^{\varepsilon}}{\partial y_3} n_j + \frac{\partial C_{ijkl}^{\varepsilon}}{\partial y_p} n_{pj}\right) n_{ok},$$

ρ_0 is mass density of unstrained body, $\mathbf{n}_o = (n_{o1}, n_{o2}, n_{o3})$ is unit vector of y_o axis.

The components $a_{il}^{\varepsilon\mathbf{n}}, a_{oil}^{\varepsilon\mathbf{n}}$ and $a_{opil}^{\varepsilon\mathbf{n}}$ in equation (3) are quantities of the same order of magnitude. However, since the body is sounding by homogeneous in its cross-section narrow beam and acoustical inhomogeneity of the body is weak, we can consider the derivatives $\partial^2 w_l / \partial y_o \partial y_3$ and $\partial^2 w_l / \partial y_o \partial y_p$ as small quantities as against $\partial^2 w_l / \partial y_3{}^2$. Similarly, the coefficients $b_{il}^{\varepsilon\mathbf{n}}$ and $b_{oil}^{\varepsilon\mathbf{n}}$ are quantities of the same order of magnitude, but derivatives $\partial w_l / \partial y_o$ are small quantities as against $\partial w_l / \partial y_3$. Hence, in the first approximation we can neglect the last three terms in the right hand side of equation (3), contained normal to **n** gradients of the disturbed wave field. This yields (using notation $y_3 \equiv y$)

$$(1+\varepsilon)\frac{\partial^2 w_i}{\partial t^2} = a_{ij}^{\varepsilon\mathbf{n}} \frac{\partial^2 w_j}{\partial y^2} + b_{ij}^{\varepsilon\mathbf{n}} \frac{\partial w_j}{\partial y}. \tag{4}$$

In the absence of initial strain we should substitute $\varepsilon_{ij}=0$ into (2). This reduces the system (4) to the system of 1-D wave equations for homogeneous elastic body

$$\frac{\partial^2 w_i}{\partial t^2} = a_{ij}^{0\mathbf{n}} \frac{\partial^2 w_j}{\partial y^2},$$

where $a_{ij}^{0\mathbf{n}}$ are components of acoustical tensor for unstrained body

$$a_{ij}^{0\mathbf{n}} = \rho_0{}^{-1} C_{ij}^{0\mathbf{n}},\ C_{ij}^{0\mathbf{n}} = C_{ijkl} n_k n_l. \tag{5}$$

For the body isotropic in its initial unstrained state

$$a_{ij}^{0\mathbf{n}} = \rho_0{}^{-1}\left((\lambda+\mu) n_i n_j + \mu\delta_{ij}\right).$$

In the basis $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}\}$ the matrix $a_{ij}^{0\mathbf{n}}$ becomes diagonal

$$a_{11}^{0\mathbf{n}} = a_{22}^{0\mathbf{n}} = \mu/\rho_0 = C_T{}^2,\ a_{33}^{0\mathbf{n}} = (\lambda+2\mu)/\rho_0 = C_L{}^2.$$

Here C_L and C_T are the phase velocities of longitudinal and transversal acoustic waves.

It is useful to represent tensors $a_{il}^{\varepsilon\mathbf{n}}$ and $b_{il}^{\varepsilon\mathbf{n}}$ in the form

$$a_{ij}^{\varepsilon\mathbf{n}} = a_{il}^{0\mathbf{n}}\left(\delta_{lj} + \kappa_{lj}^{\varepsilon\mathbf{n}}\right),\ b_{ij}^{\varepsilon\mathbf{n}} = \left(l^{\varepsilon\mathbf{n}}\right)^{-1} a_{il}^{0\mathbf{n}} \gamma_{lj}^{\varepsilon\mathbf{n}}. \tag{6}$$

Here $\kappa_{ij}^{\varepsilon\mathbf{n}} \equiv S_{ik}^{0\mathbf{n}} a_{kj}^{\varepsilon\mathbf{n}} - \delta_{ij}, \gamma_{ij}^{\varepsilon\mathbf{n}} \equiv l^{\varepsilon\mathbf{n}} S_{ik}^{0\mathbf{n}} b_{kj}^{\varepsilon\mathbf{n}}, S_{ij}^{0\mathbf{n}}$ stands for components of tensor inverse to tensor represented by components $C_{ij}^{0\mathbf{n}}$: $\left(S_{kl}^{0\mathbf{n}}\right) = \left(C_{ij}^{0\mathbf{n}}\right)^{-1}$.

Dimensionless components $\kappa_{ij}^{\varepsilon\mathbf{n}}$ and $\gamma_{ij}^{\varepsilon\mathbf{n}}$ represent material tensors responsible for strain-induced acoustical anisotropy and inhomogeneity of the body in direction **n**. In the basis of the system $\{y_1, y_2, y_3\}$ the matrix $\left(\kappa_{ij}^{\varepsilon\mathbf{n}}\right)$ looks like

$$\left(\kappa_{ij}^{\varepsilon\mathbf{n}}\right) = \begin{pmatrix} \chi_T\varepsilon - \gamma_T\varepsilon_{22}^{\mathbf{n}} & \gamma_T\varepsilon_{21}^{\mathbf{n}} & \chi_L\varepsilon_{31}^{\mathbf{n}} \\ \gamma_T\varepsilon_{12}^{\mathbf{n}} & \chi_T\varepsilon - \gamma_T\varepsilon_{11}^{\mathbf{n}} & \chi_L\varepsilon_{32}^{\mathbf{n}} \\ \chi_T\varepsilon_{13}^{\mathbf{n}} & \chi_T\varepsilon_{23}^{\mathbf{n}} & \gamma_L\varepsilon + 2\chi_L\varepsilon_{33}^{\mathbf{n}} \end{pmatrix}, \tag{7}$$

where $\varepsilon_{ij}^{\varepsilon\mathbf{n}}$ stands for initial strain components in the basis $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}\}$, $\gamma_T, \chi_T, \gamma_L, \chi_L$ are dimensionless elasticity moduli

$$\gamma_T = \frac{n}{2\mu},\quad \chi_T = \frac{m}{\mu},\quad \chi_L = \frac{2m}{(\lambda+2\mu)},\quad \gamma_L = \frac{2l}{(\lambda+2\mu)}. \tag{8}$$

3. Models for sounding by longitudinally and transversally polarized plane waves

Longitudinally and transversally polarized plane waves propagate in solids with distinct phase velocities C_L and C_T. This enables us to consider the cases of sounding of the body by longitudinal and transversal waves separately.

Let body is sounding by longitudinal plane wave $w_3^s(y,t) = W_0^s f(C_L t - y)$. In this case on the body inside surface $z = +0$ boundary conditions for displacement vector components w_i acts

$$w_1\big|_{z=0} = w_2\big|_{z=0} = 0, \qquad w_3\big|_{z=0} = W_0 f(C_L t), \tag{9}$$

where W_0 stands for an amplitude of the transmitted wave, $f(...)$ is a given function.

At these conditions, the transverse waves $w_1(z,t)$ and $w_2(z,t)$ are excited in the body volume only by the longitudinal wave transmitted into $\mathcal{V}$. Since acoustical anisotropy is weak, the coefficients $a_{13}^{\varepsilon\mathbf{n}}, b_{13}^{\varepsilon\mathbf{n}}$ and $a_{23}^{\varepsilon\mathbf{n}}, b_{23}^{\varepsilon\mathbf{n}}$ in the first and second equations (3) are small quantities of infinitesimal order α_ε. Hence, amplitudes of the transverse waves $w_1(z,t)$ and $w_2(z,t)$ will be small as compared to the longitudinal one's $w_3(z,t)$. Since the coefficients $a_{31}^{\varepsilon\mathbf{n}}, b_{31}^{\varepsilon\mathbf{n}}$ and $a_{32}^{\varepsilon\mathbf{n}}, b_{32}^{\varepsilon\mathbf{n}}$ at the terms, accounting in the third equation (3) the effect of the transverse waves on the longitudinal one, are also small quantities of infinitesimal order α_ε, the terms $a_{31}^{\varepsilon\mathbf{n}}\left(\partial^2 w_1/\partial y^2\right)$, $b_{31}^{\varepsilon\mathbf{n}}\left(\partial w_1/\partial y\right)$ and $a_{32}^{\varepsilon\mathbf{n}}\left(\partial^2 w_2/\partial y^2\right), b_{32}^{\varepsilon\mathbf{n}}\left(\partial w_2/\partial y\right)$ are small quantities of infinitesimal order α_ε^2 as compared to the term $a_{33}^{\varepsilon\mathbf{n}}\left(\partial^2 w_3/\partial y^2\right), b_{33}^{\varepsilon\mathbf{n}}\left(\partial w_3/\partial y\right)$. If to neglect them in the first approach, we will arrive from the system (5) at the equation

$$(1+\varepsilon)\frac{\partial^2 w}{\partial t^2} = a_{33}^{\varepsilon\mathbf{n}}\frac{\partial^2 w}{\partial y^2} + b_{33}^{\varepsilon\mathbf{n}}\frac{\partial w}{\partial y} \tag{10}$$

and at the system of two inhomogeneous wave equations for the components w_1, w_2

$$(1+\varepsilon)\frac{\partial^2 w_o}{\partial t^2} = a_{op}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_p}{\partial y^2} + b_{op}^{\varepsilon\mathbf{n}}\frac{\partial w_p}{\partial y} - g_o, \qquad o,p = 1,2. \tag{11}$$

Let now the body be sounding by transversal plane wave. In this case displacements w_1 and w_2 are prescribed as functions of time on the body inside surface $z = +0$ whereas the longitudinal displacement $w \equiv w_3$ equals zero

$$w_p\big|_{z=0} = W_p^0 f_p(C_T t), \qquad w\big|_{z=0} = 0. \tag{12}$$

Here W_p^0 are the amplitudes of transmitted wave, $f_p(...)$ are given functions.

Reasoning similarly as in the case of sounding by longitudinal wave, we reduce the system (5) to following homogeneous system

$$\left(1+\varepsilon\right)\frac{\partial^2 w_o}{\partial t^2}=a_{op}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_p}{\partial y^2}+b_{op}^{\varepsilon\mathbf{n}}\frac{\partial w_p}{\partial y} \tag{13}$$

and one inhomogeneous wave equation with respect to the longitudinal component w

$$\left(1+\varepsilon\right)\frac{\partial^2 w}{\partial t^2}=a_{33}^{\varepsilon\mathbf{n}}\frac{\partial^2 w}{\partial y^2}+b_{33}^{\varepsilon\mathbf{n}}\frac{\partial w}{\partial y}-g\,. \tag{14}$$

In formulae (10), (11), (13) and (14) the following denotations were used

$$g_o\equiv -a_{o3}^{\varepsilon\mathbf{n}}\frac{\partial^2 w}{\partial y^2}-b_{o3}^{\varepsilon\mathbf{n}}\frac{\partial w}{\partial y},\qquad g\equiv -a_{3p}^{\varepsilon\mathbf{n}}\frac{\partial^2 w_p}{\partial y^2}-b_{3p}^{\varepsilon\mathbf{n}}\frac{\partial w_p}{\partial y}.$$

4. Harmonic waves

In the case of longitudinally polarized harmonic wave

$$w^s\left(y,t\right)=W^s\exp\left[i\omega\left(t-y/C_L\right)\right],$$

where i is imaginary unit, ω is circular frequency of the wave, we will search a solution of the equation (10) in the form

$$w=W\left(y\right)\exp\left[i\omega\left(t-y/C_L\right)\right]. \tag{15}$$

Substituting presentation (15) into equation (10), using dimensionless coordinate $\xi=y/l^{\varepsilon\mathbf{n}}$ and taking into account formulas (6), we will come to the ordinary differential equation in unknown function $W\left(\xi\right)$

$$i\left(1+\kappa_{33}^{\varepsilon\mathbf{n}}\right)\frac{dW\left(\xi\right)}{d\xi}+\frac{\bar{\lambda}_L}{4\pi}\left[\left(1+\kappa_{33}^{\varepsilon\mathbf{n}}\right)\frac{d^2W\left(\xi\right)}{d\xi^2}+\gamma_{33}^{\varepsilon\mathbf{n}}\frac{dW\left(\xi\right)}{d\xi}\right]+$$
$$+\left[\frac{\pi}{\bar{\lambda}_L}\left(\varepsilon-\kappa_{33}^{\varepsilon\mathbf{n}}\right)+i\frac{\gamma_{33}^{\varepsilon\mathbf{n}}}{2}\right]W\left(\xi\right)=0\,. \tag{16}$$

Here $\bar{\lambda}_L=\lambda_L/l^{\varepsilon\mathbf{n}}$ — dimensionless longitudinal wavelength

Since the acoustical inhomogeneity is small, the length $l^{\varepsilon\mathbf{n}}$ is much bigger than the wavelength λ_L, hence $\bar{\lambda}_L$ is a small dimensionless parameter. We will consider it as a small quantity of infinitesimal order α_ε. It follows from formulas (7), (8) that $\gamma_{33}^{\varepsilon\mathbf{n}}$ is a dimensionless parameter of the order of unit. Function $W\left(\xi\right)$ is slowly changing — it varies on distances $\Delta\zeta\sim 1$. Hence its derivatives $dW\left(\xi\right)/d\xi,\, d^2W\left(\xi\right)/d\xi^2$ are magnitudes of the order of $W(\xi)$. Comparing three terms in the left hand side of equation (16) by their magnitudes, we can see that the first and the third ones are of the order of one, whereas the second one is of the order of $\bar{\lambda}_L \sim \alpha_\varepsilon$. Neglecting the quantities of order α_ε in equation (16), we will obtain

$$\frac{dW(\xi)}{d\xi}+\left(\frac{1}{2}\gamma_{33}^{\varepsilon\mathbf{n}}+i\frac{\pi}{\bar{\lambda}_L}\varphi_L^{\varepsilon\mathbf{n}}(\xi)\right)W(\xi)=0, \quad \varphi_L^{\varepsilon\mathbf{n}}=\left((\gamma_L-1)\varepsilon+2\chi_L\varepsilon_{33}^{e\mathbf{n}}\right). \tag{17}$$

Coefficient $1/2\gamma_{33}^{\varepsilon\mathbf{n}}$ determines the variation of the longitudinal wave amplitude, caused by acoustical inhomogeneity of the body, parameter $\varphi_L^{\varepsilon\mathbf{n}}$ is additional increment of the longitudinal wave phase, produced by strain.

Let us consider now the sounding of the body by transversally polarized harmonic plane wave $w_o^s=W_o^s\exp\left[i\omega\left(t-z/C_T\right)\right]$, $p=1,2$.

Representing the solution of the system (13) in the form

$$w_o=W_o(z)\exp\left[i\omega\left(t-z/C_T\right)\right],$$

we will arrive at a time-independent system in unknown functions $W_1(\xi)$ and $W_2(\xi)$ of the structure similar to (16). Neglecting the terms of the order of α_ε as against the terms of the order one, it will be reduced to the form

$$\frac{dW_o(\xi)}{d\xi}+\left[\frac{1}{2}\gamma_{op}^{\varepsilon\mathbf{n}}+i\frac{\pi}{\bar{\lambda}_T}\left(\varepsilon\delta_{op}-\kappa_{op}^{\varepsilon\mathbf{n}}\right)\right]W_p(\xi)=0. \tag{18}$$

Introducing 2×1-matrix $\hat{W}(\xi)=\left(W_1(\xi),W_2(\xi)\right)^{\mathrm{T}}$, we can rewrite the system (18) in a matrix form

$$\frac{d\hat{W}(\xi)}{d\xi}+\left[\hat{A}^{\mathbf{n}}-i\frac{\pi}{\bar{\lambda}_T}\left(\hat{E}^{\mathbf{n}}-\varphi_T^{\mathbf{n}}\hat{I}\right)\right]\hat{W}(\xi)=0, \tag{19}$$

where $\hat{A}^{\mathbf{n}}=\frac{1}{2}\begin{pmatrix}\gamma_{11}^{\varepsilon\mathbf{n}} & \gamma_{12}^{\varepsilon\mathbf{n}}\\ \gamma_{21}^{\varepsilon\mathbf{n}} & \gamma_{22}^{\varepsilon\mathbf{n}}\end{pmatrix}$, $\hat{E}^{\mathbf{n}}=\frac{\gamma_T}{2}\begin{pmatrix}\left(\varepsilon_{22}^{e\mathbf{n}}-\varepsilon_{11}^{e\mathbf{n}}\right) & 2\varepsilon_{12}^{e\mathbf{n}}\\ 2\varepsilon_{12}^{e\mathbf{n}} & \left(\varepsilon_{11}^{e\mathbf{n}}-\varepsilon_{22}^{e\mathbf{n}}\right)\end{pmatrix}$,

$$\varphi_T^{\mathbf{n}}=\frac{1}{2}\left[(\chi_T-1)\left(\varepsilon_{11}^{e\mathbf{n}}+\varepsilon_{22}^{e\mathbf{n}}\right)-\varepsilon_{33}^{\varepsilon\mathbf{n}}\right].$$

Matrix $\hat{A}^{\mathbf{n}}$ determines variations of the amplitudes of the transverse waves $w_1(\xi,t)$ and $w_2(\xi,t)$, owing to strain-induced acoustical inhomogeneity; the parameter $\varphi_T^{\varepsilon\mathbf{n}}$ determines an additional increment of the absolute phase each of the waves $w_1(\xi,t)$ and

$w_2(\xi,t)$, whereas matrix $\hat{E}^{\mathbf{n}}$ is responsible for an increment of phase difference between these two waves, caused by acoustical anisotropy.

5. Ray acoustoelasticity integrals

Due to (9), we should subordinate solution of equation (17) $W(y)$ to the boundary condition $W(0)=W_0$. In the issue we obtain

$$W(\xi)=W_0\exp\left[-\int_0^\xi\left(\frac{1}{2}\gamma_{33}^{\varepsilon\mathbf{n}}(\xi)+i\frac{\pi}{\lambda_L}\varphi_L^{\varepsilon\mathbf{n}}(\xi)\right)d\xi\right].$$

So, a longitudinally polarized ultrasonic beam propagating in a direction **n** crossing the strained body produces in its volume a longitudinal wave

$$w(\xi,t)=W_0\exp\left(\int_0^\xi\alpha_L^{\varepsilon\mathbf{n}}(\zeta)d\zeta\right)\exp\left[i\left(\omega t-\frac{2\pi}{\lambda_L}\xi-\frac{\pi}{\lambda_L}\int_0^\xi\varphi_L^{\varepsilon\mathbf{n}}(\zeta)d\zeta\right)\right], \qquad (20)$$

which amplitude $W_0\exp\left(\int_0^\xi 1/2\,\gamma_{33}^{\varepsilon\mathbf{n}}(\zeta)d\zeta\right)$ and phase $\varphi_L^{\varepsilon\mathbf{n}}(\xi)=\frac{2\pi}{\lambda_L}\xi+\frac{\pi}{\lambda_L}\int_0^\xi\varphi_L^{\varepsilon\mathbf{n}}(\zeta)d\zeta$ change along **n** due to the initial strain distribution on this direction.

Let $\bar{l}^{\mathbf{n}}$ be the dimensionless body's diameter in the direction **n**. Then, in compliance with solution (20), the increment of the wave phase on the segment $\left[0,\bar{l}^{\mathbf{n}}\right]$ equals $2\pi\bar{l}^{\mathbf{n}}/\lambda_L+\Delta\phi_L^{\varepsilon\mathbf{n}}$. The first term in this expression determines the phase increment in the absence of strain, whereas the second one

$$\Delta\phi_L^{\varepsilon\mathbf{n}}\equiv\int_0^{l^{\mathbf{n}}}\varphi_L^{\varepsilon\mathbf{n}}(\xi)d\xi=\frac{2\pi}{\lambda_L}\int_0^{l^{\mathbf{n}}}\frac{1}{2}\left((\gamma-1)\varepsilon^e(\xi)+2\gamma_L\varepsilon_{33}^{e\mathbf{n}}(\xi)-\varepsilon_0(\xi)\right)d\xi \qquad (21)$$

is responsible for additional phase increment caused by initial strain field.

Due to (12) the functions $W_o(\xi), o=1,2$ should be subordinated to the boundary conditions $W_o(0)=W_o^0$, where W_o^0 are the complex amplitudes of the transmitted transversally polarized wave. Their modules and the difference of arguments determine the polarization state of sounding wave at the input in the body.

Solution of the matrix equation (19) for these conditions looks like

$$\hat{w}(\xi,t)=\exp\left\{-\int_0^\xi A^{\mathbf{n}}(\zeta)d\zeta+i\left[\left(\omega t-\frac{2\pi}{\lambda_T}\xi\right)\hat{I}+\frac{\pi}{\lambda_T}\int_0^\xi\left(\hat{E}^{\mathbf{n}}(\zeta)-\varphi_T^{\mathbf{n}}(\zeta)\hat{I}\right)d\zeta\right]\right\}\cdot\hat{W}^0, \qquad (22)$$

where $\hat{w}(\xi,t)=\left(w_1(\xi,t),w_2(\xi,t)\right)^{\mathrm{T}}$, $\hat{W}=\left(W_1^0,W_2^0\right)^{\mathrm{T}}$, $\hat{I}$ is unity 2×2 matrix.

As we can see from the solution (22), both components $w_1(\xi,t)$ and $w_2(\xi,t)$ have been traveled through the body, acquire absolute phase increment $2\pi\bar{l}^{\mathbf{n}}/\bar{\lambda}_T + +\Delta\phi_T^{\varepsilon\mathbf{n}}$ on the path $\left[0,\bar{l}^{\mathbf{n}}\right]$, where

$$\Delta\phi_T^{\varepsilon\mathbf{n}} \equiv \int_0^{l^{\mathbf{n}}} \varphi_T^{\varepsilon\mathbf{n}}(\zeta)d\zeta = \frac{\pi}{\bar{\lambda}_T}\int_0^{l^{\mathbf{n}}}\Big((\gamma-1)\big(\varepsilon_{11}^{e\mathbf{n}}(\zeta)+\varepsilon_{22}^{e\mathbf{n}}(\zeta)\big)-\varepsilon_{33}^{e\mathbf{n}}(\zeta)\Big)d\zeta \tag{23}$$

determines the additional phase increment caused by initial strain field.

Besides that the additional phase difference between the components $w_1(\xi,t)$ and $w_2(\xi,t)$ arises. It is determined by two ray integrals

$$I_{1T}^{\varepsilon\mathbf{n}} = \frac{\pi}{2\bar{\lambda}_T}\gamma_T\int_0^{l^{\mathbf{n}}}\big(\varepsilon_{11}^{e\mathbf{n}}(\zeta)-\varepsilon_{22}^{e\mathbf{n}}(\zeta)\big)d\zeta, \quad I_{2T}^{\varepsilon\mathbf{n}} = \frac{\pi}{\bar{\lambda}_T}\gamma_T\int_0^{l^{\mathbf{n}}}\varepsilon_{12}^{e\mathbf{n}}(\zeta)d\zeta. \tag{24}$$

Conclusion. Mathematical models for interaction of longitudinally and transversally polarized ultrasonic beams with 3-D strain field in solids have been developed. Taking into account the weakness of strain-induced acoustical inhomogeneity and anisotropy it has been shown that the amplitude of longitudinally polarized wave changes along the direction of the wave propagation due to strain component distributions on this direction and it satisfies the ordinary differential equation (17). Cartesian components of the amplitude of transversally polarized wave, crossing the body in some direction, satisfy, in the approximation of weak acoustical inhomogeneity and anisotropy, the system (18) of equations with the coefficients dependent on initial strain's distribution.

Integral relationships (21) and (23), (24) connect line integrals of strain component distributions along any direction to measured phase and polarization parameters of longitudinally and transversally polarized waves crossing the body in this direction. So, if to sound a strained body by longitudinally polarized ultrasonic beam and measure the phase increment, has been acquired by the wave on its path, one can determine a value of the ray integral (21). Similarly, sounding the body by transversally polarized ultrasonic beam and measuring the changes of the absolute phase and polarization state, have been acquired by the wave, one can determine values of the ray integrals (23) and (24). Such measurements, carried out for a set of directions, form a posteriori data set that can be used commonly with the line integrals (21) and (23), (24) to formulate inverse problems for computing tomography of the initial strain field.

References

[1] *Hughes, D. S.* Second-order elastic deformation of solids / *D. S. Hughes, J. L. Kelly* // Phys. Rev. — 1953. — Vol. 92, No 5. — P. 1145-1149.

[2] *Toupin, R. A.* Sound waves in deformed perfectly elastic materials, acoustoelastic effect / *R. A. Toupin, B. Berstein* // Acoustic Society of America. — 1961. — Vol. 33, No 2. — P. 216-225.

[3] *Гузь, А. Н.* Введение в акустоупругостьм / *А. Н. Гузь, Ф. Г. Махорт, О. И. Гуща.* — Киев: Наук. думка, 1977. — 152 с.
[4] *Гузь, А. Н.* Упругие волны в телах с начальными (остаточными) напряжениями / *А. Н. Гузь.* — Киев: «А. С. К.», 2004. — 672 с.
[5] *Чекурін, В.* Моделі динаміки пружних збурень у неоднорідно деформованому континуумі / *В. Чекурін, О. Кравчишин* // Фіз.-мат. моделювання і інформаційні технології. — 2006. — Вип. 3. — С. 199-215.
[6] *Kravchyshyn, O. Z.* Acoustoelasticity model of inhomogeneously deformed bodies / *O. Z.Kravchyshyn, V. F. Chekurin* // Mechanics of Solid. — 2009. — Vol. 44, No 5. — P. 781-791.
[7] *Чекурін, В. Ф.* Пружні збурення в неоднорідно деформованих твердих тілах/ *В. Чекурін, О. Кравчишин.* — Львів: «Сполом», 2008. — 152 с.

Теорія інтегральної акустопружності для тривимірного напружено-деформованого стану

Василь Чекурін

Виходячи з моделі малого пружного збурення в неоднорідно деформованому тілі та беручи до уваги слабкість акустичних неоднорідності й анізотропії, індукованих деформацією, розроблено теорію інтегральної акустопружності. Сформульовані моделі взаємодії вузьких поляризованих ультразвукових пучків із тривимірним полем деформації у твердому тілі. У рамках моделей отримані інтегральні співвідношення акустопружності, що пов'язують зміни фаз коливань і стану поляризації поздовжньо та поперечно поляризованих ультразвукових хвиль, які пройшли через деформоване середовище, з інтегралами від розподілів компонент тензора початкової деформації вздовж напрямку поширення хвиль. Їх можна використати для формулювання задач обчислювальної томографії напружено-деформованого стану твердих тіл.

Теория интегральной акустоупругости для трехмерного напряженно-деформированного состояния

Василь Чекурин

Исходя из модели малого упругого возмущения в неоднородно деформированном теле и принимая во внимание, что индуцированные деформацией акустические неоднородность и анизотропия являются слабыми, разработана теория интегральной акустоупругости. Сформулированы математические модели взаимодействия узких поляризованных ультразвуковых пучков с трехмерным полем деформации в твердом теле. В рамках моделей получены лучевые интегралы акустоупругости — соотношения, устанавливающие аналитическую связь между изменениями фаз колебаний и состояния поляризации продольно и поперечно поляризованных волн, прошедших через деформированную среду, с линейными интегралами от распределений компонент начальных деформаций на направлениях распространения волн. Их можно использовать для постановки задач вычислительной томографии напряженно-деформированного состояния твердых тел.

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