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THE ROLE OF HIGHER MOMENTS ON THE DISTRIBUTION OF PARTICLES IN THE SPACE OF IMPULSES AT CYCLOTRON RESONANCES

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The results of the analysis of the dynamics of charged particles under conditions of cyclotron resonances in the field of an intense electromagnetic wave are presented. Particular attention is paid to regimes with dynamic chaos. It is shown that there are two qualitatively different regimes. The appearance of the first one is due to the overlap of nonlinear cyclotron resonances. The second mode is related to intermittency. The moments and spectra of each of these regimes are determined. It is shown that with an increase in the intensity of an external electromagnetic wave, the first regime appears at the beginning and only then the second regime appears. A characteristic feature of the second regime is intermittency. Steps appear on the time dynamics of pulses in the second mode. It is shown that the spectra in the second mode are narrower than in the first mode. A characteristic feature of the second regime (the regime with intermittency) is the fact that the higher moments turn out to be larger than the lower moments. In the first regime, the highest moments decrease rapidly. To find the particle momentum distribution function, the generalized Fokker-Planck equation was used. Solutions of this equation are written out for some important cases.

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INTRODUCTION

Moments of random processes are one of the important characteristics of these processes. The first and second moments (mean and variance) are of the greatest importance. Higher moments play a secondary role and are rarely used. In addition, in most cases, the moments decrease rapidly as their number increases. However, there are random processes, which are called intermittent regimes, and are characterized by the fact that their higher moments exceed the lower ones. Processes with intermittency are distinguished by the fact that rare, but very intense bursts appear against the background of some moderate dynamics [1, 2]. In some cases, these surges can pose a significant danger. It was found in [3] that at cyclotron resonances at sufficiently high field strengths of electromagnetic waves, regimes with intermittency can also arise. Cyclotron resonances are widely used. In particular, they are used in thermonuclear fusion facilities for plasma heating. Therefore, the study of regimes with intermittency is of both general scientific and practical interest. The first step in studying the influence of higher moments on the development of random processes can be the Fokker-Planck (FP) equation. However, the usual FP equation contains only the first and second moments. In [3], a generalization of the FP equation was written for the case when the influence of moments with any number is taken into account. Below we will consider some solutions to this equation.

The work consists of an introduction, three parts and a conclusion. In the first part, the problem statement is formulated, and the main system of equations is written out. The second part presents the results of the analysis of particle dynamics at cyclotron resonances. Two qualitatively different modes of particle dynamics are described. Spectra of particle dynamics in these regimes are determined. In the third part, the generalized FP equations are written out. This new equation takes into account all higher moments. Note that in the usual expressions of the FP equation, only the first and second moments are taken into account. Some analysis of this equation is given, in particular, the results of the stationary regime are given. In conclusion, the most important results are formulated.

1. STATEMENT OF THE PROBLEM AND BASIC EQUATIONS

Consider a charged particle that moves in an external constant magnetic field directed along the axis *z*, and in the field of a plane electromagnetic wave, which in the general case has the following components:

$$\mathbf{E} = \mathrm{Re}(E\boldsymbol{\alpha}\exp(i\omega t - i\mathbf{kr})),$$
$$\mathbf{H} = \mathrm{Re}\left(\frac{c}{\omega}[\mathbf{kE}]\exp(i\omega t - i\mathbf{kr})\right), \quad (1)$$

where $\boldsymbol{\alpha} = \{\alpha_x, i\alpha_y, \alpha_z\}$ is wave polarization vector.

Without limiting of generality, we can choose a coordinate system in which the wave vector of the wave has only two components k_x and k_z. It is also convenient to use the following dimensionless dependent and independent variables:

$$\mathbf{p} \to \mathbf{p}/mc, \ \tau \to \omega t, \ \mathbf{r} \to \frac{\omega}{c}\mathbf{r}.$$

The equations of motion in these variables will have the form:

$$\frac{d\mathbf{p}}{d\tau} = \left(1 - \frac{\mathbf{kp}}{\gamma}\right)\mathrm{Re}\left(\boldsymbol{\varepsilon}e^{i\psi}\right) + \frac{\omega_H}{\gamma}[\mathbf{ph}] + \frac{\mathbf{k}}{\gamma}\mathrm{Re}\left[(\boldsymbol{\varepsilon}\cdot\mathbf{p})e^{i\psi}\right], \quad (2)$$
$$\mathbf{v} = \frac{d\mathbf{r}}{d\tau} = \frac{\mathbf{p}}{\gamma}, \ \dot{\psi} = \frac{d\psi}{d\tau} = 1 - \frac{\mathbf{kp}}{\gamma},$$

where $\mathbf{h} = \mathbf{H}/H_0$; $\omega_H = eH/mc\omega$; $\boldsymbol{\varepsilon} = \varepsilon_0\boldsymbol{\alpha}$; $\varepsilon_0 = (eE_0/mc\omega)$; $\psi = \tau - \mathbf{kr}$; $\mathbf{k}$ – unit vector in the direction of the wave propagation; $\gamma = (1+\vec{p}^2)^{1/2}$ – dimensionless particle energy (measured in units mc^2); $\mathbf{p}$ – particle momentum.

The system of vector equations (2) can be fully analyzed only by numerical methods. However, many important features of charged particle dynamics can be discovered using new variables. We will use, similarly to [4 - 6], the following variables

$$p_x = p_\perp \cos\theta\,,\ p_y = p_\perp \sin\theta\,,\ p_z = p_\parallel\,,\ p_\perp = \sqrt{p_x^2 + p_y^2}\,,$$

$$x = \xi - \frac{p_\perp}{\omega_H}\sin\theta\,,\ y = \eta + \frac{p_\perp}{\omega_H}\cos\theta\,. \qquad (3)$$

For new variables, the system of equations (2) can be reduced to the form:

$$\frac{dp_\perp}{d\tau} = \varepsilon_0\left(1 - k_z v_z\right)\left(J_n'\right)\cos(\theta_n)\,, \qquad (4)$$

$$\frac{dp_z}{d\tau} = \varepsilon_0 k_z v_\perp J_n' \cos\theta_n\,, \qquad (5)$$

$$\frac{d\gamma}{d\tau} = \varepsilon_0 v_\perp J_n' \cos\theta_n\,, \qquad (6)$$

$$\theta_n = \tau - k_z z - k_x \xi + n\theta\,. \qquad (7)$$

where $J_n = J_n(\mu)$ $J_n' = dJ_n(\mu)/d\mu$, $\mu = k_x p_\perp / \omega_H$.

Details of obtaining system (4)-(7) can be found in [3, 5].

2. MODES WITH DYNAMIC CHAOS

Below we will consider two chaotic regimes. In the first mode, chaos arises as a result of overlapping of homoclinic trajectories (overlapping of non-linear cyclotron resonances). With an increase in the field strength, a second regime arises – the regime with intermittency.

The conditions for the emergence of the first regime are formulated, for example, in [4, 5, 7], and can be written in the form:

$$\Delta\gamma > \delta\gamma\,,\ \Delta\gamma \approx \sqrt{\varepsilon / \left|1 - k_z^2\right|}\,,\ \delta\gamma \approx \left\{\omega_H / \left|1 - k_z^2\right|\right\}, \qquad (8)$$

where $\Delta\gamma$ – nonlinear resonance width, $\delta\gamma$ – distance between cyclotron resonances.

The regime with intermittency arises at sufficiently high field strengths of the wave. It is characterized by steps in the dependence of particle momenta on time. Such a regime appears as a result of solving the Adler equation, which describes the dynamics of particles at high field strengths. This mode is described in detail in [6].

2.1. OVERLAPPING OF NONLINEAR CYCLOTRON RESONANCES

An analytical analysis of chaotic regimes in the case of overlapping cyclotron resonances was carried out in [4 - 6]. Such a regime is almost always observed at a wave strength parameter (nonlinearity parameter) of 0.26 ($\varepsilon_0 \geq 0.26$). Below we will present some results of a numerical analysis of this regime. For definiteness, we choose the following parameters: $\varepsilon_0 = 0.26$, $k_z = 0.878$, $\omega_H = 0.987$.

It can be shown that, for these parameters, the nonlinear cyclotron resonances are covered. In addition, we choose the following initial conditions: $x(0) = y(0) = 0$, $z(0) = \pi/2$, $P_x(0) = P_y(0) = P_z(0) = 0.01$.

Below, in Figs. 1 and 2, the dependence of the longitudinal momentum of the particle on time and the spectrum are presented. It can be seen that the spectrum of particle dynamics is wide.

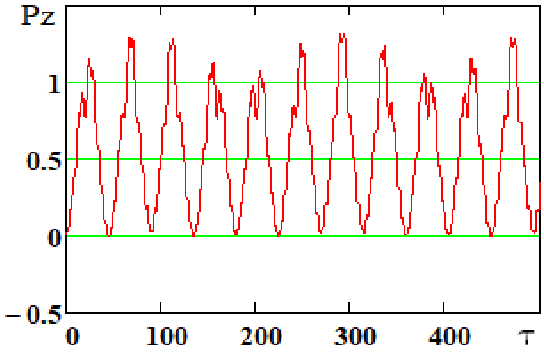

Fig. 1. Longitudinal impulse, before the appearance of steps, $\varepsilon_0 = 0.26$

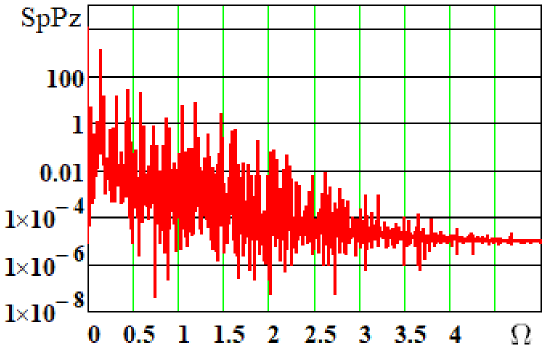

Fig. 2. Spectrum before the appearance of steps, $\varepsilon_0 = 0.26$

In the considered case, the moments have the following values:

even $\mu_0 = 1$, $\mu_2 = 0.07$, $\mu_4 = 1.6\cdot 10^{-3}$, $\mu_6 = 1.8\cdot 10^{-5}$;

odd $\mu_1 = 1.7\cdot 10^{-5}$, $\mu_3 = 0.001$, $\mu_5 = 5.2\cdot 10^{-5}$, $\mu_7 = 7.5\cdot 10^{-7}$.

It can be seen that the magnitudes of the moments rapidly decrease with increasing their number.

2.2. MODES WITH INTERMITTED

With an increase in the field strength of the wave, a regime with intermittency arises. The results of a detailed study of this mode are given in [6]. Figs. 3 and 4 present the results of a numerical analysis of this mode for the value of the wave strength parameter $\varepsilon_0 = 1.2$. The initial conditions are the same as in Figs. 1 and 2.

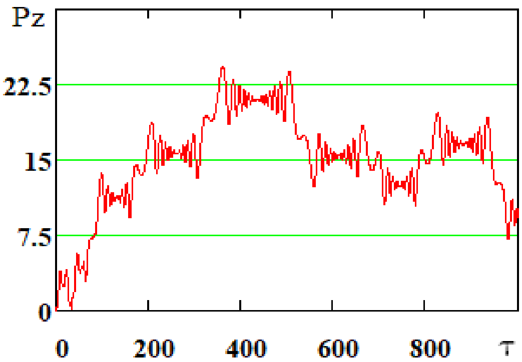

Fig. 3. Longitudinal pulse in intermittent mode, $\varepsilon_0 = 1.2$

Fig. 3 shows the steps characteristic of the intermittent mode on the time dependence of the longitudinal pulse. However, the most interesting result is shown in Fig. 4. This figure shows a narrow spectrum of particle dynamics. This result is, to some extent, unexpected. More common is the broadening of the particle dynamics spectrum with increasing wave field strength.

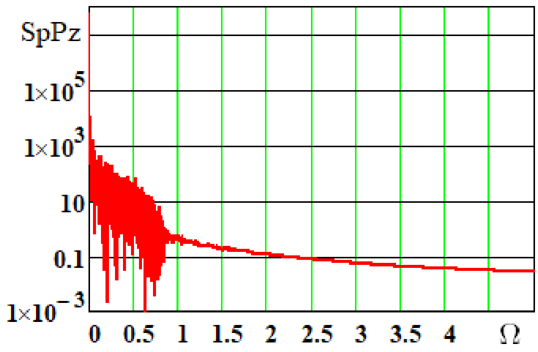

Fig. 4. Spectrum in intermittent mode, $\varepsilon_0 = 1.2$

This feature of the spectrum in the regime with intermittency is due to the fact that the particle dynamics at the steps themselves is regular. Randomness in this case is due only to the appearance of particle jumps from one stage to another stage. These jumps are random.

In this case, the moments have the following values: even $\mu_0 = 1$, $\mu_2 = 82.5$, $\mu_4 = 4\cdot 10^3$, $\mu_6 = 1.2\cdot 10^5$; odd $\mu_1 = 0.001$, $\mu_3 = 405$, $\mu_5 = 2\cdot 10^4$, $\mu_7 = 5.3\cdot 10^5$.

The main tendency in moments is that the greater the field strength of the wave, and thus the more pronounced the steps, the higher the moments are greater and the greater their magnitude.

3. ROLE OF MOMENTS IN PARTICLE DYNAMICS

Moments in the theory of random processes play a significant role. In particular, if the process is ergodic [8, 9], then the values averaged over the ensemble can be replaced by averages over time. This, in turn, makes it possible to use the results of a single-particle analysis of particle motion to find the distribution function of an ensemble of particles. This can be done using the FP equation. Indeed, in this equation, as is known, the main parameters are the first and second moments. However, as shown in [3], at cyclotron resonances, regimes appear whose characteristic feature is the fact that the higher moments turn out to be larger than the lower moments. In the same work, the generalized FP equation is presented, in which higher moments are taken into account. It should also be noted that in the FP equation (and in the generalized equation) the moment values are divided by the factorial of the moment number.

$$\frac{\partial n}{\partial \tau} = \sum_m \frac{\left\langle (p)^m \right\rangle}{m!} \frac{\partial^m n}{\partial p^m}, \quad m = 2j;\ j = \{1,2,3...\}. \quad (9)$$

To find out the role of higher moments, it is enough for us to analyze the solutions of equation (9) taking into account only the second and fourth moments:

$$\frac{\partial n}{\partial \tau} = \alpha^2 \frac{\partial^2 n}{\partial p^2} + \beta^2 \frac{\partial^4 n}{\partial p^4}. \quad (10)$$

If the parameter β is small ($\beta << 1$), then the solution of equation (10) can be sought in the form of a series in this parameter:

$$n = n_0 + \beta n_1 + \beta^2 n_2 + ... \quad (11)$$

Substituting this series into equation (10), we will find equations for finding the terms of this series. For example, to find the second term, you can get the following sequence:

$$\hat{L}n_0 = \frac{\partial n_0}{\partial \tau} - \alpha \frac{\partial^2 n_0}{\partial p^2} = 0;\ \hat{L}n_1 = \beta \frac{\partial^4 n_0}{\partial p^4};\ \hat{L}\hat{L}n_1 = \beta \frac{\partial^4 \hat{L}n_0}{\partial p^4} = 0$$

since $\hat{L}n_0 = 0$, then and $\hat{L}n_1 = 0$. The equations for the other terms of the series (11) will have an analogous form. Finally, the series (11) can be written in the form of a series of geometric progression:

$$n(p,t) = n_0\left[1 + \beta + \beta^2 + ...\right] = n_0(p,t)/(1-\beta). \quad (12)$$

This expression shows that the solutions of the FP equation are stable with respect to the influence of small higher moments.

Below we will use the results obtained in [3] and write equation (9) up to the 6th moment, which already becomes sufficient to take into account the influence of growing higher moments. We also take into account only even moments and rename $p \to x$.

Then equation (9) takes the form:

$$\frac{\partial n}{\partial \tau} = \alpha \frac{\partial^2 n}{\partial x^2} + \beta \frac{\partial^4 n}{\partial x^4} + \gamma \frac{\partial^6 n}{\partial x^6}. \quad (13)$$

The resulting equation is quite complex for both analytical and numerical analysis. Therefore, we consider the stationary case, for which the solution of Eq. (13) takes the form:

$$n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-ik(x - x_0)] \exp[-F(k)] dk, \quad (14)$$

where $F(k) = \left[\alpha k^2 - \beta k^4 + \gamma k^6\right]$.

Below are the results of numerical studies of expression (14) in the stationary case. The difference in taking into account the highest moments, namely μ_4 and μ_6, is shown in the graphs below, the red curve describes only the second moment. The blue curve was obtained taking into account additionally the fourth and sixth moments. The calculation was carried out for different values of higher moments. Fig. 5 shows the distribution function when the magnitudes of the higher moments divided by the factorial of the moment number are insignificant (order $10^{-4}…10^{-5}$). In this case, their contribution is also insignificant.

Fig. 6 shows the distribution function for the case when the higher moments are still less than the lower ones. In this case, the values of the higher moments divided by the factorial of the moment number are also not significant, and their value is several orders of magnitude smaller than the value of the lower moments. It can be noted that within the limits of the change in the field strength of the wave from $\varepsilon_0 = 0.01$ (see Fig. 5) to $\varepsilon_0 = 0.26$ (see Fig. 6) there is an insignificant but smooth increase in the higher moments, which leads to a smooth broadening of the distribution function.

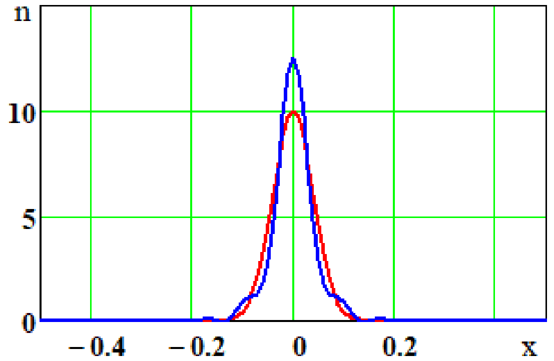

Fig. 5. Distribution function, for the case when the highest moments are less than the lowest moments, $\varepsilon_0 = 0.01$

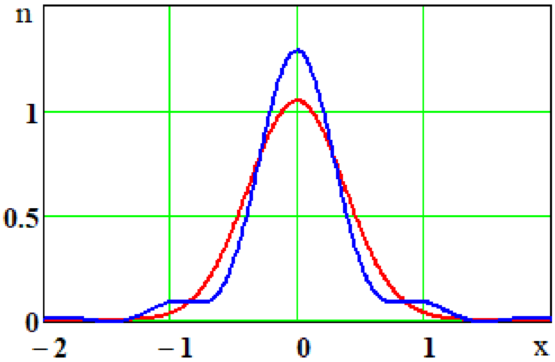

Fig. 6. Distribution function, for the case when the highest moments are less than the lowest moments, $\varepsilon_0 = 0.26$

In the future, a slight change in the field strength ($\varepsilon_0 = 0.261$) qualitatively changes the particle dynamics – the higher moments become larger than the lower ones (Fig. 7). So the second moment μ_2, divided by the factorial of the moment number, becomes $\mu_2 = 12$, $\mu_4 = 43$, $\mu_6 = 84$. This leads to a sharp broadening of the distribution function. It is also worth noting that at the same moment there are significant changes in the dynamics of particles, namely the appearance of a stepwise character of the longitudinal momentum of particles.

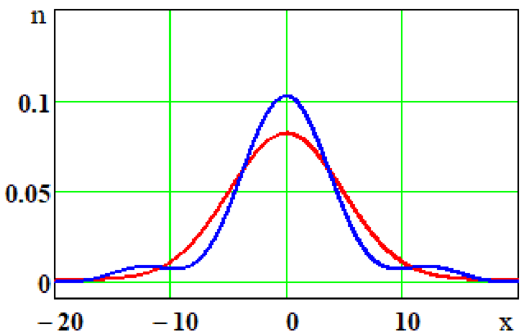

Fig. 7. Distribution function, for the case when the highest moments are greater than the lowest moments, $\varepsilon_0 = 0.261$

It should be noted that for the correct display of the distribution function, it is necessary to take the values of the moments calculated for a specific implementation. Fig. 8 shows the plots of the distribution function for the moments calculated for a specific implementation, the blue curve (the moments are equal to $\mu_2 = 0.021$, $\mu_4 = 1.1 \cdot 10^{-4}$, $\mu_6 = 2.7 \cdot 10^{-7}$) and arbitrary, red curve ($\mu_2 = 0.03$, $\mu_4 = 0.004$, $\mu_6 = 3 \cdot 10^{-4}$). It can be seen that an arbitrary choice of moments leads to a non-physical result – areas with negative particle densities appear.

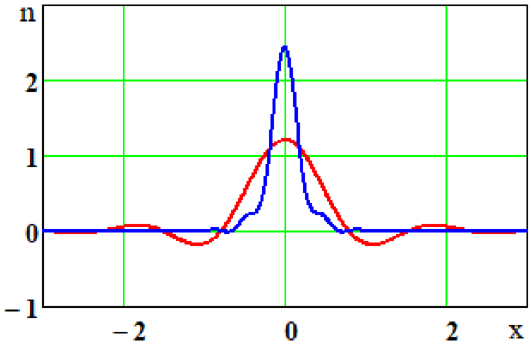

Fig. 8. Distribution function

CONCLUSIONS

1. The most interesting, unexpected and important result is that as the intensity of the electromagnetic wave increases, the particle dynamics changes qualitatively. These changes are characterized by a new regime of dynamic chaos. From chaos, which was determined by the overlap of nonlinear cyclotron resonances, chaos becomes intermittent. With such a transition, the wave amplitude increased, but the width of the particle dynamics spectrum narrowed significantly. However, at the same time, the moments began to increase. Moreover, the higher moments become larger than the lower moments. Let us try to explain such, at first glance, contradictory characteristics of the regime with intermittency. In [4], see also Fig. 3, it is shown that as a result of the phase synchronization of the wave and the particle, steps appear on the time dependence of the pulses. The dynamics of particles on the steps themselves is regular. Randomness occurs only at moments of jumps. These jumps are random in both magnitude and direction. In general, the dynamics are more regular. Therefore, the spectrum becomes much narrower (see Fig. 4). Now consider the features of the moments. Suppose we have some function $x(\tau)$. Her moment with number n will be determined by the formula $m_n = \left\langle \left(x - \langle x \rangle \right)^n \right\rangle$. Averaging is carried out over the ensemble. However, if the system under study is ergodic, then averaging can be carried out over time. Looking at Fig. 3, we see that the mean function has $\langle x \rangle$ jumps are moderated. In the function itself, the magnitude of the jumps in most cases is much larger than the average values. Therefore, the value $\left(x - \langle x \rangle \right)$ more than one. As a consequence, each next moment will be greater than the previous one.

2. Let's answer the main question of the article: an increase in higher moments leads to a more rapid broadening of the particle distribution function. Note that jumps in particle momentum can be initiators of runaway electrons.

3. We also note that the magnitudes of the moments must be determined from the actual particle dynamics. An attempt to change these values leads to erroneous results – the distribution function can become negative (see Fig. 8). This result may be useful as a diagnostic test.

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РОЛЬ ВИЩИХ МОМЕНТІВ У РОЗПОДІЛІ ЧАСТИНОК У ПРОСТОРІ ІМПУЛЬСІВ ПРИ ЦИКЛОТРОННИХ РЕЗОНАНСАХ

В.О. Буц, В.В. Кузьмін

Викладено результати аналізу динаміки заряджених частинок в умовах циклотронних резонансів у полі інтенсивної електромагнітної хвилі. Особливу увагу приділено режимам з динамічним хаосом. Показано, що існує два якісно різні режими. Виникнення першого обумовлено перекриттям нелінійних циклотронних резонансів. Другий режим пов'язаний із перемежуванням. Визначено моменти та спектри кожного з цих режимів. Показано, що зі збільшенням напруженості зовнішньої електромагнітної хвилі на початку з'являється перший режим і потім другий. Характерною рисою другого режиму є перемежування. На залежності імпульсів від часу у другому режимі з'являються сходинки. Показано, що спектри у другому режимі вужчі, ніж у першому. Характерною особливістю другого режиму (режиму з перемежуванням) є той факт, що вищі моменти виявляються більшими, ніж нижчі. У першому режимі вищі моменти швидко зменшуються. Для знаходження функції розподілу частинок за імпульсами було використано узагальнене рівняння Фоккера-Планка. Для деяких важливих випадків виписано рішення цього рівняння.