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On the derivations of Leibniz algebras of low dimension

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Let L be an algebra over a field F. Then L is called a left Leibniz algebra if its multiplication operations [·, ·] additionally satisfy the so-called left Leibniz identity: [[a,b],c] = [a,[b,c]] – [b,[a,c]] for all elements a, b, c ∈ L. In this paper, we begin the description of the algebra of derivations of Leibniz algebras having dimension 3. It is clear that the description of the algebra of derivations of all Leibniz algebras, having dimension 3, is quite large. Therefore, in this article, we will focus on the description of the nilpotent Leibniz algebra, whose nilpotency class is 3, and the nilpotent Leibniz algebra, whose center has dimension 2.

Keywords: *dimension, derivation, hypercenter, Leibniz algebra, nilpotent Leibniz algebra*

Let L be an algebra over a field F with the binary operations + and $[\cdot, \cdot]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]]$$

for all $a, b, c \in L$. We will also use another form of this identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Leibniz algebras appeared first in the paper of A. Blokh [1], but the term “Leibniz algebra” appears in the book of J.-L. Loday [2], and the article of J.-L. Loday [3]. In [4] J.-L. Loday and T. Pirashvili began the real study of the properties of Leibniz algebras. The theory of Leibniz algebras has developed very intensively in many different directions. Some of the results of this theory have been presented in the book [5].

Same as in Lie algebras, the structure of Leibniz algebras is greatly influenced by their algebras of derivations.

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Denote by $\mathrm{End}_F(L)$ the set of all linear transformations of L, then L is an associative algebra by the operation + and ∘. As usual, $\mathrm{End}_F(L)$ is a Lie algebra by the operations + and [,], where $[f, g] = f \circ g - g \circ f$ for all $f, g \in \mathrm{End}_F(L)$.

A linear transformation f of a Leibniz algebra L is called a *derivation*, if

$$f([a, b]) = [f(a), b] + [a, f(b)] \text{ for all } a, b \in L.$$

Let Der(L) be the subset of all derivations of L. It is possible to prove that Der(L) is a subalgebra of a Lie algebra $\mathrm{End}_F(L)$. Der(L) is called the *algebra of derivations* of a Leibniz algebra L.

The influence on the structure of the Leibniz algebra of their algebras of derivations can be seen from the following result: if A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of Der(A) [6, Proposition 3.2].

It is natural to start studying the algebra of derivations of Leibniz algebras, the structure of which has been studied quite extensively. A description of the structure of algebras of derivations of finite-dimensional cyclic Leibniz algebras was obtained in papers [7—9]. The question naturally arises about an algebra of derivations of Leibniz algebras, having a small dimension. In contrast to Lie algebras, the situation with Leibniz algebras of dimension 3 is very diverse. Leibniz algebras of dimension 3 are mostly described, and the description of Leibniz algebras of dimensions 4, and 5 are carried out quite intensively. Here we only note that the study of right Leibniz algebras of dimension 3 is the subject of section 3.1 of a book [5] and works [10—14].

In this paper, we begin the description of the algebra of derivations of Leibniz algebras, having dimension 3. It is clear that the description of the algebra of derivations of all Leibniz algebras, having dimension 3, is quite large. Therefore, in this article, we will focus on the description of nilpotent Leibniz algebra, whose nilpotency class is 3, and of nilpotent Leibniz algebra, whose center has dimension 2.

1. Some preliminary results. Let's start with some general properties of the algebra of derivations of Leibniz algebras. We will show in this section some basic elementary properties of derivations, which have been proved in a paper [7]. First of all, let's recall some definitions.

Every Leibniz algebra L has one specific ideal. Denote by Leib(L) the subspace, generated by the elements $[a, a]$, $a \in L$. It is possible to prove that Leib(L) is an ideal of L. The ideal Leib(L) is called the *Leibniz kernel* of algebra L. By its definition, a factor-algebra L/Leib(L) is a Lie algebra. And conversely, if K is an ideal of L such that L/K is a Lie algebra, then K includes a Leibniz kernel.

Let L be a Leibniz algebra. Define the lower central series of L as

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \geqslant \gamma_\alpha(L) \geqslant \gamma_{\alpha+1}(L) \geqslant \ldots \gamma_\delta(L) = \gamma_\infty(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \cap_{\mu < \lambda}\gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

If $\alpha = k$ is a positive integer, then $\gamma_k(L) = [L, [L, [L, ..., L] ... L]$ is the *left normed commutator* of k copies of L.

As usual, we say that a Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0\rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0\rangle$, but $\gamma_c(L) \neq \langle 0\rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule:

$$\zeta^{\text{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively,

$$\zeta^{\text{left}}(L) = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but it is not true for the right center. Moreover, $Leib(L) \leqslant z^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L, and, in general, the left and right centers are different; they even may have different dimensions (see [6]).

The *center* $\zeta(L)$ of L is defined by the rule:

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

We define now the upper central series

$$\langle 0 \rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \zeta_2(L) \leqslant \ldots \zeta_\alpha(L) \leqslant z_{\alpha+1}(L) \leqslant \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \cup_{\mu<\lambda}\zeta_\mu(L)$ for the limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the *upper hypercenter* of L. If $L = \zeta_\infty(L)$ then L is called a *hypercentral* Leibniz algebra.

Lemma 1. *Let L bea Leibniz algebra over a field F and f be a derivation of L. Then* $f(\zeta^{\text{left}}(L)) \leqslant \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \leqslant \zeta^{\text{right}}(L)$ *and* $f(\zeta(L)) \leqslant \zeta(L)$.

Corollary. *Let L bea Leibniz algebra over a field F and f be a derivation of L. Then* $f(\zeta_\alpha(L)) \leqslant \zeta_\alpha(L)$ *for every ordinal* α**.**

Lemma 2. *Let L bea Leibniz algebra over a field F and f be a derivation of L. Then* $f(\gamma_\alpha(L)) \leqslant \gamma_\alpha(L)$ *for all ordinals* α*, in particular,* $f(\gamma_\infty(L)) \leqslant \gamma_\infty(L)$.

It is natural to first give a description of the algebra of derivations of the Leibniz algebras of dimension 2. The description of Leibniz algebra, having dimension 2, is given in several papers, one of the first of which was [15]. The Leibniz algebras, having dimension 2, which are not Lie algebras, are limited to the algebras of the following two types

$$\mathrm{Lei}_1(2, F) = Fa_1 \oplus Fa_2 \text{ where } [a_1, a_1] = a_2,\ [a_1, a_2] = [a_2, a_1] = [a_2, a_2] = 0;$$

$$\mathrm{Lei}_2(2, F) = Fa_1 \oplus Fa_2 \text{ where } [a_1, a_1] = a_2,\ [a_1, a_2] = a_2,\ [a_2, a_1] = [a_2, a_2] = 0.$$

Let L be a Lie algebra. We say that L is a *semidirect sum* of an ideal A and a subalgebra B if $L = A + B$ and $A \cap B = \langle 0 \rangle$.

Proposition 1. *Let D be the algebra of derivations of the Leibniz algebra* $\mathrm{Lei}_1(2, F)$. *Then D is a semidirect sum of an ideal ofdimension* 1 *and a subalgebra of dimension* 1. *More precisely, D is* *isomorphic to a subalgebra of matrices, having the following form*

$$\begin{pmatrix} \alpha_1 & 0 \\ \alpha_2 & 2\alpha_1 \end{pmatrix}, \textit{ where } \alpha_1, \alpha_2 \in F.$$

Proposition 2. *Let D be the algebra of derivations of the Leibniz algebra* $\mathrm{Lei}_2(2, F)$. *Then D is abelian and has dimension* 1, $D = Ff$ *where* $f(a_1) = a_2, f(a_2) = a_2$.

2. Algebra of derivations of some Leibniz algebras, having dimension 3. Now, let's move on to the main part of our work, namely the consideration of the algebra of derivations of a Leibniz algebra with dimension 3. Naturally, we will only consider Leibniz algebras that are not Lie algebras, which means their Leibniz kernel is not zero. The first type of Leibniz algebras we will consider is the nilpotent Leibniz algebras, and specifically, the nilpotent Leibniz algebras of nilpotency class 3. There is only one type of such algebra, which is the following $\mathrm{Lei}_1(3, F)$:

$$\mathrm{Lei}_1(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where } [a_1, a_1] = a_2, [a_1, a_2] = a_3, [a_1, a_3] = 0,$$

$$[a_2, a_1] = [a_3, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_3] = 0.$$

It is cyclic Leibniz algebra,

$$\mathrm{Leib}(\mathrm{Lei}_1(3, F)) = \zeta^{\mathrm{left}}(\mathrm{Lei}_1(3, F)) = [\mathrm{Lei}_1(3, F), \mathrm{Lei}_1(3, F)] = Fa_2 \oplus Fa_2,$$

$$\zeta^{\mathrm{right}}(\mathrm{Lei}_1(3, F)) = \zeta(\mathrm{Lei}_1(3, F)) = \gamma_3(\mathrm{Lei}_1(3, F)) = Fa_3.$$

Theorem 1. *Let D be the algebra of derivations of the Leibniz algebra* $\mathrm{Lei}_1(3, F)$. *Then D is a semidirect sum of an ideal N of dimension* 1 *and a subalgebra of dimension* 1, *generated by derivation* f_1 *such that* $f_1(a_1) = a_1, f_1(a_2) = 2a_2, f_1(a_3) = 3a_3$. *Furthermore, N is abelian,* $N = Ff_2 \oplus Ff_3$, *where* $f_2(a_1) = a_2,\ f_2(a_2) = a_3, f_2(a_3) = 0,\ f_3(a_1) = a_3,\ f_3(a_2) = 0,\ f_3(a_3) = 0$. *An algebra D is isomorphic to a Lie subalgebra of matrices, having the following form*

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ \alpha_2 & 2\alpha_1 & 0 \\ \alpha_3 & \alpha_2 & 3\alpha_1 \end{pmatrix}, \textit{ where } \alpha_1, \alpha_2, \alpha_3 \in F.$$

Theorem 2. *Let D be the algebra of derivations of the Leibniz algebra* $\mathrm{Lei}_2(3, F)$. *Then D has a series of ideals* $\langle 0 \rangle \leqslant N \leqslant C \leqslant A \leqslant D$ *such that N is abelian,* $N = Ff_3 \oplus Ff_4$, $C = N \oplus Ff_2$, $A = C \oplus Ff_1$, $D = A \oplus Ff_0$, *where* f_0, f_1, f_2, f_3, f_4 *are the derivation, defined by the rules:*

$$f_0(a_1) = a_1,\ f_0(a_2) = 2a_2,\ f_0(a_3) = 0;$$

$$f_1(a_1) = 0,\ f_1(a_2) = 0,\ f_1(a_3) = a_3;$$

$$f_2(a_1) = a_3,\ f_2(a_2) = 0,\ f_2(a_3) = 0;$$

$$f_3(a_1) = a_2,\ f_3(a_2) = 0,\ f_3(a_3) = 0;$$

$$f_4(a_1) = 0,\ f_4(a_2) = 0,\ f_4(a_3) = a_2.$$

Moreover,

$$f_3 \circ f_4 = f_4 \circ f_3,\ f_3 \circ f_2 = f_2 \circ f_3,$$

$$[f_4, f_2] = f_3,\ [f_1, f_2] = f_2,$$

$$f_3 \circ f_1 = f_1 \circ f_3,\ [f_4, f_1] = f_4,$$

$$f_0 \circ f_1 = f_1 \circ f_0,$$

$$[f_2, f_0] = f_2,\ [f_0, f_3] = f_3,\ [f_0, f_4] = 2f_4.$$

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ПРО ПОХІДНІ АЛГЕБР ЛЕЙБНІЦА МАЛОЇ ВИМІРНОСТІ

Нехай L — це алгебра над полем F. Тоді L називається лівою алгеброю Лейбніца, якщо її операції множення $[\cdot, \cdot]$ задовольняють так звану ліву тотожність Лейбніца: $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ для всіх елементів $a, b, c \in L$. У статті започатковано опис алгебри похідних алгебр Лейбніца, що мають вимірність 3. Зрозуміло, що опис алгебри похідних всіх алгебр Лейбніца вимірності 3 є досить великим. Тому тут наведено опис нільпотентних алгебр Лейбніца, клас нільпотентності яких дорівнює 3, та нільпотентних алгебр Лейбніца, центр яких має розмірність 2.

Ключові слова: *вимірність, похідна, гіперцентр, алгебра Лейбніца, нільпотентна алгебра Лейбніца*.