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RESEARCH ARTICLE

Diagonal torsion matrices associated with modular data

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Abstract. Modular data are commonly studied in mathematics and physics. A modular datum defines a finite-dimensional representation of the modular group $SL_2(\mathbb{Z})$. Cuntz (2007) defined isomorphic integral modular data. Here we discuss isomorphic integral and non-integral modular data as well as non-isomorphic but closely related modular data. In this paper, we give some insights into diagonal torsion matrices associated to modular data.

Introduction

Diagonal torsion matrices are a fundamental ingredient of modular data. Modular data is a basic component of rational conformal field theory. Further, rational conformal field theory has important applications in physics. In particular, it has nice applications to string theory, statistical mechanics, and condensed matter physics, see [4] and [7]. Modular data give rise to fusion rings, C-algebras and C^*-algebras, see [2], [3], [5] and [6]. These rings and algebras are interesting topics of study in their own right.

A modular datum is a pair (S, T), where S is a Fourier matrix and T is a diagonal torsion matrix that satisfy certain properties. In this article, we investigate diagonal torsion matrices associated with integral modular data and non-integral modular data as well as isomorphic modular data and non-isomorphic but closely related modular data in certain cases. This

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article is inspired from Cuntz's example of non-isomorphic integral Fourier matrices that are related to the character table of an elementary abelian group of order 4, see [2, pp. 365].

In section 2, we collect the definitions and notations. In section 3, we show how to find the diagonal torsion matrices associated to Fourier matrices from the diagonal torsion matrices that either form isomorphic modular data or closely related modular data. We show how to find the whole chain of modular data from just one modular datum of that chain. Also, we determine the number of diagonal torsion matrices associated to certain type of modular data.

1. Preliminaries

To keep the generality, in the following definition of Modular data we assume the structure constants to be integers instead of nonnegative integers, see [2].

Definition 1. Let $r \in \mathbb{Z}^+$ and I an $r \times r$ identity matrix. A pair (S, T) of $r \times r$ complex matrices is called modular datum if

(1) S is a unitary and symmetric matrix, that is, $S\bar{S}^T = 1, S = S^T$;
(2) T is diagonal matrix and of finite multiplicative order;
(3) $S_{i0} > 0$, for $0 \leqslant i \leqslant r-1$, where S is indexed by $\{0, 1, 2, \ldots, r-1\}$;
(4) $(ST)^3 = S^2$;
(5) $N_{ijk} = \sum_l S_{li} S_{lj} \bar{S}_{lk} S_{l0}^{-1} \in \mathbb{Z},$ for all $0 \leqslant i, j, k \leqslant r-1$.

Definition 2. A matrix S satisfying the axioms $(i), (iii)$ and (v) of Definition 1 is called a *Fourier matrix*. A matrix T satisfying the axioms (ii) and (iv) of Definition 1 is called a diagonal torsion matrix.

Let S be a Fourier matrix. Let $s = [s_{ij}]$ be the matrix with entries $s_{ij} = S_{ij}/S_{i0}$, for all i, j, and we call it an *s-matrix associated to S* (briefly, s-matrix). If an s-matrix has integral entries then the s-matrix is called integral Fourier matrix and the pair (s, T) is called an integral modular datum, see [2, Definition 3.1]. A Fourier matrix S is called a homogeneous Fourier matrix if all the entries of first row of its associated s-matrix are equal to 1, otherwise, S is called a non-homogenous Fourier matrix, see [5] and [6]. Let (S, T) and (S_0, T_0) be two modular data. Then (S, T) is called isomorphic to (S_0, T_0) if there exists a permutation matrix P with $P_{00} = 1$ such that $P^t S P = S_0$ and $P^t T P = T_0$, otherwise (S, T) and (S_0, T_0) are called non-isomorphic modular data, also see the definition of isomorphic integral modular data [2, Definfition 3.1]. Throughout the paper, M^t denotes the transposed matrix of a matrix M, and ζ_k denotes the kth primitive root of unity.

2. Diagonal torsion matrices associated to modular data

In this section, we establish some criteria to find the diagonal torsion matrices associated to certain type of a Fourier matrix whose rows and/or columns permutations result in a Fourier matrix that has an associated diagonal torsion matrix. We show that the permutations of diagonal entries of diagonal torsion matrix associated to a Fourier matrix result again a diagonal torsion matrix associated to same Fourier matrix under some conditions. Also, we determine number of diagonal torsion matrices associated to certain type of Fourier matrices.

The matrix consists of the entries of the character table of a finite dimensional group algebra is called the first eigenmatrices of the group algebra. So, we use the terms character table of a group and the first eigenmatrix of the group algebra interchangeably. Bannai and Bannai classify the diagonal torsion matrices corresponding to Fourier matrices whose associated s-matrices are the first eigenmatrices of group algebras of finite cyclic groups, see [1, Theorem 1]. The tensor product of two modular data is a modular datum, [2]. Let S be a Fourier matrix whose associated s-matrix is a character table of an abelian group. Note that, character table of an abelain group can be written as a tensor product of character tables of cyclic groups. Therefore, the tensor products of the diagonal torsion matrices corresponding to tensor factors of S are diagonal torsion matrices corresponding to the Fourier matrix S. Our results enable us to find the diagonal torsion matrices associated to Fourier matrices of any type, not only integral homogeneous Fourier matrices that are tensor product of Fourier matrices of smaller rank.

Let (S, T) be a modular datum. Since $(S\zeta_3 T)^3 = (ST)^3 = S^2$, $(S, \zeta_3 T)$ is also a modular datum. Hence, in a modular datum, corresponding to a Fourier matrix there must be at least three different corresponding diagonal torsion matrices. For the remaining section we investigate the properties of Fourier matrices and permutation matrices to establish the results that help to find the additional diagonal torsion matrices from a given diagonal torsion matrix T in a modular datum (S, T). In particular, the following results are useful when an s-matrix is not a tensor product of the s-matrices of lower rank but can be obtained from such a matrix by permuting its rows and/or columns.

In the following theorem we see if a permutation matrix commute with a Fourier matrix then conjugation of the associated diagonal torsion matrix with the permutation matrix results in diagonal torsion matrix associated with the given Fourier matrix. Also, we see that simultaneous permutation of rows and columns of a Fourier matrix and associated diagonal matrix result in modular datum provided the first row and column are fixed. Note that, we don't require s-matrix to be an integral Fourier matrix.

Theorem 1. *Let (S, T) be a modular datum. Let P be a permutation matrix.*

(1) *If $SP = PS$ then (S, P^tTP) is a modular datum.*
(2) *If $P_{00} = 1$ then (P^tSP, P^tTP) is a modular datum.*

Proof. (1) (S, T) is a modular datum, therefore $(ST)^3 = S^2$. Since $SP = PS$, $(S(P^tTP))^3 = P^t(ST)^3P = P^tS^2P = S^2$. Note that, P^tTP is a diagonal torsion matrix of multiplicative order equal to the multiplicative order of T. Hence (S, P^tTP) is a modular datum.

(2) The simultaneous row and column permutation of a unitary and symmetric matrix results in a unitary and symmetric matrix. Thus P^tSP is a unitary and symmetric matrix. The assumption $P_{00} = 1$ assures that the entries of the first row and column of P^tSP matrix are positive real numbers. Note that, the set of structure constants generated by the columns of P^tSP matrix under entrywise multiplication is same as the set of structure constants generated by the columns of S matrix under entrywise multiplication. Therefore, P^tSP is a Fourier matrix.

Since (S, T) is a modular datum, $(ST)^3 = S^2$. The inverse of a permutation matrix is its transpose, therefore $((P^tSP)(P^tTP))^3 = P^t(ST)^3P$ $= P^tS^2P = P^tSPP^tSP = (P^tSP)^2$. Also, P^tTP is a diagonal matrix with finite multiplicative order. Thus (P^tSP, P^tTP) is a modular datum. □

Note that, if S is a homogeneous Fourier matrix of rank r then $s = \sqrt{r}S$. Therefore, s-matrix is a symmetric matrix. Let P be a permutation matrix of rank r. Then SP (PS, respectively) is a symmetric matrix if and only if sP (Ps, respectively) is a symmetric matrix. Also, $SP = PS$ if and only if $sP = Ps$. In the following example, we apply Theorem 1 and Proposition 1 to demonstrate how to relate the diagonal torsion matrices of two non-isomorphic modular data.

Example 1. Consider the Fourier matrix $S = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Let $T = \operatorname{diag}(x, y)$ be the corresponding diagonal matrix, where $x, y \in \mathbb{C}$ and have finite multiplicative order. Since $(ST)^3 = S^2$, we have

$$\begin{aligned} T \quad \in \quad & \{\mathrm{diag}(\zeta_{24}^{7}, \zeta_{24}^{13}), \mathrm{diag}(\zeta_{24}^{15}, \zeta_{24}^{21}), \mathrm{diag}(\zeta_{24}^{23}, \zeta_{24}^{5}), \\ & \mathrm{diag}(\zeta_{24}, \zeta_{24}^{19}), \mathrm{diag}(\zeta_{24}^{9}, \zeta_{24}^{3}), \mathrm{diag}(\zeta_{24}^{17}, \zeta_{24}^{11})\}. \end{aligned}$$

The $\tilde{s}$-matrix, the character table of an elementary abelian group of order 4, is obtained from the s-matrix as follows.

$$\tilde{s} = s \otimes s = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The tensor product of two modular data is a modular datum, see [2]. Thus, corresponding Fourier matrix $\tilde{S}$ has associated diagonal torsion matrices $\tilde{T}$ in the set below obtained from the tensor product of T matrices from the above set. Therefore,

$$\begin{aligned} \tilde{T} \in \{ & \mathrm{diag}(\zeta_{24}^{14}, \zeta_{24}^{20}, \zeta_{24}^{20}, \zeta_{24}^{2}), \mathrm{diag}(\zeta_{24}^{22}, \zeta_{24}^{4}, \zeta_{24}^{4}, \zeta_{24}^{10}), \mathrm{diag}(\zeta_{24}^{6}, \zeta_{24}^{12}, \zeta_{24}^{12}, \zeta_{24}^{18}), \\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{2}, \zeta_{24}^{14}, \zeta_{24}^{8}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{10}, \zeta_{24}^{22}, \zeta_{24}^{16}), \mathrm{diag}(1, \zeta_{24}^{18}, \zeta_{24}^{6}, 1), \\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{14}, \zeta_{24}^{2}, \zeta_{24}^{8}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{22}, \zeta_{24}^{10}, \zeta_{24}^{16}), \mathrm{diag}(1, \zeta_{24}^{6}, \zeta_{24}^{18}, 1), \\ & \mathrm{diag}(\zeta_{24}^{2}, \zeta_{24}^{20}, \zeta_{24}^{20}, \zeta_{24}^{14}), \mathrm{diag}(\zeta_{24}^{10}, \zeta_{24}^{4}, \zeta_{24}^{4}, \zeta_{24}^{22}), \\ & \mathrm{diag}(\zeta_{24}^{18}, \zeta_{24}^{12}, \zeta_{24}^{12}, \zeta_{24}^{6})\}. \end{aligned}$$

Let $P_{(ijk)}$ ($P_{(ij)}$, respectively) be a permutation matrix that permutes rows i, j, k (i and j, respectively) of a matrix on left multiplication to it. Therefore, $P_{(243)}\tilde{s} = \tilde{s}P_{(243)}$. Thus, by Theorem 1(1), we obtain the following associated diagonal torsion matrices T' from $\tilde{T}$.

$$\begin{aligned} T' \in \{ & \mathrm{diag}(\zeta_{24}^{14}, \zeta_{24}^{2}, \zeta_{24}^{20}, \zeta_{24}^{20}), \mathrm{diag}(\zeta_{24}^{22}, \zeta_{24}^{10}, \zeta_{24}^{4}, \zeta_{24}^{4}), \mathrm{diag}(\zeta_{24}^{6}, \zeta_{24}^{18}, \zeta_{24}^{12}, \zeta_{24}^{12}), \\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{8}, \zeta_{24}^{2}, \zeta_{24}^{14}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{16}, \zeta_{24}^{10}, \zeta_{24}^{22}), \mathrm{diag}(1, 1, \zeta_{24}^{6}, \zeta_{24}^{18}), \\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{8}, \zeta_{24}^{14}, \zeta_{24}^{2}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{16}, \zeta_{24}^{22}, \zeta_{24}^{10}), \mathrm{diag}(1, 1, \zeta_{24}^{18}, \zeta_{24}^{6}), \\ & \mathrm{diag}(\zeta_{24}^{2}, \zeta_{24}^{14}, \zeta_{24}^{20}, \zeta_{24}^{20}), \mathrm{diag}(\zeta_{24}^{10}, \zeta_{24}^{22}, \zeta_{24}^{4}, \zeta_{24}^{4}), \\ & \mathrm{diag}(\zeta_{24}^{18}, \zeta_{24}^{6}, \zeta_{24}^{12}, \zeta_{24}^{12})\}. \end{aligned}$$

Also, $P_{(234)}\tilde{s} = \tilde{s}P_{(234)}$. Therefore, by Theorem 1(1), we obtain the following associated diagonal torsion matrices T_1' from T'.

$$\begin{aligned} T_1' \in \big\{ & \mathrm{diag}(\zeta_{24}^{14}, \zeta_{24}^{20}, \zeta_{24}^{2}, \zeta_{24}^{20}), \mathrm{diag}(\zeta_{24}^{22}, \zeta_{24}^{4}, \zeta_{24}^{10}, \zeta_{24}^{4}), \mathrm{diag}(\zeta_{24}^{6}, \zeta_{24}^{12}, \zeta_{24}^{18}, \zeta_{24}^{12}),\\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{14}, \zeta_{24}^{8}, \zeta_{24}^{2}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{22}, \zeta_{24}^{16}, \zeta_{24}^{10}), \mathrm{diag}(1, \zeta_{24}^{18}, 1, \zeta_{24}^{6}),\\ & \mathrm{diag}(\zeta_{24}^{8}, \zeta_{24}^{2}, \zeta_{24}^{8}, \zeta_{24}^{14}), \mathrm{diag}(\zeta_{24}^{16}, \zeta_{24}^{10}, \zeta_{24}^{16}, \zeta_{24}^{22}), \mathrm{diag}(1, \zeta_{24}^{6}, 1, \zeta_{24}^{18}),\\ & \mathrm{diag}(\zeta_{24}^{2}, \zeta_{24}^{20}, \zeta_{24}^{14}, \zeta_{24}^{20}), \mathrm{diag}(\zeta_{24}^{10}, \zeta_{24}^{4}, \zeta_{24}^{22}, \zeta_{24}^{4}),\\ & \mathrm{diag}(\zeta_{24}^{18}, \zeta_{24}^{12}, \zeta_{24}^{6}, \zeta_{24}^{12}) \big\}. \end{aligned}$$

Note that, $P_{(23)}\tilde{s} = \tilde{s}P_{(23)}$, thus, by Theorem 1(1), on switching the second and third entry of each of the diagonal torsion matrix associated to $\tilde{S}$ we obtain a diagonal torsion matrix that is also associated to $\tilde{S}$. Therefore, $\tilde{S}$ has above 36 associated diagonal torsion matrices.

The following s_1-matrix cannot be obtained from the tensor product of integral Fourier matrices of smaller rank.

$$s_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

However, $s_1 = P_{(24)}^t \tilde{s} P_{(24)}$. Therefore, by applying Theorem 1(2), we can obtain the diagonal torsion matrices associated to the Fourier matrix S_1 from the above diagonal torsion matrices. Also, an application of Theorem 1(2) gives the diagonal torsion matrices associated to Fourier matrix $P_{(34)}^t \tilde{S} P_{(34)}$. Note that, $s_1 = P_{(243)}\tilde{s}$. Therefore, we can also apply Proposition 1 to find the diagonal torsion matrices.

In the next lemma, we show that the multiplication of a Fourier matrix with a permutation matrix result in a Fourier matrix, provided the first row and column of the Fourier matrix are not permuted.

Lemma 1. *Let S be a non-singular symmetric matrix and P a permutation matrix. Let SP be a symmetric matrix. If S is a Fourier matrix and $P_{00} = 1$ then PS and SP are Fourier matrices.*

Proof. Since S is a unitary matrix and $P^{-1} = P^t$, $(PS)\overline{(PS)}^t = PS\overline{S}^t P^t = I$, where I is the identity matrix. Therefore, PS is a unitary matrix. Since $P_{00} = 1$, the entries of first row and column of PS are positive real numbers. Also, the set of the structure constants of generated by the columns of PS under entrywise multiplication is equal to the set of the structure constants of generated by the columns of S under entrywise multiplication. Therefore, PS is a Fourier matrix. Similarly, SP is a Fourier matrix. □

In the following proposition, we prove that for a permutation matrix P of order 3 and modular datum (S, T) if PS is symmetric matrix then $(PS, P^T TP)$ is a modular datum.

Proposition 1. *Let (S, T) be a modular datum. Let P be a permutation matrix of order 3 such that $P_{00} = 1$.*

(1) *If PS is a symmetric matrix then $(PS, P^t TP)$ is a modular datum.*
(2) *If SP is a symmetric matrix then (SP, PTP^t) is a modula datum.*

Proof. (1) Since S is a Fourier matrix, by Lemma 1, PS is a Fourier matrix. Also $(ST)^3 = S^2$ and $(PS)^t = PS$ imply $(PSP^t TP)^3 = P^2 S^2 P = (PS)^2$. The matrix $P^t TP$ is a diagonal torsion matrix whose multiplicative order is equal to the multiplicative order of T. Hence $(PS, P^t TP)$ is a modular datum.

(2) Proof is similar to part (1) above. □

An application of the above proposition can be found in Example 1.

Definition 3. Let S be a symmetric matrix of rank r and $P_1, P_2, \ldots, P_n$ be permutation matrices of rank r and multiplicative order k, where n is a positive integer determined by r. Let SP_1, $SP_1P_2, \ldots, SP_1 \ldots P_n$, and P_1S, $P_1P_2S, \ldots, P_1 \ldots P_nS$ be symmetric matrices. Then we call the set

1) $V := \{P_1^t SP_1,\ P_2^t P_1^t SP_1P_2, \ldots, P_n^t \ldots P_1^t SP_1 \ldots P_n\}$ a *chain of symmetric matrices* under the action of k-cycles;
2) $V_1 := \{SP_1,\ SP_1P_2, \ldots, SP_1 \ldots P_n\}$ a *left chain of symmetric matrices* under the action of k-cycles;
3) $V_2 := \{P_1S,\ P_1P_2S, \ldots,\ P_1 \ldots P_nS\}$, a *right chain of symmetric matrices* under the action of k-cycles.

The character table of an abelian group can be written as a tensor product of character tables of cyclic groups of smaller order. Therefore, for each non-prime rank there exists a chain (see Definition 3) that has a Fourier matrix which is a tensor product of s-matrices of smaller rank. However, it is not necessary that a chain corresponds uniquely to order k, see examples 1 and 2. Also, a Fourier matrix can be a member of more than one chain. For example, for the Fourier matrix S whose s-matrix is a character table of an elementary abelian group of order 8, there are

5 different chains for 28 matrices obtained from the character table by permutating its rows and/or columns.

In the next theorem, we find the number of diagonal torsion matrices corresponding to a Fourier matrix S whose associated s-matrix is the character table of an elementary abelian group.

Theorem 2. *Let n be a positive integer. Let S be a Fourier matrix of rank 2^n whose associated s-matrix is the character table of an elementary abelian group. Then S has $3^n \times 2^n$ associated diagonal torsion matrices.*

Proof. We prove the result by induction on n. For $n = 1$ and 2, the result is true, see Example 1. Suppose the result is true for $n = k$. Let $n = k + 1$. The s-matrix is a character table of an elementary abelian group. Therefore, Fourier matrix S can be expressed as $S_1 \otimes S_2$, a tensor product of two Fourier matrices S_1 and S_2 of rank 2^k and 2, respectively. By induction, there are $3^k \times 2^k$ and 3×2 diagonal torsion matrices associated to S_1 and S_2, respectively. (Note that, the number of matrices are multiple of 3, because if (S, T) is a modular datum then $(S, \zeta_3 T)$ is a modular datum.) The tensor product of two modular data is a modular datum. Therefore, corresponding to Fourier matrix S there are $(3^k \times 2^k) \times (3 \times 2) = 3^{k+1} \times 2^{k+1}$ diagonal matrices. Hence, the theorem is proved. □

The following corollary is immediate from the above theorem.

Corollary 1. *Let (S, T) be a modular datum. Let s-matrix be the character table of an elementary abelian group of order 2^n. Then the chain has $(2^n - 1)!$ Fourier matrices and $(2^n - 1)! \times 3^n \times 2^n$ associated diagonal torsion matrices.*

The following proposition has an application in Example 2.

Proposition 2. *Let S be a Fourier matrix of rank r whose associated s-matrix is a rows and/or columns permutation of the character table of an elementary abelian group. Let $T_i := diag(1, \ldots, -1, \ldots, 1)$ be a diagonal matrix with -1 at the ith position on its diagonal and all the other diagonal entries equal to 1, where $i \in \{1, 2, \ldots, r\}$.*

(1) *If (S, T_i) is a modular datum for an i then $r = 4$.*
(2) *If $r = 4$ then (S, T_i) is a modular datum for an i if and only if $s_{jj} = 1$, for all j.*
(3) *If $r = 4$ then (S, xT_i) is a modular datum if and only if $s_{jj} = 1$, for all j, and $x^3 = 1$, that is, $x \in \{1, \zeta_3, \zeta_3^2\}$.*

(4) *If $r = 4$ then $(S, y(-T_i))$ is a modular datum if and only if $s_{jj} = 1$, for all j, and $y^3 = -1$, that is, $y \in \{\zeta_6, \zeta_6^3, \zeta_6^5,\}$.*

Proof. (1) Suppose T is a diagonal torsion matrix associated to Fourier matrix S. Then $(ST)^3 = S^2 = I$, where I is an identity matrix of rank r. Since $S = r^{-1/2}s$, $S^2 = I$ if and only if $s^{-1} = r^{-1}s$. Therefore, $(ST)^3 = I$ if and only if $(r^{-1/2}sT)^3 = I$, that is, $(sT)^2 = r^{3/2}T^{-1}s^{-1} = r^{1/2}T^{-1}s$. Note that, s-matrix is a symmetric matrix and $s^2 = rI$. Therefore, if T_i is an associated diagonal torsion matrix then all the non-diagonal entries of $(sT_i)^2$ are ± 2 and the diagonal entries are $\pm(r-2)$. Since the entries of s-matrix are only ± 1, $(sT_i)^2 = r^{1/2}T_i^{-1}s$ implies $r = 4$, where $T_i^{-1} = T_i$.

(2) Note that, by part (1), $r = 4$ and $(sT_i)^2 = 2T_is$. We consider the following two cases.

Case 1. Let $i > 1$.

Then $(sT_i)^2)_{0j} = 2$ for all j. The entries of s-matrix are ± 1. Thus $(sT_i)_{0i} = -1$ implies $(sT_i)^2_{0i} = 2$ if and only if $s_{ii} = 1$. Since s-matrix is a symmetric matrix, $(sT_i)^2_{jj} = 2$ for all $j \neq i$. Therefore, $(sT_i)^2 = 2T_is$ if and only if $s_{jj} = 1$ for all j. Hence (S, T_i) is a modular datum if and only if $s_{jj} = 1$ for all j.

Case 2. Let $i = 1$.

Obviously, $s_{00} = 1$. The s-matrix and $(sT_1)^2$ have ± 1 and ± 2 entries, respectively. Also, s-matrix is a symmetric matrix. Thus, $(sT_1)^2_{jj} = 2$ for all $j \neq 0$. Therefore, $(sT_1)^2 = 2T_1s$ if and only if $s_{jj} = 1$ for all j. Hence (S, T_1) is a modular datum if and only if $s_{jj} = 1$ for all j.

(3) By part (2), (S, T_i) is modular datum if and only if $s_{jj} = 1$. Now $(S(xT_i))^3 = x^3(ST_i)^3 = x^3I$. Therefore, (S, xT_i) is modular datum if and only if $s_{jj} = 1$ and $x^3 = 1$, that is, $x \in \{1, \zeta_3, \zeta_3^2\}$.

(4) By part (2), (S, T_i) is modular datum if and only if $s_{jj} = 1$. Now $(Sy(-T_i))^3 = -y^3(ST_i)^3 = -y^3I$. Therefore, $(S, y(-T_i))$ is modular datum if and only if $s_{jj} = 1$ and $y^3 = -1$, that is, $y \in \{\zeta_6, \zeta_6^3, \zeta_6^5,\}$. □

In the next proposition, we show that for a Fourier matrix S and a permutation matrix P of order 2, SP is a Fourier matrix only if the rank of S is 4.

Proposition 3. *Let S be a Fourier matrix whose s-matrix is the character table of an elementary abelian group of order r and P a permutation matrix of order 2. If SP is a Fourier matrix then $r = 4$.*

Proof. By Lemma 1, SP is a Fourier matrix. Let $R := SP$. The symmetry of R and S imply $R_{ij} = R_{ik}$, $R_{jj} = R_{kk}$, for all $i \neq j, k$. Note that,

$j, k \geqslant 1$ and rows/columns of s-matrix are orthogonal, thus $R_{jk} \neq R_{kk}$ and $R_{kj} \neq R_{jj}$. The entries of s-matrix are ± 1. Therefore, the orthogonality of columns of s-matrix gives $(r-2)-2=0$ implies $r=4$. $\square$

Example 2. Consider the $\tilde{s}$-matrix as described in Example 1. The following matrix $s_2 = P_{(23)}\tilde{s}$ is obtained from $\tilde{s}$-matrix by interchanging its second and third rows.

$$s_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Note that, the Fourier matrix S_2 is not a tensor product of any lower rank Fourier matrices. Let T_i be the diagonal matrices of rank 4 as defined in Proposition 2. Since $(s_2)_{jj} = 1$, by Proposition 2, xT_1, xT_2, xT_3, xT_4, $y(-T_1), y(-T_2), y(-T_3)$, and $y(-T_4)$ are the diagonal torsion matrices associated to Fourier matrix S_2, where $x \in \{1, \zeta_3, \zeta_3^2\}$ and $y \in \{\zeta_6, \zeta_6^3, \zeta_6^5\}$.

The following corollary is immediate from the proof of the above results and the definition.

Corollary 2. *Let V be a chain (left chain, right chain) of symmetric matrices under the action of k-cycles, where $k \in \{2, 3\}$.*

(1) *If V has a Fourier matrix then every element of the chain is a Fourier matrix.*
(2) *If V has a homogenous Fourier matrix then every element of the chain is a homogeneous Fourier matrix.*
(3) *If an element S in the chain V forms a modular datum. Then each element of V forms a modular datum and they have equal number of associated diagonal torsion matrices that are completely determined by the diagonal matrices for S.*

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