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MODEL OF FINITE INHOMOGENEOUS CAVITY CHAIN AND APPROXIMATE METHODS OF ITS ANALYSIS

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A new approach to the description of an inhomogeneous chain of coupled resonators (inhomogeneous disk waveguides) is proposed. New matrix difference equations based on the technique of coupled integral equations and the decomposition method are obtained. Various approximate approaches have been developed, including the WKB approximation.

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INTRODUCTION

Inhomogeneous Travelling-Wave Accelerating Sections (ITWAS) have been (and are) the workhorse of accelerating technology for more than half a century. Several thousand different sections were manufactured and used in linacs. Only one linac (SLAC) included 960 sections [1]. ITWASs are in fact chains of coupled resonators connected with two external waveguides. This apparent simplicity of structure is very deceiving. The reason is that the homogeneous periodic waveguide has the infinite number of eigen waves, most of which do not propagate (evanescent waves). Any inhomogeneity leads to the appearance such fields that decay exponentially from the interface at which they are formed. In ITWAS there are many small discontinuities with small field disturbance. Developing an electrodynamic model that combines propagation and evanescence is not an easy task. This is proved by the fact that before "computer age", as we know, only one mathematical model that could rigorously describe characteristics of ITWASs was developed [2].

Today, using various computer programs, we can simulate almost any accelerating sections (see, for example, [3, 4]). However, the complexity of the results obtained, their strong dependence on the grid parameters and impossibility of using approximate analysis still make the development and use of semi-analytical approaches actual.

Two approximate approaches were mainly used to describe ITWASs: a coupled cavity model [5 - 17] and a waveguide approximation [18 - 25]. While the first approach is based on the strict physical and mathematical foundation, the necessities of use many eigen modes, difficulties of coupling coefficient calculation and taking into account the losses in walls made the definition of parameters of coupled cavity models very approximate. Nevertheless, these models were useful in practice and together with developer skills gave good results.

The second approach is based on assumption that there are such slow parameter changes under which there are no practical differences between equations of homogeneous and inhomogeneous waveguides. Under such assumption, we can transform the definition for homogeneous waveguide $R_{ser}=E_0^2/P$ (R_{ser} − serious impedance, E_0 − amplitude of the principal space harmonic) into an equation $R_{ser}(z)=E_0^2(z)/P(z)$, which is the base of this approach. It was a useful assumption, but nobody knows accuracy of the obtained results.

Smooth approximate models are widely used, especially in the study of beam current loading and transient effects, but so far the notion of spatial averaged electric field in the model equations (together with model equations) has not yet been rigorously defined.

Approximate model equations are used with parameters that are the slow functions of coordinate. Under assumptions of these models in the considered structures there are only two independent (forward and backward) waves which characteristics slowly change along the waveguide. Evanescent wave are ignored in these models. These features arise in mathematic physics when we use asymptotic expansions. It is obvious that approximate models are based on the several first equations of the asymptotic expansion chain of the solutions of the exact equations (if such equations exist). But at what level: on the equations of the zero (Eikonal) or first (WKB) order?

The possibility of using the WKB approach to describe the ITWAS gives not only a simplification of the calculation. It also allows the use of simpler physical models of transient processes. Using the traveling wave concept simplifies the understanding of pulsed-excited ITWAS transients and the development of methods to mitigate their effect on beam parameters.

Difficulties in describing the ITWASs arise from the fact that there were not obtained closed and rigorous equations (except the Maxwell equations with boundary conditions) for parameters of the ITWAS from which we could obtain approximate models by using different mathematic methods.

There are works that study waves in slowly varying band-gap media on the base of analyses of differential operators without assumption that the wavelength is long compared with the size of the repeating cell (see, for example, [26 - 30] and cited there literature). Results obtained in these works cannot be used for description ITWASs as there are no suitable smooth differential operators. Taking into account this circumstance it was proposed to use difference equations to describe ITWASs [31]. The first attempt was made on the base of the coupled cavities model that was developed with using many eigen modes and rigorous calculation of coupling coefficients, but without losses in the walls [16]. Obtained difference equations that connect the values of electric field in different points of resonators correctly describe the main waves but also contain different spurious oscillations. The reason of appearance of spurious oscillations and its influence on the solutions are not quite clear.

To explore other possibilities of using difference equations and approximate methods, we propose a simple but rigorous model of ITWAS [32]. This model is based on the method of Coupled Integral Equations (CIE) [33]. In paper we also present the results of the development of approximate methods for the analysis of this model. Using the theory of solving matrix equations [34, 35] and the decomposition method [36], we obtained new matrix difference equations, on the basis of which various approximate approaches, including the WKB approach, can be developed.

It is worth to note that the unknowns in the matrix difference equations are vectors which components are the moments of electric fields on the surfaces that divide the chain resonators. Determining these moments gives possibility to calculate electromagnetic fields in any point of resonator. Therefore, proposed equations are not direct equations for the electric field. This circumstance makes it difficult to analyze the foundations of the equations that are currently used.

1. CHAIN OF THE FINITE NUMBER OF RESONATORS. BASIC EQUATIONS

Consider a chain of cylindrical resonators with annular discs of zero thickness. The first and last resonators are connected through cylindrical openings to semi-infinite cylindrical waveguides. The geometry of the chain is shown in Fig. 1. All resonators are filled with dielectric ($\varepsilon = \varepsilon' + i\varepsilon''$, $\varepsilon'' > 0$). We will consider only axially symmetric fields with E_z, E_r, H_φ components (TM). Time dependence is $\exp(-i\omega t)$. In each resonator we expand the electromagnetic field with the waveguide modes

$$\vec{H}^{(k)} = \sum_s \left(h_s^{(k)} \vec{\mathcal{H}}_s^{(k)} + h_{-s}^{(k)} \vec{\mathcal{H}}_{-s}^{(k)} \right), \quad (1)$$

$$\vec{E}^{(k)} = \sum_s \left(h_s^{(k)} \vec{\mathcal{E}}_s^{(k)} + h_s^{(-k)} \vec{\mathcal{E}}_{-s}^{(k)} \right), \quad (2)$$

where

$$\mathcal{E}_{s,z}^{(k)} = J_0\left(\frac{\lambda_s}{b_k} r\right) \exp\left\{\gamma_s^{(k)} (z - z_k)\right\}, \quad (3)$$

$$\mathcal{H}_{s,\varphi}^{(k)} = -i\omega \frac{\varepsilon_0 \varepsilon b_k}{\lambda_s} J_1\left(\frac{\lambda_s}{b_k} r\right) \exp\left\{\gamma_s^{(k)} (z - z_k)\right\}, \quad (4)$$

$$\mathcal{E}_{s,r}^{(k)} = -\frac{b_k}{\lambda_s} \gamma_s^{(k)} J_1\left(\frac{\lambda_s}{b_k} r\right) \exp\left\{\gamma_s^{(k)} (z - z_k)\right\}, \quad (5)$$

$$\gamma_s^{(k)2} = \left(\frac{\lambda_s}{b_k}\right)^2 - \frac{\varepsilon\omega^2}{c^2} = \frac{1}{b_k^2}\left(\lambda_s^2 - \frac{\varepsilon' b_k^2 \omega^2}{c^2}\right) - \frac{i\varepsilon''\omega^2}{c^2}, \quad (6)$$

$\mathrm{Im}\gamma_s^{(k)} > 0$, $\mathrm{Re}\gamma_s^{(k)} < 0$, $\gamma_{-s}^{(k)} = -\gamma_s^{(k)}$, $J_0(\lambda_m) = 0$,
$z \in [0, d_k]$, $r \in [0, b_k]$.

In the waveguides the electromagnetic field can also be decompose in terms of TM modes ($k = 1, 2$)

$$\vec{H}^{(w,k)} = \sum_s \left(G_s^{(k)} \vec{\mathcal{H}}_s^{(w,k)} + G_{-s}^{(k)} \vec{\mathcal{H}}_{-s}^{(w,k)} \right), \quad (7)$$

$$\vec{E}^{(w,k)} = \sum_s \left(G_s^{(k)} \vec{\mathcal{E}}_s^{(w,k)} + G_s^{(-k)} \vec{\mathcal{E}}_{-s}^{(w,k)} \right), \quad (8)$$

where $z_{w,1} = z_1$, $z_{w,2} = z_{N_R+1}$,

$$\mathcal{E}_{s,z}^{(w,k)} = J_0\left(\frac{\lambda_s}{b_{w,k}} r\right) \exp\left\{\gamma_s^{(w,k)} (z - z_{w,k})\right\}, \quad (9)$$

$$\mathcal{H}_{s,\varphi}^{(w,k)} = -i\omega \frac{\varepsilon_0 b_{w,k}}{\lambda_s} J_1\left(\frac{\lambda_s}{b_{w,k}} r\right) \exp\left\{\gamma_s^{(w,k)} (z - z_{w,k})\right\}, \quad (10)$$

$$\mathcal{E}_{s,r}^{(w,k)} = -\frac{b_{w,k}}{\lambda_s} \gamma_s^{(w,k)} J_1\left(\frac{\lambda_s}{b_{w,k}} r\right) \exp\left\{\gamma_s^{(w,k)} (z - z_{w,k})\right\}, \quad (11)$$

$$\gamma_s^{(w,k)2} = \frac{1}{b_{w,k}^2}\left(\lambda_s^2 - \frac{b_{w,k}^2 \omega^2}{c^2}\right). \quad (12)$$

The boundary conditions at the interface $z = z_k$ require the continuity of the tangential magnetic fields across the apertures at $z = z_k$ ($k = 2, 3, \ldots N_R$)

$$\sum_s \left(h_s^{(k)} \mathcal{H}_{s,\varphi}^{(k)} + h_{-s}^{(k)} \mathcal{H}_{-s,\varphi}^{(k)} \right) = \\ = \sum_s \left(h_s^{(k-1)} \mathcal{H}_{s,\varphi}^{(k-1)} + h_{-s}^{(k-1)} \mathcal{H}_{-s,\varphi}^{(k-1)} \right), \; z = z_k, 0 \le r < a_k. \quad (13)$$

Substituting $\mathcal{H}_{s,\varphi}^{(k)}$ from (4), we get

$$\sum_s \frac{b_k}{\lambda_s} \left(h_s^{(k)} + h_{-s}^{(k)} \right) J_1\left(\frac{\lambda_s}{b_k} r\right) = \\ = \sum_s \frac{b_{k-1}}{\lambda_s} \begin{pmatrix} h_s^{(k-1)} \exp\left(\gamma_s^{(k-1)} d_{k-1}\right) + \\ + h_{-s}^{(k-1)} \exp\left(\gamma_{-s}^{(k-1)} d_{k-1}\right) \end{pmatrix} J_1\left(\frac{\lambda_s}{b_{k-1}} r\right), \quad (14)$$

$0 \le r < a_k$.

We will use the Moment Method to solve the system of coupled equations. Multiplying the right and left sides of this relation by a testing function $\psi_{s'}\left(r / a_k\right)$, $s' = 1, 2 \ldots, N_m$ and integrating with respect to r from 0 to a_k, we get N_m equations

$$\sum_s \frac{b_k}{\lambda_s} R_{s',s}^{\psi(k,1)} \left(h_s^{(k)} + h_{-s}^{(k)} \right) = \\ = \sum_s \frac{b_{k-1}}{\lambda_s} R_{s',s}^{\psi(k,2)} \begin{pmatrix} h_s^{(k-1)} \exp\left(\gamma_s^{(k-1)} d_{k-1}\right) + \\ + h_{-s}^{(k-1)} \exp\left(\gamma_{-s}^{(k-1)} d_{k-1}\right) \end{pmatrix}, \quad (15)$$

$k = 2, 3, \ldots N_R$, $s' = 1, 2 \ldots, N_m$,

$$R_{s',s}^{\psi(k,1)} = \int_0^1 \psi_{s'}(x) J_1\left(a_k \lambda_s x / b_k\right) x dx, \\ R_{s',s}^{\psi(k,2)} = \int_0^1 \psi_{s'}(x) J_1\left(a_k \lambda_s x / b_{k-1}\right) x dx. \quad (16)$$

For the first and last resonators we have

$$\sum_s \left(G_s^{(1)} \mathcal{H}_{s,\varphi}^{(w,1)} + G_{-s}^{(1)} \mathcal{H}_{-s,\varphi}^{(w,1)} \right) = \\ = \sum_s \left(h_s^{(1)} \mathcal{H}_{s,\varphi}^{(1)} + h_{-s}^{(1)} \mathcal{H}_{-s,\varphi}^{(1)} \right), \; z = z_1, 0 \le r < a_1, \quad (17)$$

$$\sum_s \left(h_s^{(N_R)} \mathcal{H}_{s,\varphi}^{(N_R)} + h_{-s}^{(N_R)} \mathcal{H}_{-s,\varphi}^{(N_R)} \right) = \\ = \sum_s G_s^{(2)} \mathcal{H}_{s,\varphi}^{(w,2)}, \; z = z_{N_R+1}, \; 0 \le r < a_{N_R+1}, \quad (18)$$

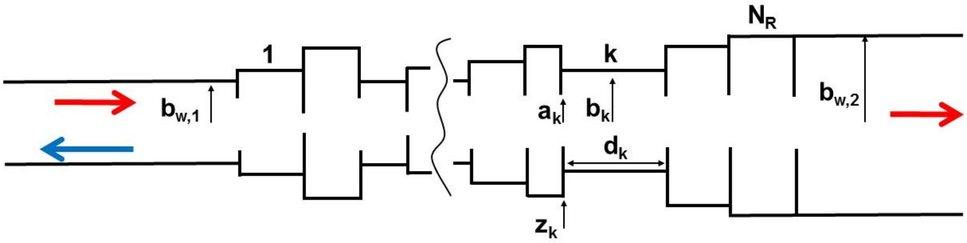

Fig. 1. Chain of resonators with two waveguides

Using the same procedure, we get

$$\sum_s \frac{b_{w,1}}{\lambda_s} R_{s',s}^{\psi(w,1)}\left(G_s^{(1)}+G_{-s}^{(1)}\right)=\varepsilon\sum_s \frac{b_1}{\lambda_s} R_{s',s}^{\psi(1,1)}\left(h_s^{(1)}+h_{-s}^{(1)}\right), s'=1,2...,N_m, \quad (19)$$

$$\sum_s \frac{b_{w,2}}{\lambda_s} R_{s',s}^{\psi(w,2)} G_s^{(2)}=\varepsilon\sum_s (-1)^n \frac{b_{N_R}}{\lambda_s} R_{s',s}^{\psi(w,2)}\begin{pmatrix} h_s^{(N_R)}\exp\left(\gamma_s^{(N_R)} d_{N_R}\right)+ \\ +h_{-s}^{(N_R)}\exp\left(\gamma_{-s}^{(N_R)} d_{N_R}\right)\end{pmatrix}, \quad (20)$$

$s'=1,2...,N_m$.

The tangential electromagnetic field $E_r^{(k)}(r,z=d_k)$ we expand in terms of a set of basis functions $\varphi_s\left(r/a_k\right)$

$$E_r^{(k)}=\sum_{s=1}^{N_m} C_s^{(k)}\varphi_s\left(r/a_k\right). \quad (21)$$

The boundary condition for electric field at the junction $z=z_k$ can be written as

$$\sum_{s'}\left(h_{s'}^{(k)}\mathcal{E}_{s',r}^{(k)}+h_{-s'}^{(k)}\mathcal{E}_{-s',r}^{(k)}\right)=\begin{cases}\sum_{s'=1}^{N_m} C_{s'}^{(k)}\varphi_{s'}\left(r/a_k\right), 0\le r<a_k, \\ 0, \quad a_k\le r<b_k,\end{cases} \quad (22)$$

$$\sum_{s'}\left(h_{s'}^{(k-1)}\mathcal{E}_{s',r}^{(k-1)}+h_{-s'}^{(k-1)}\mathcal{E}_{-s',r}^{(k-1)}\right)=\begin{cases}\sum_{s'=1}^{N_m} C_{s'}^{(k)}\varphi_{s'}\left(r/a_k\right), 0\le r<a_k, \\ 0, \quad a_k\le r<b_{k-1}.\end{cases}$$

Using the completeness and orthogonality of Bessel functions $J_1\left(\lambda_s r/b\right)$, we obtain ($k = 1,\dots,N_R$)

$$\frac{b_k^3 J_1^2\left(\lambda_s\right)}{\lambda_s}\gamma_s^{(k)} h_s^{(k)} sh\left(\gamma_s^{(k)} d_k\right)=$$

$$=-a_{k+1}^2\sum_{s'=1}^{N_m} R_{s,s'}^{\varphi(k+1,2)} C_{s'}^{(k+1)}+\exp\left(\gamma_{-s}^{(k)} d_k\right) a_k^2\sum_{s'=1}^{N_m} R_{s,s'}^{\varphi(k,1)} C_{s'}^{(k)} \quad (23)$$

$$\frac{b_k^3 J_1^2\left(\lambda_s\right)}{\lambda_s}\gamma_s^{(k)} h_{-s}^{(k)} sh\left(\gamma_s^{(k)} d_k\right)=$$

$$=-a_{k+1}^2\sum_{s'=1}^{N_m} R_{s,s'}^{\varphi(k+1,2)} C_{s'}^{(k+1)}+\exp\left(\gamma_s^{(k)} d_k\right) a_k^2\sum_{s'=1}^{N_m} R_{s,s'}^{\varphi(k,1)} C_{s'}^{(k)},$$

where

$$R_{m,s}^{\varphi(k,1)}=\int_0^1 \varphi_s\left(x\right) J_1\left(a_k\lambda_m x/b_k\right) x dx,$$

$$R_{m,s}^{\varphi(k+1,2)}=\int_0^1 \varphi_s\left(x\right) J_1\left(a_{k+1}\lambda_m x/b_k\right) x dx. \quad (24)$$

Consider the case when the dimensions of two waveguides are chosen such that only the dominant mode TM_{01} propagates, and the higher-order modes are all evanescent. We will suppose that there is an incident wave that travels from $z=-\infty$ with amplitude $G_1^{(1)}=1$ ($G_s^{(1)}=0, s\ge 2$).

Then the boundary condition for electric field at the junction of the first waveguide and the first resonator ($z=z_1$) gives relations

$$-1+G_{-1}^{(1)}=2\frac{a_1^2\lambda_1}{J_1^2\left(\lambda_1\right) b_{w,1}^2\gamma_1^{(1)} b_{w,1}}\sum_{s'}^{N_m} R_{s',1}^{\varphi(1,1)} C_{s'}^{(1)},$$

$$G_{-s}^{(1)}=2\frac{a_1^2\lambda_s}{J_1^2\left(\lambda_s\right) b_{w,1}^2\gamma_s^{(1)} b_{w,1}}\sum_{s'}^{N_m} R_{s',s}^{\varphi(1,1)} C_{s'}^{(1)}. \quad (25)$$

Using the same procedure at the junction $z=z_{N_R+1}$, we get ($s=1,2,\dots,N_m$)

$$G_s^{(2)}=-2\frac{a_{N_R+1}^2\lambda_s}{J_1^2\left(\lambda_s\right) b_{w,2}^2\gamma_s^{(2)} b_{w,2}}\sum_{s'}^{N_m} R_{s',s}^{\varphi(N_R+1,2)} C_{s'}^{(N_R+1)}. \quad (26)$$

For the case of one diaphragm between two circular waveguides we obtain

$$\sum_s \frac{b_{w,1}}{\lambda_s} R_{s',s}^{\psi(w,1)}\left(G_s^{(1)}+G_{-s}^{(1)}\right)=\sum_s \frac{b_{w,2}}{\lambda_s} R_{s',s}^{\psi(w,2)} G_s^{(2)}. \quad (27)$$

Substitution (25) and (26) into (27) gives such system

$$\sum_{s'}^{N_m} C_{s'}^{(1)}\sum_{s''}\left[\frac{a_1^2 R_{s,s''}^{\psi(w,1)} R_{s',s''}^{\varphi(1,1)}}{b_{w,1}^2 J_1^2\left(\lambda_{s''}\right)\gamma_{s''}^{(1)} b_{w,1}}+\frac{b_{w,2}}{b_{w,1}}\frac{a_1^2 R_{s,s''}^{\psi(w,2)} R_{s',s''}^{\varphi(1,2)}}{b_{w,2}^2 J_1^2\left(\lambda_{s''}\right)\gamma_{s''}^{(2)} b_{w,2}}\right]=-\frac{1}{\lambda_1} R_{s,1}^{\psi(w,1)}. \quad (28)$$

2. COUPLED INTEGRAL EQUATION MODEL

Substitution (23) into (15) gives (N_R-1) systems from $\left(N_R+1\right)$ necessary systems[1] of the CIE technique [33].

$$\sum_{s'=1}^{N_m} T_{s,s'}^{(k,1)} C_{s'}^{(k)}+\sum_{s'=1}^{N_m} T_{s,s'}^{(k,2)} C_{s'}^{(k)}-\sum_{s'=1}^{N_m} T_{s,s'}^{(k,3)} C_{s'}^{(k+1)}-\sum_{s'=1}^{N_m} T_{s,s'}^{(k,4)} C_{s'}^{(k-1)}=0, \quad (29)$$

where $k=2,3,\dots,N_R$, $s=1,2,\dots,N_m$,

$$T_{s,s'}^{(k,1)}=\frac{a_k}{b_k}\sum_m \frac{\Lambda_m^{(k)}(d_k)}{J_1^2(\lambda_m)} R_{s,m}^{\psi(k,1)} R_{m,s'}^{\varphi(k,1)},$$

$$T_{s,s'}^{(k,2)}=\frac{a_k}{b_{k-1}}\sum_m \frac{\Lambda_m^{(k-1)}(d_{k-1})}{J_1^2(\lambda_m)} R_{s,m}^{\psi(k,2)} R_{m,s'}^{\varphi(k,2)},$$

$$T_{s,s'}^{(k,3)}=\frac{a_{k+1}^2}{a_k b_k}\sum_m \frac{\Lambda_m^{(k)}(0)}{J_1^2(\lambda_m)} R_{s,m}^{\psi(k,1)} R_{m,s'}^{\varphi(k+1,2)}, \quad (30)$$

$$T_{s,s'}^{(k,4)}=\frac{a_{k-1}^2}{a_k b_{k-1}}\sum_m \frac{\Lambda_m^{(k-1)}(0)}{J_1^2(\lambda_m)} R_{s,m}^{\psi(k,2)} R_{m,s'}^{\varphi(k-1,1)},$$

$$\Lambda_m^{(k)}(z)=\frac{ch\left(b_k\gamma_m^{(k)}\frac{z}{d_k}\frac{d_k}{b_k}\right)}{\gamma_m^{(k)} b_k sh(b_k\gamma_m^{(k)}\frac{d_k}{b_k})}. \quad (31)$$

From (19) and (20) we obtain two additional systems

[1] We have (N_R+1) interfaces.

$$\sum_{s'=1}^{N_m}\left(\varepsilon T^{(1,1)}_{s,s'} - W^{(1)}_{s,s'}\right) - \sum_{s'=1}^{N_m}\varepsilon T^{(1,3)}_{s,s'}C^{(2)}_{s'} = \frac{b_{w,1}}{a_1\lambda_1}R^{\psi(w,1)}_{s',1},$$
$$\sum_{s'=1}^{N_m}\varepsilon T^{(N_R+1,4)}_{s,s'}C^{(N_R)}_{s'} + \sum_{s'=1}^{N_m}\left(W^{(2)}_{s,s'} - \varepsilon T^{(N_R+1,2)}_{s,s'}\right)C^{(N_R+1)}_{s'} = 0, \quad (32)$$

where

$$W^{(1)}_{s',s} = \frac{a_1}{b_{w,1}}\sum_m \frac{R^{\psi(w,1)}_{s,m}R^{\varphi(1,1)}_{s',m}}{J_1^2(\lambda_m)\gamma_m^{(w,1)}b_{w,1}}, \quad (33)$$

$$W^{(2)}_{s',s} = \frac{a_{N_R+1}}{b_{w,2}}\sum_m \frac{R^{\psi(w,2)}_{s,m}R^{\varphi(N_R+1,1)}_{s',m}}{J_1^2(\lambda_m)\gamma_m^{(w,2)}b_{w,2}}. \quad (34)$$

The reflection and transmission coefficients are given by

$$R_w = G^{(1)}_{-1} = 1 + 2\frac{a_1^2\lambda_1}{J_1^2(\lambda_1)b_{w,1}^2\gamma_1^{(1)}b_{w,1}}\sum_{s'}^{N_m}R^{\varphi(1,1)}_{s',1}C^{(1)}_{s'}, \quad (35)$$

$$T_w = G^{(2)}_1 = -\frac{2a^2_{N_R+1}\lambda_1}{J_1^2(\lambda_1)b_{w,2}^2\gamma_1^{(2)}b_{w,2}}\sum_{s'}^{N_m}R^{\varphi(N_R+1,2)}_{s',1}C^{(N_R+1)}_{s'}. \quad (36)$$

Electric field in the k-th resonator can be calculated by summing the relevant sequences

$$E_z^{(k)}(z_k, r=0) = \sum_{s=1}^{N_m}T^{E(k)}_{s,1}C^{(k)}_s - \sum_{s''=1}^{N_m}T^{E(k)}_{s,2}C^{(k+1)}_s, \; 0 < z_k < d_k, \quad (37)$$

where

$$T^{E(k)}_{s,1} = 2\frac{a_k^2}{b_k^2}\sum_m \frac{\lambda_m R^{\varphi(k,1)}_{m,s}\Lambda^{(k)}_m(z_k - d_k)}{J_1^2(\lambda_m)},$$
$$T^{E(k)}_{s,2} = 2\frac{a_{k+1}^2}{b_k^2}\sum_m \frac{\lambda_m R^{\varphi(k+1,2)}_{m,s}\Lambda^{(k)}_m(z_k)}{J_1^2(\lambda_m)}. \quad (38)$$

Therefore, the set of systems (in matrix form) of the CIE Model are ($k = 2,3,...,N_R$)

$$\left(T^{(k,1)} + T^{(k,2)}\right)C^{(k)} - T^{(k,3)}C^{(k+1)} - T^{(k,4)}C^{(k-1)} = 0,$$
$$\left(\varepsilon T^{(1,1)} - W^{(1)}\right)C^{(1)} - \varepsilon T^{(1,3)}C^{(2)} = \frac{b_{w,1}}{a_1\lambda_1}R_1^{\psi(w,1)}, \quad (39)$$
$$\varepsilon T^{(N_R+1,4)}C^{(N_R)} + \left(W^{(2)} - \varepsilon T^{(N_R+1,2)}\right)C^{(N_R+1)} = 0,$$

where $T^{(k)}, W$ are $N_m \times N_m$ complex matrices, $C^{(k)} = \left(C^{(k)}_1, C^{(k)}_2, ..., C^{(N_R+1)}_{N_m}\right)^T$, $R_1^{\psi(w,1)} = \left(R^{\psi(w,1)}_{1,1}, R^{\psi(w,1)}_{2,1}, ..., R^{\psi(w,1)}_{N_m,1}\right)^T$.

We can rewrite (39) as

$$T^{\Sigma}C^{\Sigma} = R^{\Sigma}, \quad (40)$$

where T^{Σ} is a block-tridiagonal matrix, $C^{\Sigma} = \left(C^{(1)}, C^{(2)}, ..., C^{(N_R+1)}\right)$. Block tridiagonal systems of linear equations are of great interest since they are encountered in a wide variety of problems, in particular, in discrete differential equations (see, for example, [37] and the literature cited there).

3. NUMERICAL IMPLEMENTATIONS OF THE MODEL

In our models we have to make several choices: the kind and the number N_m of the basis functions φ_n and the testing functions ψ_n, and the upper limit L_m of summation in the sums for calculation of matrix elements $T_{s,s'}$.

In this work we used the entire-domain basis and testing functions. We considered several sets that give analytical expressions for the Hankel transform (24) (coefficients $R^{\varphi(\psi)}$). The simplest case (J-J) is the use of the Bessel functions $\varphi_s(x) = \psi_s(x) = J_1(\lambda_s x)$, $x \in [0,1]$

$$R^{\psi(k,1)}_{m',m} = R^{\varphi(k,1)}_{m',m} = \int_0^1 \psi_{m'}(x)J_1\left(\frac{a_k\lambda_m}{b_k}x\right)xdx = -\frac{a_k\lambda_m}{b_k}\frac{J_0\left(\frac{a_k\lambda_m}{b_k}\right)J_1(\lambda_{m'})}{\left(\frac{a_k\lambda_m}{b_k}\right)^2 - (\lambda_{m'})^2}, \quad (41)$$
$$R^{\psi(k,2)}_{m',m} = R^{\varphi(k,2)}_{m',m} = \int_0^1 \psi_{m'}(x)J_1\left(\frac{a_k\lambda_m}{b_{k-1}}x\right)xdx = -\frac{a_k\lambda_m}{b_{k-1}}\frac{J_0\left(\frac{a_k\lambda_m}{b_{k-1}}\right)J_1(\lambda_{m'})}{\left(\frac{a_k\lambda_m}{b_{k-1}}\right)^2 - (\lambda_{m'})^2}.$$

In this case the edge behavior of electric field is not incorporated into the algorithm. The second case (M-J) is the use the Bessel functions as the testing functions $\psi_s(x) = J_1(\lambda_s x)$, $x \in [0,1]$ and the complete set of functions that fulfil the edge condition on the diaphragm rims as the basis functions (the Meixner basis). We use such Meixner basis [38]

$$\varphi_s(r) = 2\sqrt{\pi}\frac{\Gamma(s+1)}{\Gamma(s-0.5)}\frac{1}{\sqrt{1-r^2}}P^{-1}_{2s-1}\left(\sqrt{1-r^2}\right), \quad (42)$$

where $P_n^m(x)$ are Legendre functions (or spherical functions) of the first kind [39]. The first three functions are:

$$\varphi_1(r) = \frac{r}{\sqrt{1-r^2}}, \quad (43)$$

$$\varphi_2(r) = \frac{r}{\sqrt{1-r^2}}\left\{-5r^2 + 4\right\}, \quad (44)$$

$$\varphi_3(r) = \frac{r}{\sqrt{1-r^2}}\left\{21r^4 - 28r^2 + 8\right\}. \quad (45)$$

There exist useful integral for our consideration

$$\int_0^1 \varphi_s(t)J_1(xt)tdt = j_{2s-1}(x) = \sqrt{\frac{\pi}{2x}}J_{2s-0.5}(x), \quad (46)$$

$$R^{\varphi(k,1)}_{m',m} = \int_0^1 \varphi_{m'}(x)J_1\left(\frac{a_k\lambda_m}{b_k}\right)xdx = j_{2m'-1}\left(\frac{a_k\lambda_m}{b_k}\right) = \sqrt{\frac{\pi b_k}{2a_k\lambda_m}}J_{m'-0.5}\left(\frac{a_k\lambda_m}{b_k}\right),$$
$$R^{\varphi(k,1)}_{m',m} = \int_0^1 \varphi_{m'}(x)J_1\left(\frac{a_k\lambda_m}{b_{k-1}}\right)xdx = j_{2m'-1}\left(\frac{a_k\lambda_m}{b_{k-1}}\right) = \sqrt{\frac{\pi b_{k-1}}{2a_k\lambda_m}}J_{m'-0.5}\left(\frac{a_k\lambda_m}{b_{k-1}}\right). \quad (47)$$

The third case (M-M) is the use the Meixner basis as the basis and testing functions.

The simplest geometrical configuration that can give estimations about the "quality" of the chosen sets of functions is the one thin diaphragm in the cylindrical waveguide. Few calculations were performed to obtain the characteristics of the scattering TM waves on the circular diaphragm [40 - 43], so we studied the numerical convergence of the results that was obtained with using the Moment Method. It is known that the Moment Method can lead to ill-conditioned systems of linear equations (see, for example, [44 - 47]).

We studied diffraction of TM_{01} wave on the circular diaphragm (frequency $f = 2.856$ GHz, waveguide radius $b_{w1} = b_{w2} = 4.2$ cm, aperture radius $a = 1.5$ cm). From calculation results we can make such conclusions:

- there is a wide range of parameters for which the system of linear equations (39) is not ill-conditioned and we can get results with acceptable accuracy;

- using all three sets of functions gives similar results;
- accuracy of J-J sets is worse than M-J and M-M;
- accuracy of M-J sets is the same as M-M sets;
- accuracy of amplitude calculations in the fourth sign and hundredths of a degree in phase is achieved at $N_m = 2$ and $L_m = 500$ for the M-J and M-M cases.

The correctness of the calculation results is confirmed by comparison with the experimental results [40].

Bellow we will be use the M-J representation with $N_m = 2$ and $L_m = 500$.

Analysis of more complicated system (wave diffraction on the two coupled resonators $b_{w1} = b_{w2} = b_1 = b_2 = 4.2$ cm, aperture radius $a = 1.5$ cm) shows that chosen values of the number of functions give acceptable accuracy of field calculation too.

4. TRANSFORMATION OF THE BASIC EQUATIONS

Matrix equations (39), that describe the finite chain of resonators, we can rewrite as:

$$\begin{aligned}
&\left(\varepsilon T^{(1,1)} - W^{(1)}\right)C^{(1)} - \varepsilon T^{(1,3)}C^{(2)} = \frac{b_{w,1}}{a_1\lambda_1} R_1^{\psi(w,1)},\\
&\left(T^{(2,1)} + T^{(2,2)}\right)C^{(2)} - T^{(2,3)}C^{(3)} - T^{(k,4)}C^{(1)} = 0,\\
&C^{(k+1)} + \tilde{T}^{(k,4)}C^{(k-1)} = \tilde{T}^{(k)}C^{(k)},\ k = 3,4,...,N_R - 1,\\
&\left(T^{(N_R,1)} + T^{(N_R,2)}\right)C^{(N_R)} - T^{(N_R,3)}C^{(N_R+1)} - T^{(N_R,4)}C^{(N_R-1)} = 0,\\
&\varepsilon T^{(N_R+1,4)}C^{(N_R)} + \left(W^{(2)} - \varepsilon T^{(N_R+1,2)}\right)C^{(N_R+1)} = 0,
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
T^{(k)} &= T^{(k,1)} + T^{(k,2)},\\
\tilde{T}^{(k)} &= T^{(k,3)-1}T^{(k)},\\
\tilde{T}^{(k,4)} &= T^{(k,3)-1}T^{(k,4)}.
\end{aligned} \tag{49}$$

We separated the equations for the first two and the last two resonators from the others, since when the waveguides are matched to the chain, the first and last resonators can be very different from the rest.

WKB asymptotic approximation theory (see [48] and the literature sited there) was developed for a class of almost-diagonal ('asymptotically diagonal') linear second-order matrix difference equations

$$C^{(k+2)} + A^{(k)}C^{(k+1)} + B^{(k)}C^{(k)} = 0\ , \tag{50}$$

by transforming them into the form

$$C^{(k+2)} - 2C^{(k+1)} + C^{(k)} + G^{(k)}C^{(k)} = 0\ . \tag{51}$$

We shall transform (50) into the other form

$$C^{(k+2)} + C^{(k)} + G^{(k+1)}C^{(k+1)} = 0\ . \tag{52}$$

We use the procedure similar to that was used in [35].

In equation ($k = 3,4,...,N_R - 1$)

$$C^{(k+1)} + \tilde{T}^{(k,4)}C^{(k-1)} = \tilde{T}^{(k)}C^{(k)} \tag{53}$$

we put ($k = 2,3,...,N_R$)

$$C^{(k)} = \Xi^{(k)}\tilde{C}^{(k)}, \tag{54}$$

with invertible matrices $\Xi^{(k)}$.

Suppose that the matrices $\Xi^{(k)}$ satisfy the equation

$$\Xi^{(k+2)} = \tilde{T}^{(k+1,4)}\Xi^{(k)}, \tag{55}$$

with $\Xi^{(2)} = \Xi^{(3)} = I$, I – the unit matrix.

Solution of equation (55) is ($n = 2,3,...$):

$$\Xi^{(2n)} = \prod_{s=2}^{n}\tilde{T}^{(2s-1,4)},\qquad \Xi^{(2n+1)} = \prod_{s=2}^{n}\tilde{T}^{(2s,4)}. \tag{56}$$

Equation (53) takes then the form ($k = 3,4,...,N_R - 1$)

$$\tilde{C}^{(k+1)} + \tilde{C}^{(k-1)} = \tilde{\tilde{T}}^{(k)}\tilde{C}^{(k)}, \tag{57}$$

where $\tilde{\tilde{T}}^{(k)} = \Xi^{(k+1)-1}\tilde{T}^{(k)}\Xi^{(k)}$

As the equation (57) is of the second order, we represent the solution of the matrix difference equation (57) as the sum of two new vectors [36] ($k = 2,3,...,N_R$)

$$\tilde{C}^{(k)} = \tilde{C}^{(k,1)} + \tilde{C}^{(k,2)}. \tag{58}$$

By introducing two new unknowns $C^{(k,i)}$ instead of the one $C^{(k)}$, we can impose an additional condition. This condition we write in the form ($k = 2,3,...,N_R - 1$)

$$\tilde{C}^{(k+1)} = M^{(k,1)}\tilde{C}^{(k,1)} + M^{(k,2)}\tilde{C}^{(k,2)}, \tag{59}$$

where $M^{(k,i)}$ are the arbitrary invertible matrices.

Using (58) and (59) we can rewrite (57) as ($k = 2,4,...,N_R - 2$)

$$\begin{aligned}
&\left(M^{(k+1,1)} - M^{(k+1,2)}\right)\tilde{C}^{(k+1,1)} = \left\{\left(\tilde{\tilde{T}}^{(k+1)} - M^{(k+1,2)}\right)M^{(k,1)} - I\right\}\tilde{C}^{(k,1)} +\\
&\qquad + \left\{\left(\tilde{\tilde{T}}^{(k+1)} - M^{(k+1,2)}\right)M^{(k,2)} - I\right\}\tilde{C}^{(k,2)},\\
&\left(M^{(k+1,2)} - M^{(k+1,1)}\right)\tilde{C}^{(k+1,2)} = \left\{\left(\tilde{\tilde{T}}^{(k+1)} - M^{(k+1,1)}\right)M^{(k,1)} - I\right\}\tilde{C}^{(k,1)} +\\
&\qquad + \left\{\left(\tilde{\tilde{T}}^{(k+1)} - M^{(k+1,1)}\right)M^{(k,2)} - I\right\}\tilde{C}^{(k,2)}.
\end{aligned} \tag{60}$$

Let's choose matrices $M^{(k,i)}$ ($i = 1, 2$) so that they satisfy quadratic matrix equations ($k = 2,4,...,N_R - 2$)

$$\left(\tilde{\tilde{T}}^{(k+1)} - M^{(k+1,i)}\right)M^{(k+1,i)} = I\ . \tag{61}$$

It should be noted that these equations do not define $M^{(2,i)}$. As $M^{(k,i)}$ can be chosen arbitrary, we shall take $M^{(2,i)} = M^{(3,i)}$.

Then (60) transforms into

$$\begin{aligned}
&M^{(k+1,2)}\left(M^{(k+1,1)} - M^{(k+1,2)}\right)\tilde{C}^{(k+1,1)} = \left(M^{(k+1,1)} - M^{(k+1,2)}\right)\tilde{C}^{(k,1)} +\\
&\qquad + \left(M^{(k,2)} - M^{(k+1,2)}\right)C^{(k,2)} + \left(M^{(k,1)} - M^{(k+1,1)}\right)\tilde{C}^{(k,1)},\\
&M^{(k+1,1)}\left(M^{(k+1,2)} - M^{(k+1,1)}\right)\tilde{C}^{(k+1,2)} = \left(M^{(k+1,2)} - M^{(k+1,1)}\right)\tilde{C}^{(k,2)} +\\
&\qquad + \left(M^{(k,1)} - M^{(k+1,1)}\right)\tilde{C}^{(k,1)} + \left(M^{(k,2)} - M^{(k+1,2)}\right)\tilde{C}^{(k,2)}.
\end{aligned} \tag{62}$$

It can be shown that in our case[2] the matrix $\tilde{\tilde{T}}^{(k)}$ is nondefective, and can be decomposed as

$$\tilde{\tilde{T}}^{(k)} = U^{(k)}\Theta^{(k)}U^{(k)-1}, \tag{63}$$

where $U^{(k)}$ is the matrix of eigen vectors and $\Theta^{(k)} = diag(\theta_1^{(k)}, \theta_2^{(k)}, ..., \theta_{N_m}^{(k)})$, $\theta_i^{(k)}$ – eigen values.

Then the solutions of quadratic equations (61) are

$$M^{(k,i)} = U^{(k)}\Lambda^{(k,i)}U^{(k)-1}, \tag{64}$$

where $\Lambda^{(k,i)} = diag(\lambda_1^{(k,i)}, \lambda_2^{(k,i)}, ..., \lambda_{N_m}^{(k,i)})$ and $\lambda_s^{(k.i)}$ are the solutions of the characteristic equations

[2] The infinitive uniform disk-loaded waveguide has $2N_m$ different independent solutions (waves). We can expect that this property will be correct for inhomogeneous waveguide too, at least for the case of slowly varying parameters.

$$\lambda_s^{(k,i)2} - \theta_s^{(k)}\lambda_s^{(k,i)} + 1 = 0,$$
$$\lambda_s^{(k,1)} = \theta_s^{(k)}/2 + \sqrt{\left(\theta_s^{(k)}/2\right)^2 - 1}, \quad (65)$$
$$\lambda_s^{(k,2)} = \theta_s^{(k)}/2 - \sqrt{\left(\theta_s^{(k)}/2\right)^2 - 1}.$$

The matrices $M^{(k,i)}$ have the same eigen vectors, therefore they are commutative. As $\lambda_s^{(k,1)}\lambda_s^{(k,2)} = 1$, the matrices $M^{(k,i)}$ satisfy the condition

$$M^{(k,1)}M^{(k,2)} = I. \quad (66)$$

Using these properties we transform (62) into ($k = 2,4,...,N_R - 2$)

$$\tilde{C}^{(k+1,1)} = M^{(k,1)}\tilde{C}^{(k,1)} + \left(\tilde{M}^{(k+1,1)} - I\right)\left(M^{(k,1)} - M^{(k+1,1)}\right)\tilde{C}^{(k,1)} + \\ + \tilde{M}^{(k+1,1)}\left(M^{(k,2)} - M^{(k+1,2)}\right)\tilde{C}^{(k,2)}, \quad (67)$$
$$\tilde{C}^{(k+1,2)} = M^{(k,2)}\tilde{C}^{(k,2)} + \left(\tilde{M}^{(k+1,2)} - I\right)\left(M^{(k,2)} - M^{(k+1,2)}\right)\tilde{C}^{(k,2)} + \\ + \tilde{M}^{(k+1,2)}\left(M^{(k,1)} - M^{(k+1,1)}\right)\tilde{C}^{(k,1)},$$

where

$$\tilde{M}^{(k,1)} = \left[M^{(k,2)}\left(M^{(k,1)} - M^{(k,2)}\right)\right]^{-1} = U^{(k)}\tilde{\Lambda}^{(k,1)}U^{(k)-1},$$
$$\tilde{M}^{(k,2)} = \left[M^{(k,1)}\left(M^{(k,2)} - M^{(k,1)}\right)\right]^{-1} = U^{(k)}\tilde{\Lambda}^{(k,2)}U^{(k)-1}, \quad (68)$$
$$\tilde{\Lambda}^{(k,1)} = diag\left(\left(1 - \lambda_1^{(k,2)2}\right)^{-1}, ..., \left(1 - \lambda_{N_m}^{(k,2)2}\right)^{-1}\right),$$
$$\tilde{\Lambda}^{(k,2)} = diag\left(\left(1 - \lambda_1^{(k,1)2}\right)^{-1}, ..., \left(1 - \lambda_{N_m}^{(k,1)2}\right)^{-1}\right).$$

As $\tilde{M}^{(k+1,1)} + \tilde{M}^{(k+1,2)} = I$, then from (67) we get

$$\tilde{C}^{(k+1)} = \tilde{C}^{(k+1,1)} + \tilde{C}^{(k+1,2)} = M^{(k,1)}\tilde{C}^{(k,1)} + M^{(k,2)}\tilde{C}^{(k,2)}. \quad (69)$$

This matches the condition (59).

If elements of matrices $M^{(k,i)}$ vary sufficiently slowly with k, then the differences $\left|M_{s,m}^{(k+1,i)} - M_{s,m}^{(k,i)}\right|$ are the small values and we can neglect them (Eikonal approximation)

$$\tilde{C}^{(k+1,1)} = M^{(k,1)}\tilde{C}^{(k,1)},$$
$$\tilde{C}^{(k+1,2)} = M^{(k,2)}\tilde{C}^{(k,2)}. \quad (70)$$

If we neglect only nondiagonal terms in (67) we get the WKB approximation

$$\tilde{C}^{(k+1,1)} = \tilde{\tilde{M}}^{(k+1,1)}\tilde{C}^{(k,1)} =$$
$$= \left\{M^{(k+1,1)} + \tilde{M}^{(k+1,1)}\left(M^{(k,1)} - M^{(k+1,1)}\right)\right\}\tilde{C}^{(k,1)},$$
$$\tilde{C}^{(k+1,2)} = \tilde{\tilde{M}}^{(k+1,2)}\tilde{C}^{(k,2)} = \quad (71)$$
$$= \left\{M^{(k+1,2)} + \tilde{M}^{(k+1,2)}\left(M^{(k,2)} - M^{(k+1,2)}\right)\right\}\tilde{C}^{(k,2)}.$$

Finally, we can wright the transformed system

$$\left(\varepsilon T^{(1,1)} - W^{(1)}\right)C^{(1)} - \varepsilon T^{(1,3)}\left(\tilde{C}^{(2,1)} + \tilde{C}^{(2,2)}\right) = \frac{b_{w,1}}{a_1\lambda_1}R_1^{\psi(w,1)}, \quad (72)$$
$$T^{(2)}\left(\tilde{C}^{(2,1)} + \tilde{C}^{(2,2)}\right) - T^{(2,3)}\left(\tilde{C}^{(3,1)} + \tilde{C}^{(3,2)}\right) - T^{(2,4)}C^{(1)} = 0,$$
$$k = 2,3,...,N_R - 2,$$
$$\tilde{C}^{(k+1,1)} = \tilde{\tilde{M}}^{(k+1,1)}\tilde{C}^{(k,1)} + \tilde{M}^{(k+1,1)}\left(M^{(k,2)} - M^{(k+1,2)}\right)\tilde{C}^{(k,2)}, \quad (73)$$
$$\tilde{C}^{(k+1,2)} = \tilde{\tilde{M}}^{(k+1,2)}\tilde{C}^{(k,2)} + \tilde{M}^{(k+1,2)}\left(M^{(k,1)} - M^{(k+1,1)}\right)\tilde{C}^{(k,1)},$$
$$T^{(N_R)}\Xi^{(N_R)}\left(M^{(N_R-1,1)}\tilde{C}^{(N_R-1,1)} + M^{(N_R-1,2)}\tilde{C}^{(N_R-1,2)}\right) -$$
$$-T^{(N_R,4)}\Xi^{(N_R-1)}\left(\tilde{C}^{(N_R-1,1)} + \tilde{C}^{(N_R-1,2)}\right) - T^{(N_R,3)}C^{(N_R+1)} = 0, \quad (74)$$
$$\varepsilon T^{(N_R+1,4)}\Xi^{(N_R)}\left(M^{(N_R-1,1)}\tilde{C}^{(N_R-1,1)} + M^{(N_R-1,2)}\tilde{C}^{(N_R-1,2)}\right) +$$
$$+\left(W^{(2)} - \varepsilon T^{(N_R+1,2)}\right)C^{(N_R+1)} = 0.$$

Electric field in the k-th resonator can be calculated by summing the relevant sequences ($0 < z_k < d_k$)

$$E_z^{(k)}(z_k, r = 0) = \sum_{s=1}^{N_m} T_{s,1}^{E(k)}C_s^{(k)} - \sum_{s''=1}^{N_m} T_{s,2}^{E(k)}C_s^{(k+1)}, \quad (75)$$

where

$$C^{(k)} = C^{(k,1)} + C^{(k,2)} = \Xi^{(k)}\tilde{C}^{(k,1)} + \Xi^{(k)}\tilde{C}^{(k,2)}. \quad (76)$$

As in the WKB and Eikonal approximations the matrix equations (70) and (71) have the analytic solutions, we can greatly simplify the system (72). Under this, it should be borne in mind that one part of the field is described by growing solutions when moving in the positive direction of the waveguide axis; therefore, it is necessary to calculate the field by moving in the negative direction. It is also convenient to use vectors $C^{(k,i)}$ as they determine the values of electric field (see (75)).

Such simplified system of equations is

$$C^{(k+1,1)} = \Upsilon^{(k,1)}C^{(k,1)}, \quad k = 2,3,...,N_R - 2,$$
$$C^{(k,2)} = \Upsilon^{(k+1,2)}C^{(k+1,2)}, k = N_R - 2,...,3,2, \quad (77)$$

where

$$\Upsilon^{(k,1)} = \Xi^{(k+1)}\tilde{\tilde{M}}^{(k+1,1)}\Xi^{(k)-1},$$
$$\Upsilon^{(k+1,2)} = \Xi^{(k)}\left(\tilde{\tilde{M}}^{(k+1,2)}\right)^{-1}\Xi^{(k+1)-1}. \quad (78)$$

Boundary values of vectors $C^{(k,1)}\left(C^{(2,1)}\right)$ and $C^{(k,2)}\left(C^{(N_R-1,2)}\right)$ are defined by equations

$$\left(\varepsilon T^{(1,1)} - W^{(1)}\right)C^{(1)} - \varepsilon T^{(1,3)}C^{(2,1)} -$$
$$-\varepsilon T^{(1,3)}M^{(-)}\Xi^{(N_R-1)-1}C^{(N_R-1,2)} = \frac{b_{w,1}}{a_1\lambda_1}R_1^{\psi(w,1)},$$
$$-T^{(2,4)}C^{(1)} + \left(T^{(2)} - T^{(2,3)}M^{(3,1)}\right)C^{(2,1)} + \quad (79)$$
$$+\left(T^{(2)} - T^{(2,3)}M^{(3,2)}\right)M^{(-)}\Xi^{(N_R-1)-1}C^{(N_R-1,2)} = 0,$$
$$\left(T^{(N_R)}\Xi^{(N_R)}M^{(N_R-1,1)} - T^{(N_R,4)}\Xi^{(N_R-1)}\right)M^{(+)}C^{(2,1)} +$$
$$+\left(T^{(N_R)}\Xi^{(N_R)}M^{(N_R-1,2)}\Xi^{(N_R-1)-1} - T^{(N_R,4)}\right)C^{(N_R-1,2)} -$$
$$-T^{(N_R,3)}C^{(N_R+1)} = 0, \quad (80)$$
$$\varepsilon T^{(N_R+1,4)}\Xi^{(N_R)}M^{(N_R-1,1)}M^{(+)}C^{(2,1)} +$$
$$+\varepsilon T^{(N_R+1,4)}\Xi^{(N_R)}M^{(N_R-1,2)}\Xi^{(N_R-1)-1}C^{(N_R-1,2)} +$$
$$+\left(W^{(2)} - \varepsilon T^{(N_R+1,2)}\right)C^{(N_R+1)} = 0,$$

where

$$M^{(+)} = \prod_{k=3}^{N_R-1}\tilde{\tilde{M}}^{(k,1)}, M^{(-)} = \prod_{k=N_R-1}^{3}\left(\tilde{\tilde{M}}^{(k,2)}\right)^{-1}. \quad (81)$$

This system of equation is more suitable for simulation as we have to solve a system of linear equations which dimension is fixed and equals to 8. This makes possible to consider any number of resonators N_R. Comparison of results of calculation by using this system and the one based on the solving the full system of linear equations (72) - (74) in the WKB approximation shows their good coincidence (error is up to 1.E-7)

5. INFINITE HOMOGENEOUS CHAIN

If we omit the presence of boundaries for the uniform chain of resonators ($b_k = b$, $a_k = a$), we obtain the equations describing an infinite homogeneous disk-loaded waveguide

$$\tilde{C}^{(k+1,1)} = M^{(1)}\tilde{C}^{(k,1)}, \quad \tilde{C}^{(k+1,2)} = M^{(2)}\tilde{C}^{(k,2)}. \tag{82}$$

For the uniform chain of resonators matrices $\Xi^{(k)} = I$ and $C^{(k,i)} = \tilde{C}^{(k,i)}$.

It can be shown that the general solutions of the difference matrix equations (82) are

$$C^{(k,i)} = \sum_{s=1}^{N_m} B_s^{(i)} \lambda_s^{(i)k} U_s \,, \tag{83}$$

where $B_s^{(i)}$ – are constants, λ_s (characteristic or Floquet multipliers) are the solutions of the characteristic equations

$$\lambda_s^2 - \theta_s \lambda_s + 1 = 0 \,, \tag{84}$$

with θ_s and U_s that are eigen values and eigen vectors of matrix $\tilde{T}$

$$\tilde{T} U_s = \theta_s U_s \,. \tag{85}$$

From (84) it follows that

$$\lambda_{s,1} \lambda_{s,2} = 1 \,. \tag{86}$$

This property of the Floquet multipliers (along with the assumption that $\varepsilon'' \neq 0$) guarantees that problem (72) - (74) is well-conditioned, at least in the case when matrix elements are slowly changing [48].

Analysis of the solution (83) shows that representation (58) is not a trivial decomposition into forward and backward waves. Decomposition (58) with the conditions (59) and (61) divide the solution of matrix difference equation (57) into two parts each of which is generalization of concepts forward and backward waves, especially in the case of inhomogeneous waveguides.

Starting from the N_m-dimensional system, in the case of homogeneous waveguide we can obtain the difference equation that describes the behavior of one component of the variables $C^{(k)}$, say $C_1^{(k)}$. The above solution of equation for $C^{(k)}$ (83) shows that the characteristic equation of this difference equation must have roots that coincide with the $2N_m$ eigenvalues λ_s.

The general form of this equation can be prompted by considering the simplest case $N_m = 2$ in system (48) for infinite chain

$$\left(C_1^{(k+1)} + C_1^{(k-1)}\right) - \tilde{T}_{1,1} C_1^{(k)} = \tilde{T}_{1,2} C_2^{(k)}, \quad \left(C_2^{(k+1)} + C_2^{(k-1)}\right) - \tilde{T}_{2,2} C_2^{(k)} = \tilde{T}_{2,1} C_1^{(k)}. \tag{87}$$

We introduce the commutative operators $\hat{L}_i$

$$\hat{L}_i = \hat{\sigma}^+ + \hat{\sigma}^- - \tilde{T}_{i,i} \,, \tag{88}$$

where $\hat{\sigma}^+ \left(\hat{\sigma}^+ b^{(k)} = b^{(k+1)}\right)$ and $\hat{\sigma}^- \left(\hat{\sigma}^- b^{(k)} = b^{(k-1)}\right)$ – are shift operators. From (87) we can get such equation

$$\widehat{\det} \begin{pmatrix} \hat{L}_1 & -\tilde{T}_{1,2} \\ -\tilde{T}_{2,1} & \hat{L}_2 \end{pmatrix} C_1^{(k)} = 0 \,, \tag{89}$$

where the operator $\widehat{\det}$ is defined on the base of rules of common determinants[3]

$$\widehat{\det} \begin{pmatrix} \hat{L}_1 & -\tilde{T}_{1,2} \\ -\tilde{T}_{2,1} & \hat{L}_2 \end{pmatrix} = \hat{L}_1 \hat{L}_2 - \tilde{T}_{1,2} \tilde{T}_{2,1} \,. \tag{90}$$

It can be shown that in general case we get the equation

$$\widehat{\det} \begin{pmatrix} \hat{L}_1 & -T_{1,2} & \dots & -T_{1,N_m} \\ -T_{2,1} & \hat{L}_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -T_{N_m,1} & -T_{N_m,1=2} & \dots & \hat{L}_{N_m} \end{pmatrix} C_1^{(k)} = 0 \,, \tag{91}$$

with the characteristic equation

$$\begin{vmatrix} \lambda^2 - T_{1,1}\lambda + 1 & -\lambda T_{1,2} & \dots & -\lambda T_{1,N_m} \\ -\lambda T_{2,1} & \lambda^2 - T_{2,2}\lambda + 1 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ -\lambda T_{N_m,1} & -\lambda T_{N_m,1=2} & \dots & \lambda^2 - T_{N_m,N_m}\lambda + 1 \end{vmatrix} = 0 \,. \tag{92}$$

Numerical calculations show that (84) and (92) give the values of λ_s which coincide with good accuracy.

It can be shown that equations for other components of vector $C^{(k)}$ are the same as (91).

6. FINITE CHAIN OF RESONATORS

We wrote two computer codes. The first is based on the system (39), the second – on the transformed system (72) - (74). All results that are given bellow were calculated with $N_m = 2^4$, $L_m = 500$. These codes give practically the same results. It is confirmed by results of calculation that are presented in Fig. 2 ($\varepsilon = 1^5$), where differences between amplitudes of electric fields at the centers of resonators calculated on the base of systems (39) and (72) - (74) for homogeneous and inhomogeneous waveguides with 60 resonators are given ($d_{1-60} = 3.4989$ cm, $b_{2-59} = 4.16595$ cm, $b_1 = b_{60} = 4.19825$ cm, $a_1 = a_{61} = 1.7661$ cm, $f = 2.856$ GHz, changes in the size of the apertures are shown in Fig. 3). Here and below we consider the propagation of an incident TM_{01} wave with a unit amplitude through the disk-loaded waveguide (DLW), shown schematically in Fig. 1.

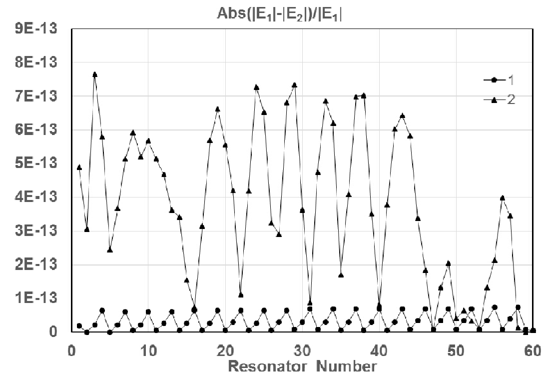

Fig. 2. Differences between amplitudes of electric fields at the centers of resonators calculated on the base of systems (39) and (72) - (72) for two DLW: homogeneous (1) and inhomogeneous (2)

[3] We have deal with commutative matrices.

[4] Taking such value we include in consideration one propagation wave and one evanescent oscillation.

[5] If $\mathrm{Im}\,\varepsilon \neq 0$ the differences become greater, but less than 1.E-6.

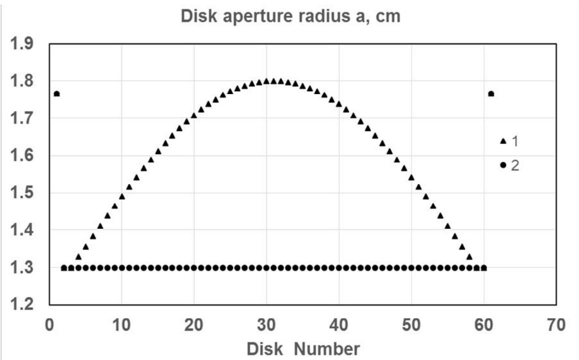

Fig. 3. Dimensions of the apertures considered waveguides

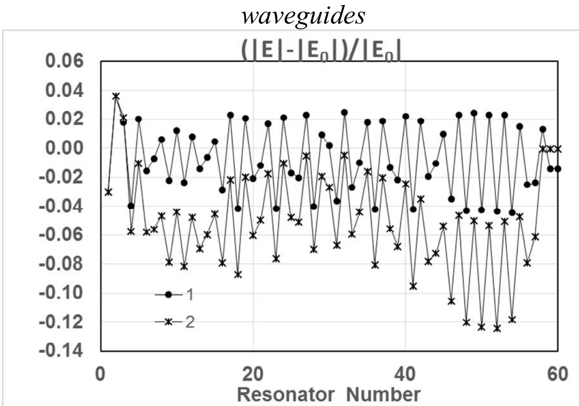

Fig. 4. Comparisons of electric field amplitude distributions calculated on the base of the initial system (39) and WKB approximation (71) - (1), the initial system (39) and Eikonal approximation (70) - (2)

At the selected frequency and for a homogeneous DLW with $a_{2-59} = 1.3$ cm (the first disk aperture distribution (1) in Fig. 3) the phase shift per cell[6] in the DLW equals $2\pi/3$, the reflection coefficient is $R =$ 7.86E-04 ($T = 0.9999$).

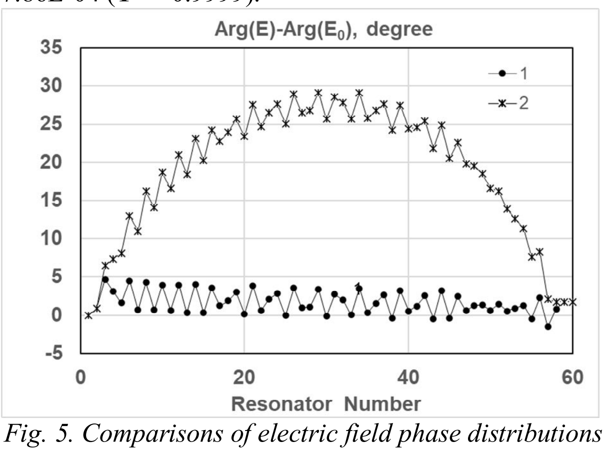

Fig. 5. Comparisons of electric field phase distributions calculated on the base of the initial system (39) and WKB approximation (71) - (1), the initial system (39) and Eikonal approximation (70) - (2)

Consider the accuracy of WKB approximation in the case of IDLW with the geometric dimensions indicated above (the second disk aperture distribution (2) in Fig. 3). Parameters of this IDLW change along the waveguide at a moderate gradient. Results of comparison of electric field distributions calculated on the base of systems (39) and (71) are presented in Figs. 4 and 5 – (1). We also present a comparison of the electric field distributions calculated on the base of systems (39) and (70) (see Figs. 4 and 5 – (2)).

Results of comparisons show that for moderate gradient of IDLW parameters the WKB approximation gives suitable accuracy, while the results of the Eikonal approximation differ from the exact ones more significantly, especially in the phase distribution.

CONCLUSIONS

The presented approach to the description of inhomogeneous resonator chains (inhomogeneous disk-loaded waveguides) can be a useful tool in studying the properties of slow wave system. On its basis, various approximate approaches have been developed, including the WKB approximation.

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МОДЕЛЬ КОНЕЧНОЙ НЕОДНОРОДНОЙ ЦЕПОЧКИ РЕЗОНАТОРОВ И ПРИБЛИЖЕННЫЕ МЕТОДЫ ЕЕ АНАЛИЗА

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Предложен новый подход к описанию неоднородной цепи связанных резонаторов (неоднородных диафрагмированных волноводов). Получены новые матричные разностные уравнения, основанные на технике связанных интегральных уравнений и методе декомпозиции. Разработаны различные приближенные подходы, включая приближение WKB.

МОДЕЛЬ КІНЦЕВОГО НЕОДНОРІДНОГО ЛАНЦЮГА РЕЗОНАТОРІВ І НАБЛИЖЕНІ МЕТОДИ ЇЇ АНАЛІЗУ

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Запропоновано новий підхід до опису неоднорідного ланцюга зв'язаних резонаторів (неоднорідних діафрагмованих хвилеводів). Отримані нові матричні різницеві рівняння, які засновані на техніці зв'язаних інтегральних рівнянь та методі декомпозиції. Розроблені різні наближені підходи, включаючи наближення WKB.