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S-second submodules of a module

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Abstract. Let R be a commutative ring with identity and let M be an R-module. The main purpose of this paper is to introduce and study the notion of S-second submodules of an R-module M as a generalization of second submodules of M.

1. Introduction

Throughout this paper, R will denote a commutative ring with identity and $\mathbb{Z}$ will denote the ring of integers.

Consider a nonempty subset S of R. We call S a multiplicatively closed subset (briefly, m.c.s.) of R if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $s\acute{s} \in S$ for all $s, \acute{s} \in S$ [15]. Note that $S = R - P$ is a m.c.s. of R for every prime ideal P of R. Let M be an R-module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [9]. A non-zero submodule N of M is said to be *second* if for each $a \in R$, the homomorphism $N \stackrel{a}{\to} N$ is either surjective or zero [16].

Let S be a m.c.s. of R and P a submodule of an R-module M with $(P :_R M) \cap S = \varnothing$. Then the submodule P is said to be an *S-prime submodule* of M if there exists an $s \in S$, and whenever $am \in P$, then $sa \in (P :_R M)$ or $sm \in P$ for each $a \in R$, $m \in M$ [14]. Particularly, an ideal I of R is said to be an *S-prime ideal* if I is an S-prime submodule of the R-module R.

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Let S be a m.c.s. of R and M be an R-module. The main purpose of this paper is to introduce the notion of S-second submodules of an R-module M as a generalization of second (dual notion of S-prime) submodules of M and provide some useful information concerning this class of modules. Moreover, we obtain some results analogous to those for S-prime submodules considered in [14].

2. Main results

Let M be an R-module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i\in I} N_i$, where $\{N_i\}_{i\in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [11].

Remark 2.1. Let N and K be two submodules of an R-module M. To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$ [4].

Theorem 2.2. *Let S be a m.c.s. of R. For a submodule N of an R-module M with $\mathrm{Ann}_R(N) \cap S = \varnothing$ the following statements are equivalent:*

(a) *There exists an $s \in S$ such that $srN = sN$ or $srN = 0$ for each $r \in R$;*

(b) *There exists an $s \in S$ and whenever $rN \subseteq K$, where $r \in R$ and K is a submodule of M, implies either that $rsN = 0$ or $sN \subseteq K$;*

(c) *There exists an $s \in S$ and whenever $rN \subseteq L$, where $r \in R$ and L is a completely irreducible submodule of M, implies either that $rsN = 0$ or $sN \subseteq L$.*

(d) *There exists an $s \in S$, and $JN \subseteq K$ implies $sJ \subseteq \mathrm{Ann}_R(N)$ or $sN \subseteq K$ for each ideal J of R and submodule K of M.*

Proof. $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are clear.

$(c) \Rightarrow (a)$ By part (c), there exists an $s \in S$. Assume that $srN \neq 0$ for some $r \in R$. Then $s^2rN \neq 0$. If $rsN \subseteq L$ for some completely irreducible submodule L of M, then by assumption, $sN \subseteq L$. Hence, by Remark 2.1, $sN \subseteq rsN$, as required.

$(b) \Rightarrow (d)$ Suppose that $JN \subseteq K$ for some ideal J of R and submodule K of M. By part (b), there is an $s \in S$ so that $rN \subseteq K$ implies $sr \in \mathrm{Ann}_R(N)$ or $sN \subseteq K$ for each $r \in R$. Assume that $sN \not\subseteq K$. Then by Remark 2.1, there exists a completely irreducible submodule L of M such that $K \subseteq L$ but $sN \not\subseteq L$. Then note that for each $a \in J$, we have $aN \subseteq L$. By part (b), we can conclude that $sa \in \mathrm{Ann}_R(N)$ and so $sJ \subseteq \mathrm{Ann}_R(N)$.

$(d) \Rightarrow (b)$ Take $a \in R$ and K a submodule of M with $aN \subseteq K$. Now, put $J = Ra$. Then we have $JN \subseteq K$. By assumption, there is an $s \in S$ such that $sJ = Ras \subseteq \mathrm{Ann}_R(N)$ or $sN \subseteq K$ and so either $sa \in \mathrm{Ann}_R(N)$ or $sN \subseteq K$ as needed. □

Definition 2.3. Let S be a m.c.s. of R and N be a submodule of an R-module M such that $\mathrm{Ann}_R(N) \cap S = \varnothing$. We say that N is an *S-second submodule of M* if satisfies the equivalent conditions of Theorem 2.2. By an *S-second module*, we mean a module which is an S-second submodule of itself.

The following lemma is known, but we write it here for the sake of reference.

Lemma 2.4. Let M be an R-module, S a m.c.s. of R, and N be a finitely generated submodule of M. If $S^{-1}N \subseteq S^{-1}K$ for a submodule K of M, then there exists an $s \in S$ such that $sN \subseteq K$.

Proof. This is straightforward. □

Let S be a m.c.s. of R. Recall that the saturation S^* of S is defined as $S^* = \{x \in R : x/1 \textit{ is a unit of } S^{-1}R\}$. It is obvious that S^* is a m.c.s. of R containing S [12].

Proposition 2.5. Let S be a m.c.s. of R and M be an R-module. Then we have the following.

(a) If N is a second submodule of M such that $S \cap \mathrm{Ann}_R(N) = \varnothing$, then N is an S-second submodule of M. In fact if $S \subseteq u(R)$ and N is an S-second submodule of M, then N is a second submodule of M.

(b) If $S_1 \subseteq S_2$ are m.c.s.s of R and N is an S_1-second submodule of M, then N is an S_2-second submodule of M in case $\mathrm{Ann}_R(N) \cap S_2 = \varnothing$.

(c) N is an S-second submodule of M if and only if N is an S^*-second submodule of M

(d) If N is a finitely generated S-second submodule of M, then $S^{-1}N$ is a second submodule of $S^{-1}M$

Proof. (a) and (b) These are clear.

(c) Assume that N is an S-second submodule of M. We claim that $\mathrm{Ann}_R(N) \cap S^* = \varnothing$. To see this assume that there exists an $x \in \mathrm{Ann}_R(N) \cap S^*$ As $x \in S^*$, $x/1$ is a unit of $S^{-1}R$ and so $(x/1)(a/s) = 1$ for some $a \in R$ and $s \in S$. This yields that $us = uxa$ for some $u \in S$. Now we have that $us = uxa \in \mathrm{Ann}_R(N) \cap S$, a contradiction. Thus, $\mathrm{Ann}_R(N) \cap S^* = \varnothing$. Now as $S \subseteq S^*$, by part (b), N is an S^*-second submodule of M. Conversely, assume that N is an S^*-second submodule of M. Let $rN \subseteq K$. As N is an S^*-second submodule of M, there is an $x \in S^*$ such that $xr \in \mathrm{Ann}_R(N)$ or $xN \subseteq K$. As $x/1$ is a unit of $S^{-1}R$, there exist $u, s \in S$ and $a \in R$ such that $us = uxa$. Then note that $(us)r = uaxr \in \mathrm{Ann}_R(N)$ or $us(xN) \subseteq K$. Therefore, N is an S-second submodule of M.

(d) If $S^{-1}N = 0$, then as N is finitely generated, there is an $s \in S$ such that $s \in \mathrm{Ann}_R(N)$ by Lemma 2.4. This implies that $\mathrm{Ann}_R(N) \cap S \neq \varnothing$, a contradiction. Thus $S^{-1}N \neq 0$. Now let $r/t \in S^{-1}R$. As N is an S-second submodule of M, there is an $s \in S$ such that $rsN = sN$ or $rsN = 0$. If $rsN = sN$, then $(r/s)S^{-1}N = S^{-1}N$. If $rsN = 0$, then $(r/s)S^{-1}N = 0$, as needed. □

Corollary 2.6. Let M be an R-module and set $S = \{1\}$. Then every second submodule of M is an S-second submodule of M.

Proof. Let N be a second submodule of M. Then as $N \neq 0$, we have $1 \notin \mathrm{Ann}_R(N)$. Hence $S \cap \mathrm{Ann}_R(N) = \varnothing$ and the result follows from Proposition 2.5 (a). □

The following examples show that the converses of Proposition 2.5 (a) and (d) are not true in general.

Example 2.7. Take the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_2$ for a prime number p. Then $2(\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_2) = \mathbb{Z}_{p^\infty} \oplus 0$ implies that M is not a second $\mathbb{Z}$-module. Now, take the m.c.s. $S = \mathbb{Z} \setminus \{0\}$ and put $s = 2$. Then $2rM = \mathbb{Z}_{p^\infty} \oplus 0 = 2M$ for all $r \in \mathbb{Z}$ and so M is an S-second $\mathbb{Z}$-module.

Example 2.8. Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Q}$, where $\mathbb{Q}$ is the field of rational numbers. Take the submodule $N = \mathbb{Z} \oplus 0$ and the m.c.s. $S = \mathbb{Z} \setminus \{0\}$. Then one can see that N is not an S-second submodule of M. Since $S^{-1}\mathbb{Z} = \mathbb{Q}$ is a field, $S^{-1}(\mathbb{Q} \oplus \mathbb{Q})$ is a vector space so that a non-zero submodule $S^{-1}N$ is a second submodule of $S^{-1}(\mathbb{Q} \oplus \mathbb{Q})$.

An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M, we have $N = (0 :_M Ann_R(N))$ [2].

Proposition 2.9. Let M be an R-module and S be a m.c.s. of R. Then the following statements hold.

(a) If N is an S-second submodule of M, then $\mathrm{Ann}_R(N)$ is an S-prime ideal of R.

(b) If M is a comultiplication R-module and $\text{Ann}_R(N)$ is an S-prime ideal of R, then N is an S-second submodule of M.

Proof. (a) Let $ab \in \text{Ann}_R(N)$ for some $a, b \in R$. As N is an S-second submodule of M, there exists an $s \in S$ such that $asN = sN$ or $asN = 0$ and $bsN = sN$ or $bsN = 0$. If $asN = 0$ or $bsN = 0$ we are done. If $asN = sN$, then $0 = basN = bsN$, a contradiction. If $bsN = sN$, then $0 = absN = asN$, a contradiction. Thus in any case, $asN = 0$ or $bsN = 0$, as needed.

(b) Assume that M is a comultiplication R-module and $\text{Ann}_R(N)$ is an S-prime ideal of R. Let $r \in R$ and K be a submodule of M with $rN \subseteq K$. Then $\text{Ann}_R(K)rN = 0$. As $\text{Ann}_R(N)$ is an S-prime ideal of R, by [14, Corollary 2.6], there is an $s \in S$ such that $sAnn_R(K) \subseteq \text{Ann}_R(N)$ or $sr \in \text{Ann}_R(N)$. If $sr \in \text{Ann}_R(N)$, we are done. If $sAnn_R(K) \subseteq \text{Ann}_R(N)$, then $\text{Ann}_R(K) \subseteq \text{Ann}_R(sN)$. Now as M is a comultiplication R-module, we have $sN \subseteq K$, as desired. □

An R-module M satisfies the *double annihilator conditions* (DAC for short) if for each ideal I of R we have $I = \text{Ann}_R(0 :_M I)$ [10]. An R-module M is said to be a *strong comultiplication module* if M is a comultiplication R-module and satisfies the DAC conditions [6].

Theorem 2.10. *Let M be a strong comultiplication R-module and N be a submodule of M such that $\text{Ann}_R(N) \cap S = \varnothing$, where S is a m.c.s. of R. Then the following are equivalent:*

(a) *N is an S-second submodule of M;*
(b) *$\text{Ann}_R(N)$ is an S-prime ideal of R;*
(c) *$N = (0 :_M I)$ for some S-prime ideal I of R with $\text{Ann}_R(N) \subseteq I$.*

Proof. $(a) \Rightarrow (b)$ This follows from Proposition 2.9.

$(b) \Rightarrow (c)$ As M is a comultiplication R-module, $N = (0 :_M Ann_R(N))$. Now the result is clear.

$(c) \Rightarrow (a)$ As M satisfies the DAC conditions, $\text{Ann}_R((0 :_M I)) = I$. Now the result follows from Proposition 2.9. □

Let R_i be a commutative ring with identity, M_i be an R_i-module for each $i = 1, 2, \ldots, n$, and $n \in \mathbb{N}$. Assume that $M = M_1 \times M_2 \times \cdots \times M_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Then M is clearly an R-module with componentwise addition and scalar multiplication. Also, if S_i is a multiplicatively closed subset of R_i for each $i = 1, 2, \ldots, n$, then $S = S_1 \times S_2 \times \cdots \times S_n$ is a multiplicatively closed subset of R. Furthermore, each submodule N of M is of the form $N = N_1 \times N_2 \times \cdots \times N_n$, where N_i is a submodule of M_i.

Theorem 2.11. *Let $M = M_1 \times M_2$ be an $R = R_1 \times R_2$-module and $S = S_1 \times S_2$ be a m.c.s. of R, where M_i is an R_i-module and S_i is a m.c.s. of R_i for each $i = 1, 2$. Let $N = N_1 \times N_2$ be a submodule of M. Then the following are equivalent:*

(a) *N is an S-second submodule of M;*

(b) *N_1 is an S_1-second submodule of M_1 and $\mathrm{Ann}_{R_2}(N_2) \cap S_2 \neq \varnothing$ or N_2 is an S_2-second submodule of M_2 and $\mathrm{Ann}_{R_1}(N_1) \cap S_1 \neq \varnothing$.*

Proof. $(a) \Rightarrow (b)$ Let $N = N_1 \times N_2$ be an S-second submodule of M. Then $\mathrm{Ann}_R(N) = \mathrm{Ann}_{R_1}(N_1) \times \mathrm{Ann}_{R_2}(N_2)$ is an S-prime ideal of R by Proposition 2.9. By [14, Lemma 2.13], either $\mathrm{Ann}_R(N_1) \cap S_1 \neq \varnothing$ or $\mathrm{Ann}_R(N_2) \cap S_2 \neq \varnothing$. We may assume that $\mathrm{Ann}_R(N_1) \cap S_1 \neq \varnothing$. We show that N_2 is an S_2-second submodule of M_2. To see this, let $r_2 N_2 \subseteq K_2$ for some $r_2 \in R_2$ and a submodule K_2 of M_2. Then $(1, r_2)(N_1 \times N_2) \subseteq M_1 \times K_2$. As N is an S-second submodule of M, there exists an$(s_1, s_2) \in S$ such that $(s_1, s_2)(N_1 \times N_2) \subseteq M_1 \times K_2$ or $(s_1, s_2)(1, r_2)(N_1 \times N_2) = 0$. It follows that $s_2 N_2 \subseteq K_2$ or $s_2 r_2 N_2 = 0$ and so N_2 is an S_2-second submodule of M_2. Similarly, if $\mathrm{Ann}_{R_2}(N_2) \cap S_2 \neq \varnothing$, one can see that N_1 is an S_1-second submodule of M_1.

$(b) \Rightarrow (a)$ Assume that N_1 is an S_1-second submodule of M_1 and $\mathrm{Ann}_{R_2}(N_2) \cap S_2 \neq \varnothing$. Then there exists an $s_2 \in \mathrm{Ann}_{R_2}(N_2) \cap S_2$. Let $(r_1, r_2)(N_1 \times N_2) \subseteq K_1 \times K_2$ for some $r_i \in R_i$ and submodule K_i of M_i, where $i = 1, 2$. Then $r_1 N_1 \subseteq K_1$. As N_1 is an S_1-second submodule of M_1, there exists an $s_1 \in S_1$ such that $s_1 N_1 \subseteq K_1$ or $s_1 r_1 N_1 = 0$. Now we set $s = (s_1, s_2)$. Then $s(N_1 \times N_2) \subseteq K_1 \times K_2$ or $s(r_1, r_2)(N_1 \times N_2) = 0$. Therefore, N is an S-second submodule of M. Similarly one can show that if N_2 is an S_2-second submodule of M_2 and $\mathrm{Ann}_{R_1}(N_1) \cap S_1 \neq \varnothing$, then N is an S-second submodule of M. □

Theorem 2.12. *Let $M = M_1 \times M_2 \times \cdots \times M_n$ be an $R = R_1 \times R_2 \times \cdots \times R_n$-module and $S = S_1 \times S_2 \times \cdots \times S_n$ be a m.c.s. of R, where M_i is an R_i-module and S_i is a m.c.s. of R_i for each $i = 1, 2, \ldots, n$. Let $N = N_1 \times N_2 \times \cdots \times N_n$ be a submodule of M. Then the following are equivalent:*

(a) *N is an S-second submodule of M;*

(b) *N_i is an S_i-second submodule of M_i for some $i \in \{1, 2, \ldots, n\}$ and $\mathrm{Ann}_{R_j}(N_j) \cap S_j \neq \varnothing$ for all $j \in \{1, 2, \ldots, n\} - \{i\}$.*

Proof. We apply induction on n. For $n = 1$, the result is true. If $n = 2$, then the result follows from Theorem 2.11. Now assume that parts (a) and (b) are equal when $k < n$. We shall prove $(b) \Leftrightarrow (a)$ when $k = n$. Let $N = N_1 \times N_2 \times \cdots \times N_n$. Put $\acute{N} = N_1 \times N_2 \times \cdots \times N_{n-1}$ and $\acute{S} = S_1 \times S_2 \times \cdots \times S_{n-1}$. Then by Theorem 2.11, the necessary and sufficient condition for N is an S-second submodule of M is that $\acute{N}$ is an $\acute{S}$-second submodule of $\acute{M}$ and $\mathrm{Ann}_{R_n}(N_n) \cap S_n \neq \varnothing$ or N_n is an S_n-second submodule of M_n and $\mathrm{Ann}_{\acute{R}}(\acute{N}) \cap \acute{S} \neq \varnothing$, where $\acute{R} = R_1 \times R_2 \times \cdots \times R_{n-1}$. Now the result follows from the induction hypothesis. □

Lemma 2.13. Let S be a m.c.s. of R and N be an S-second submodule of an R-module M. Then the following statements hold for some $s \in S$.

(a) $sN \subseteq \acute{s}N$ for all $\acute{s} \in S$.

(b) $(\mathrm{Ann}_R(N) :_R \acute{s}) \subseteq (\mathrm{Ann}_R(N) :_R s)$ for all $\acute{s} \in S$.

Proof. (a) Let N be an S-second submodule of M. Then there is an $s \in S$ such that $rN \subseteq K$ for each $r \in R$ and a submodule K of M implies that $sN \subseteq K$ or $srN = 0$. Let L be a completely irreducible submodule of M such that $\acute{s}N \subseteq L$. Then $sN \subseteq L$ or $\acute{s}sN = 0$. As $\mathrm{Ann}_R(N) \cap S = \varnothing$, we get that $sN \subseteq L$. Thus $sN \subseteq \acute{s}N$ by Remark 2.1.

(b) This follows from Proposition 2.9 (a) and [14, Lemma 2.16 (ii)]. □

Proposition 2.14. Let S be a m.c.s. of R and N be a finitely generated submodule of M such that $\mathrm{Ann}_R(N) \cap S = \varnothing$. Then the following are equivalent:

(a) N is an S-second submodule of M;

(b) $S^{-1}N$ is a second submodule of $S^{-1}M$ and there is an $s \in S$ satisfying $sN \subseteq \acute{s}N$ for all $\acute{s} \in S$.

Proof. $(a) \Rightarrow (b)$ This follows from Proposition 2.5 (d) and Lemma 2.13.

$(b) \Rightarrow (a)$ Let $aN \subseteq K$ for some $a \in R$ and a submodule K of M. Then $(a/1)(S^{-1}N) \subseteq S^{-1}K$. Thus by part (b), $S^{-1}N \subseteq S^{-1}K$ or $(a/1)(S^{-1}N) = 0$. Hence by Lemma 2.4, $s_1N \subseteq K$ or $s_2aN = 0$ for some $s_1, s_2 \in S$. By part (b), there is an $s \in S$ such that $sN \subseteq s_1N$ and $sN \subseteq s_2N \subseteq (0 :_M a)$. Therefore, $sN \subseteq K$ or $asN = 0$, as desired. □

Theorem 2.15. *Let S be a m.c.s. of R and N be a submodule of an R-module M such that $\mathrm{Ann}_R(N) \cap S = \varnothing$. Then N is an S-second submodule of M if and only if sN is a second submodule of M for some $s \in S$.*

Proof. Let sN be a second submodule of M for some $s \in S$. Let $aN \subseteq K$ for some $a \in R$ and a submodule K of M. As $asN \subseteq K$ and sN is a second submodule of M, we get that $sN \subseteq K$ or $asN = 0$, as needed. Conversely, assume that N is an S-second submodule of M. Then there is an $s \in S$

such that if $aN \subseteq K$ for some $a \in R$ and a submodule K of M, then $sN \subseteq K$ or $saN = 0$. Now we show that sN is a second submodule of M. Let $a \in R$. As $asN \subseteq asN$, by assumption, $sN \subseteq asN$ or $as^2N = 0$. If $sN \subseteq asN$, then there is nothing to show. Assume that $sN \not\subseteq asN$. Then $as^2N = 0$ and so $a \in (\mathrm{Ann}_R(N) :_R s^2) \subseteq (\mathrm{Ann}_R(N) :_R s)$ by Lemma 2.13. Thus, we can conclude that $asN = 0$, as desired. □

The set of all maximal ideals of R is denoted by $Max(R)$.

Theorem 2.16. *Let S be a m.c.s. of R and N be a submodule of an R-module M such that $\mathrm{Ann}_R(N) \subseteq Jac(R)$, where $Jac(R)$ is the Jacobson radical of R. Then the following statements are equivalent:*

(a) *N is a second submodule of M;*

(b) *$\mathrm{Ann}_R(N)$ is a prime ideal of R and N is an $(R \setminus \mathfrak{M})$-second submodule of M for each $\mathfrak{M} \in Max(R)$.*

Proof. $(a) \Rightarrow (b)$ Let N be a second submodule of M. Clearly, $\mathrm{Ann}_R(N)$ is a prime ideal of R. Since $\mathrm{Ann}_R(N) \subseteq Jac(R)$, $\mathrm{Ann}_R(N) \subseteq \mathfrak{M}$ for each $\mathfrak{M} \in Max(R)$ and so $\mathrm{Ann}_R(N) \cap (R \setminus \mathfrak{M}) = \varnothing$. Now the result follows from Proposition 2.5 (a).

$(b) \Rightarrow (a)$ Let $\mathrm{Ann}_R(N)$ be a prime ideal of R and N be an $(R \setminus \mathfrak{M})$-second submodule of M for each $\mathfrak{M} \in Max(R)$. Let $a \in R$ and $a \notin \mathrm{Ann}_R(N)$. We show that $aN = N$. Let $\mathfrak{M} \in Max(R)$. Then as $aN \subseteq aN$, there exists an $s_{\mathfrak{M}} \in R \setminus \mathfrak{M}$ such that $s_{\mathfrak{M}}N \subseteq aN$ or $s_{\mathfrak{M}}aN = 0$. As $\mathrm{Ann}_R(N)$ is a prime ideal of R and $s_{\mathfrak{M}} \notin \mathrm{Ann}_R(N)$, we have $as_{\mathfrak{M}} \notin \mathrm{Ann}_R(N)$ and so $s_{\mathfrak{M}}N \subseteq aN$. Now consider the set

$$\Omega = \{s_{\mathfrak{M}} : \exists\, \mathfrak{M} \in Max(R), s_{\mathfrak{M}} \notin \mathfrak{M} \text{ and } s_{\mathfrak{M}}N \subseteq aN\}.$$

Then we claim that $\Omega = R$. To see this, take any maximal ideal $\acute{\mathfrak{M}}$ containing Ω. Then the definition of Ω requires that there exists an $s_{\acute{\mathfrak{M}}} \in \Omega$ and $s_{\acute{\mathfrak{M}}} \notin \acute{\mathfrak{M}}$. As $\Omega \subseteq \acute{\mathfrak{M}}$, we have $s_{\acute{\mathfrak{M}}} \in \Omega \subseteq \acute{\mathfrak{M}}$, a contradiction. Thus, $\Omega = R$ and this yields

$$1 = r_1 s_{\mathfrak{M}_1} + r_2 s_{\mathfrak{M}_2} + \cdots + r_n s_{\mathfrak{M}_n}$$

for some $r_i \in R$ and $s_{\mathfrak{M}_i} \in R \setminus \mathfrak{M}_i$ with $s_{\mathfrak{M}_i}N \subseteq aN$, where $\mathfrak{M}_i \in Max(R)$ for each $i = 1, 2, \ldots, n$. This yields that

$$N = (r_1 s_{\mathfrak{M}_1} + r_2 s_{\mathfrak{M}_2} + \cdots + r_n s_{\mathfrak{M}_n})N \subseteq aN.$$

Therefore, $N \subseteq aN$ as needed. □

Now we determine all second submodules of a module over a quasilocal ring in terms of S-second submodules.

Corollary 2.17. Let S be a m.c.s. of a quasilocal ring $(R, \mathfrak{M})$ and N be a submodule of an R-module M. Then the following statements are equivalent:

(a) N is a second submodule of M;

(b) $\mathrm{Ann}_R(N)$ is a prime ideal of R and N is an $(R\backslash\mathfrak{M})$-second submodule of M.

Proof. This follows from Theorem 2.16. □

Proposition 2.18. Let S be a m.c.s. of R and $f : M \to \acute{M}$ be a monomorphism of R-modules. Then we have the following.

(a) If N is an S-second submodule of M, then $f(N)$ is an S-second submodule of $\acute{M}$.

(b) If $\acute{N}$ is an S-second submodule of $\acute{M}$ and $\acute{N} \subseteq f(M)$, then $f^{-1}(\acute{N})$ is an S-second submodule of M.

Proof. (a) As $\mathrm{Ann}_R(N) \cap S = \varnothing$ and f is a monomorphism, we have $\mathrm{Ann}_R(f(N)) \cap S = \varnothing$. Let $r \in R$. Since N is an S-second submodule of M, there exists an $s \in S$ such that $srN = sN$ or $srN = 0$. Thus $srf(N) = sf(N)$ or $srf(N) = 0$, as needed.

(b) $\mathrm{Ann}_R(\acute{N}) \cap S = \varnothing$ implies that $\mathrm{Ann}_R(f^{-1}(\acute{N})) \cap S = \varnothing$. Now let $r \in R$. As $\acute{N}$ is an S-second submodule of M, there exists an $s \in S$ such that $sr\acute{N} = s\acute{N}$ or $sr\acute{N} = 0$. Therefore $srf^{-1}(\acute{N}) = sf^{-1}(\acute{N})$ or $srf^{-1}(\acute{N}) = 0$, as requested. □

Proposition 2.19. Let S be a m.c.s. of R, M a comultiplication R-module, and let N be an S-second submodule of M. Suppose that $N \subseteq K + H$ for some submodules K, H of M. Then $sN \subseteq K$ or $sN \subseteq H$ for some $s \in S$.

Proof. As $N \subseteq K + H$, we have $\mathrm{Ann}_R(K)\,\mathrm{Ann}_R(H) \subseteq \mathrm{Ann}_R(N)$. This implies that there exists an $s \in S$ such that $sAnn_R(K) \subseteq \mathrm{Ann}_R(N)$ or $sAnn_R(H) \subseteq \mathrm{Ann}_R(N)$ since by Proposition 2.9, $\mathrm{Ann}_R(N)$ is an S-prime ideal of R. Therefore, $\mathrm{Ann}_R(K) \subseteq \mathrm{Ann}_R(sN)$ or $\mathrm{Ann}_R(H) \subseteq \mathrm{Ann}_R(sN)$. Now as M is a comultiplication R-module, we have $sN \subseteq K$ or $sN \subseteq H$ as needed. □

Let M be an R-module. The idealization $R(+)M = \{(a, m) : a \in R, m \in M\}$ of M is a commutative ring whose addition is componentwise and whose multiplication is defined as $(a, m)(b, \acute{m}) = (ab, a\acute{m} + bm)$ for each $a, b \in R$, $m, \acute{m} \in M$ [13]. If S is a m.c.s. of R and N is a submodule of M, then $S(+)N = \{(s, n) : s \in S, n \in N\}$ is a m.c.s. of $R(+)M$ [1].

Proposition 2.20. Let M be an R-module and let I be an ideal of R such that $I \subseteq \mathrm{Ann}_R(M)$. Then the following are equivalent:

(a) I is a second ideal of R;

(b) $I(+)0$ is a second ideal of $R(+)M$.

Proof. This is straightforward. □

Theorem 2.21. *Let S be a m.c.s. of R, M be an R-module, and I be an ideal of R such that $I \subseteq \mathrm{Ann}_R(M)$ and $I \cap S = \varnothing$. Then the following are equivalent:*

(a) *I is an S-second ideal of R;*

(b) *$I(+)0$ is an $S(+)0$-second ideal of $R(+)M$;*

(c) *$I(+)0$ is an $S(+)M$-second ideal of $R(+)M$.*

Proof. $(a) \Rightarrow (b)$ Let $(r, m) \in R(+)M$. As I is an S-second ideal of R, there exists an $s \in S$ such that $rsI = sI$ or $rsI = 0$. If $rsI = 0$, then $(r, m)(s, 0)(I(+)0) = 0$. If $rsI = sI$, then we claim that $(r, m)(s, 0)(I(+)0) = (s, 0)(I(+)0)$. To see this let $(sa, 0) = (s, 0)(a, 0) \in (s, 0)(I(+)0)$. As $rsI = sI$, we have $sa = rsb$ for some $b \in I$. Thus as $b \in I \subseteq \mathrm{Ann}_R(M)$,

$$(sa, 0) = (srb, 0) = (srb, smb) = (sr, sm)(b, 0) = (s, 0)(r, m)(b, 0).$$

Hence $(s, 0)(a, 0) \in (r, m)(s, 0)(I(+)0)$, and so

$$(s, 0)(I(+)0) \subseteq (r, m)(s, 0)(I(+)0).$$

Since the inverse inclusion is clear we reach the claim.

$(b) \Rightarrow (c)$ Since $S(+)0 \subseteq S(+)M$, the result follows from Proposition 2.5(b).

$(c) \Rightarrow (a)$ Let $r \in R$. As $I(+)0$ is an $S(+)M$-second ideal of $R(+)M$, there exists an $(s, m) \in S(+)M$ such that $(r, 0)(s, m)(I(+)0) = (s, m)(I(+)0)$ or $(r, 0)(s, m)(I(+)0) = 0$. If $(r, 0)(s, m)(I(+)0) = 0$, then

$$0 = (r, 0)(s, m)(a, 0) = (rs, rm)(a, 0) = (rsa, rma) = (rsa, 0)$$

far each $a \in I$. Thus $rsI = 0$. If $(r, 0)(s, m)(I(+)0) = (s, m)(I(+)0)$, then we claim that $rsI = sI$. To see this let $sa \in sI$. Then for some $b \in I$, we have

$$\begin{aligned}(sa, 0) &= (sa, am) = (s, m)(a, 0) = (s, m)(r, 0)(b, 0) \\ &= (srb, rmb) = (srb, 0).\end{aligned}$$

Hence, $sa \in rsI$ and so $sI \subseteq srI$, as needed. □

Let P be a prime ideal of R and N be a submodule of an R-module M. *The P-interior of N relative to M* is defined (see [3, 2.7]) as the set

$$I_P^M(N) = \bigcap\{L \mid L \text{ is a completely irreducible submodule of } M \text{ and } rN \subseteq L \text{ for some } r \in R - P\}.$$

Let R be an integral domain. A submodule N of an R-module M is said to be a *cotorsion-free submodule of M* (the dual of torsion-free) if $I_0^M(N) = N$ and is a *cotorsion submodule of M* (the dual of torsion) if $I_0^M(N) = 0$. Also, M said to be *cotorsion* (resp. *cotorsion-free*) if M is a cotorsion (resp. cotorsion-free) submodule of M [5].

One can see that if M is a cotorsion-free R-module, then R is an integral domain and M is a faithful R-module. In [5, Proposition 2.9 (e)], it is shown that if M is a comultiplication R-module the reverse is true. The following example shows that sometimes the reverse of this statement may not be true.

Example 2.22. Consider the $\mathbb{Z}$-module $M = \prod_{i=1}^{\infty} \mathbb{Z}_{p^i}$, where p is a prime number. Then it is easy to see that M is a faithful $\mathbb{Z}$-module. But the $\mathbb{Z}$-module M is not second since $(\bar{1}, \bar{0}, \bar{0}, \dots) \notin pM$ and so $M \neq pM$. Therefore, by [5, Theorem 2.10], $I_0^M(M) \neq M$ and so the $\mathbb{Z}$-module M is not a cotorsion-free module.

Definition 2.23. Let M be an R-module and S be a m.c.s. of R with $\mathrm{Ann}_R(M) \cap S = \varnothing$. We say that M is an *S-cotorsion-free module* in the case that we can find $s \in S$ such that if $rM \subseteq L$, where $r \in R$ and L is a completely irreducible submodule of M, then $sM \subseteq L$ or $rs = 0$.

Proposition 2.24. Let M be an R-module and S be a m.c.s. of R. Then the following statements are equivalent.

(a) M is an S-second R-module.

(c) $P = \mathrm{Ann}_R(M)$ is an S-prime ideal of R and the R/P-module M is an S-cotorsion-free module.

Proof. $(a) \Rightarrow (b)$. We can assume that $P = 0$. By Proposition 2.9 (a), $\mathrm{Ann}_R(M)$ is an S-prime ideal of R. Now let L be a completely irreducible submodule of M and $r \in R$ such that $rM \subseteq L$. Then there exists an $s \in S$ such that $sM \subseteq L$ or $srM = 0$ because M is S-second. Therefore, $sM \subseteq L$ or $rs \in \mathrm{Ann}_R(M) = 0$, as needed.

$(b) \Rightarrow (a)$. As $\mathrm{Ann}_R(M)$ is an S-prime ideal of R, $\mathrm{Ann}_R(M) \cap S = \varnothing$. Suppose that there exist $r \in R$ and completely irreducible submodule L of M such that $rM \subseteq L$. By assumption, there is an $s \in S$ such that $sM \subseteq L$ or $rs = 0_{R/P}$. Thus $sM \subseteq L$ or $rs \in P = \mathrm{Ann}_R(M)$, as desired. □

Theorem 2.25. *Let M be a module over an integral domain R. Then the following are equivalent:*
(a) *M is a cotorsion-free R-module;*
(b) *M is an $(R \setminus P)$-cotorsion-free for each prime ideal P of R;*
(c) *M is an $(R \setminus \mathfrak{M})$-cotorsion-free for each maximal ideal $\mathfrak{M}$ of R.*

Proof. $(a) \Rightarrow (b)$ This is clear.

$(b) \Rightarrow (c)$ This is obvious.

$(c) \Rightarrow (a)$ Let M be $(R \setminus \mathfrak{M})$-cotorsion-free for each maximal ideal $\mathfrak{M}$ of R. Let $aM \subseteq L$ for some $a \in R$ and a completely irreducible submodule L of M. Assume that $a \neq 0$. Take a maximal ideal $\mathfrak{M}$ of R. As M is $(R \setminus \mathfrak{M})$-cotorsion-free, there exists an $s_{\mathfrak{M}} \in R \setminus \mathfrak{M}$ such that $s_{\mathfrak{M}}M \subseteq L$ or $as_{\mathfrak{M}} = 0$. As R is an integral domain, $as_{\mathfrak{M}} \neq 0$ and so $s_{\mathfrak{M}}M \subseteq L$. Now set

$$\Omega = \{s_{\mathfrak{M}} : \exists\, \mathfrak{M} \in Max(R), s_{\mathfrak{M}} \notin \mathfrak{M} \; and \; s_{\mathfrak{M}}M \subseteq L\}.$$

A similar argument as in the proof of Theorem 2.16 shows that $\Omega = R$. Thus we have $\langle s_{\mathfrak{M}_1} \rangle + \langle s_{\mathfrak{M}_2} \rangle + \cdots + \langle s_{\mathfrak{M}_n} \rangle = R$ for some $s_{\mathfrak{M}_i} \in \Omega$. This implies that $M = (\langle s_{\mathfrak{M}_1} \rangle + \langle s_{\mathfrak{M}_2} \rangle + \cdots + \langle s_{\mathfrak{M}_n} \rangle)M \subseteq L$ and hence $M = L$. This means that M is a cotorsion-free R-module. □

Let M be an R-module. The dual notion of $Z_R(M)$, the set of zero divisors of M, is denoted by $W_R(M)$ and defined by

$$W(M) = \{a \in R : aM \neq M\}.$$

Theorem 2.26. *Let S be a m.c.s. of R and M be a finitely generated comultiplication R-module with $\mathrm{Ann}_R(M) \cap S = \varnothing$. Then the following statements are equivalent:*
(a) *Each non-zero submodule of M is S-second;*
(b) *M is a simple R-module.*

Proof. $(a) \Rightarrow (b)$ Assume that every non-zero submodule of M is an S-second submodule of M. First we show that $W_R(M) = \mathrm{Ann}_R(M)$. Let $a \in W_R(M)$. Then $aM \neq M$. Since M is S-second, there exists an $s \in S$ such that $saM = sM$ or $saM = 0$. If $saM = sM$, then $s \in (saM :_R M)$. Now put $N = (0 :_M (saM :_R M))$ and note that $s \in S \cap \mathrm{Ann}_R(N) \neq \varnothing$. Thus, N is not S-second and so by part (a), we have $N = (0 :_M (saM :_R M)) = 0$. Now as M is a comultiplication R-module, one can see that $M(saM :_R M) = M$. By [7, Corollary 2.5], $1 - x \in \mathrm{Ann}_R(M) \subseteq (saM :_R M)$ for some $x \in (saM :_R M)$ since M is a finitely generated R-module. This implies that $(saM :_R M) = R$

and so $saM = M$. It follows that $aM = M$, which is a contradiction. Therefore, $saM = 0$. Then $s \in \mathrm{Ann}_R(aM)$ and so $S \cap \mathrm{Ann}_R(aM) \neq \varnothing$. Hence by assumption, $aM = 0$. Thus, we get $W_R(M) = \mathrm{Ann}_R(M)$. Let $a \notin W_R(M)$. Now we will show that $(0 :_M a) = 0$. If $(0 :_M a^2) = 0$, then $(0 :_M a) = 0$. Suppose $(0 :_M a^2) \neq 0$. Since $(0 :_M a^2)$ is an S-second submodule of M and $a(0 :_M a^2) \subseteq (0 :_M a)$, there is an $s \in S$ such that $sa(0 :_M a^2) = 0$. This implies that $(0 :_M a^2) \subseteq (0 :_M as)$. Let $m_1 \in (0 :_M a)$. Then $am_1 = 0$. We have $m_1 = am_2$ since $aM = M$. Thus $am_1 = a^2m_2 = 0$. Hence $m_2 \in (0 :_M a^2) \subseteq (0 :_M as)$. This implies that $0 = sam_2 = sm_1$ and so $m_1 \in (0 :_M s)$. Therefore, $(0 :_M a) \subseteq (0 :_M s)$ and so $s \in \mathrm{Ann}_R((0 :_M a))$. Hence $S \cap \mathrm{Ann}_R((0 :_M a)) \neq \varnothing$. So by assumption, we have $(0 :_M a) = 0$. Now take a submodule H of M. If $\mathrm{Ann}_R(H) = \mathrm{Ann}_R(M)$, then $H = M$ since M is a comultiplication R-module. Take an element $a \in \mathrm{Ann}_R(H) \setminus \mathrm{Ann}_R(M)$. As $W_R(M) = \mathrm{Ann}_R(M)$, $a \notin W_R(M)$ and so $(0 :_M a) = 0$. Then we get

$$H = (0 :_M \mathrm{Ann}_R(H)) \subseteq (0 :_M a) = 0.$$

Therefore, M is a simple R-module.

$(b) \Rightarrow (a)$ Note that every simple R-module M is a second submodule of M. Since $\mathrm{Ann}_R(M) \cap S = \varnothing$, by Proposition 2.5 (a), M is an S-second submodule of M. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

Corollary 2.27. Let S be a m.c.s. of R. If M is a finitely generated multiplication and comultiplication R-module with $\mathrm{Ann}_R(M) \cap S = \varnothing$, then the following statements are equivalent:

(a) Each non-zero submodule of M is S-second;
(b) M is a simple R-module;
(c) Each proper submodule of M is an S-prime submodule of M.

Proof. This follows from Theorem 2.26 and [14, Theorem 2.26]. □

Example 2.28. Consider the $\mathbb{Z}$-module $\mathbb{Z}_n$. Take $S = \mathbb{Z} - 0$. We know that $\mathbb{Z}_n$ is a finitely generated multiplication and comultiplication $\mathbb{Z}$-module. Then by Corollary 2.27, if n is not a prime number, the $\mathbb{Z}$-module $\mathbb{Z}_n$ has a non-zero submodule which is not S-second and a proper submodule which is not S-prime.

References

[1] Anderson D.D., Winders M.,*Idealization of a module*, Journal of Commutative Algebra **1** (1) 2009, 3-56.

[2] Ansari-Toroghy H., Farshadifar F., *The dual notion of multiplication modules*, Taiwanese J. Math., **11** (4), (2007), 1189–1201.

[3] Ansari-Toroghy H., Farshadifar F., *On the dual notion of prime submodules*, Algebra Colloq., **19** (Spec 1), (2012), 1109-1116.

[4] Ansari-Toroghy H., Farshadifar F., *The dual notion of some generalizations of prime submodules*, Comm. Algebra, **39** (2011), 2396-2416.

[5] Ansari-Toroghy H., Farshadifar F., *On the dual notion of prime submodules (II)*, Mediterr. J. Math., **9** (2) (2012), 329-338.

[6] Ansari-Toroghy H., Farshadifar F., *Strong comultiplication modules*, CMU. J. Nat. Sci. **8** (1) (2009), 105–113.

[7] Atiyah M.F., Macdonald I.G., *Introduction to commutative algebra*, Addison-Wesley, 1969.

[8] A. Barnard, *Multiplication modules*, J. Algebra, **71** (1981), 174–178.

[9] Dauns J., *Prime submodules*, J. Reine Angew. Math., **298**, (1978), 156–181.

[10] Faith C., *Rings whose modules have maximal submodules*, Publ. Mat. **39** (1995), 201-214.

[11] Fuchs L., Heinzer W., Olberding B., *Commutative ideal theory without finiteness conditions: Irreducibility in the quotient filed*, in : Abelian Groups, Rings, Modules, and Homological Algebra, Lect. Notes Pure Appl. Math., **249**, (2006), 121–145.

[12] Gilmer R., *Multiplicative Ideal Theory*, Queen's Papers in Pure and Applied Mathematics, No. 90. Kingston, Canada: Queen's University, 1992.

[13] Nagata M., *Local Rings*, New York, NY, USA: Interscience, 1962.

[14] Sevim E.S., Arabaci T., Tekir *Ü*., Koc S., *On S-prime submodules*, Turkish Journal of Mathematics, **43** (2) (2019), 1036-1046.

[15] Wang F., Kim H., *Foundations of Commutative Rings and Their Modules*, Singapore: Springer, 2016.

[16] Yassemi S., *The dual notion of prime submodules*, Arch. Math. (Brno), **37**, (2001), 273–278.

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