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Structure of relatively free trioids

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Dedicated to the 60th anniversary of the Department of Algebra and Mathematical Logic of Taras Shevchenko National University of Kyiv

Abstract. Loday and Ronco introduced the notions of a trioid and a trialgebra, and constructed the free trioid of rank 1 and the free trialgebra. This paper is a survey of recent developments in the study of free objects in the varieties of trioids and trialgebras. We present the constructions of the free trialgebra and the free trioid, the free commutative trioid, the free n-nilpotent trioid, the free left (right) n-trinilpotent trioid, and the free rectangular trioid. Some of these results can be applied to constructing relatively free trialgebras.

1. Introduction

An *associative trialgebra* (an *associative trioid*) is a vector space (a set, respectively) equipped with three binary associative operations $\dashv$, $\vdash$, and $\perp$ satisfying the following eight axioms:

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{T1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{T2}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z), \tag{T3}$$

$$(x \dashv y) \dashv z = x \dashv (y \perp z), \tag{T4}$$

$$(x \perp y) \dashv z = x \perp (y \dashv z), \tag{T5}$$

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$$(x \dashv y) \perp z = x \perp (y \vdash z), \tag{T6}$$
$$(x \vdash y) \perp z = x \vdash (y \perp z), \tag{T7}$$
$$(x \perp y) \vdash z = x \vdash (y \vdash z). \tag{T8}$$

In this paper, for short, we will use the terms "trialgebra" and "trioid" to refer to an associative trialgebra and an associative trioid, respectively.

Trialgebras appeared first in the paper of Loday and Ronco [6] as a non-commutative version of Poisson algebras. The operad associated with trialgebras is Koszul dual to the operad associated with dendriform trialgebras [6]. Trialgebras appeared to be naturally related to several areas such as algebraic K-theory, Leibniz algebra theory, dialgebra theory, dimonoid theory and semigroup theory. They are linear analogues of trioids introduced also in [6]. If all operations of a trialgebra (trioid) coincide, we obtain the notion of an associative algebra (semigroup), and if two concrete operations of a trialgebra (trioid) coincide, we obtain the notion of a dialgebra (dimonoid) and so, trialgebras (trioids) are a generalization of associative algebras (semigroups) and dialgebras (dimonoids). Trioid theory found some applications in trialgebra theory (see, e.g., [1–4]). Trioids are also associated with n-tuple semigroups [13, 15, 22, 26], dimonoids [5, 9, 10, 12, 14, 17, 19, 32] and g-dimonoids [7, 25, 31, 33]. Now trioid theory are developed quite intensively. On one hand, some analogues of important results from semigroup theory were proven. On the other hand, natural questions about the structure of trioids are not considered. For example, till now, free idempotent trioids, free products of trioids were not described.

In this survey, we want to gather and to systematize the main results that belong to the variety theory of trioids and trialgebras. We will not touch issues related to the study of congruences on trioids, we will not focus on the properties and connections of trialgebras, as well as issues related to the applications of trialgebras. Note, that most of the results obtained to date relate to the theory of free systems in a trioid variety. It happened thanks to the fact that relatively free objects in any variety of algebras are important in the study of that variety. We will focus on the results clarifying the structure of free objects in the varieties of trioids and trialgebras. Our goal is to see which free objects have already been constructed, and this will allow us to see which free objects should be constructed further. First we give examples of trioids. Then recall and summarize the results obtained by Loday and Ronco, the author as well as others on the structure of free objects in the varieties of trialgebras and trioids, namely, we give explicit structure theorems for the free trialgebra and the free trioid, the free commutative trioid, the free n-nilpotent trioid, the free left (right) n-trinilpotent trioid, and the free rectangular trioid. These results develop the variety theory of algebraic structures and some of them can be applied to constructing relatively free trialgebras.

2. Examples of trioids

First we give examples of trioids.

a) Let S be a semigroup. A transformation φ of S is called an *averaging operator* on S if φ is an endomorphism of S and

$$(x\varphi\, y)\varphi = (x(y\varphi))\varphi = x\varphi\, y\varphi$$

for all $x, y \in S$. A transformation ψ of a trioid $(T, \dashv, \vdash, \perp)$ is called an *averaging operator* on $(T, \dashv, \vdash, \perp)$ [21] if ψ is an averaging operator on $(T, \dashv)$, $(T, \vdash)$ and $(T, \perp)$.

Let f be an idempotent endomorphism of S. Define operations $\dashv$, $\vdash$, and $\perp$ on S by

$$x \dashv y = x(yf), \qquad x \vdash y = (xf)y, \qquad x \perp y = (xy)f$$

for all $x, y \in S$.

Proposition 2.1 ([21] , Proposition 3.1)**.** $(S, \dashv, \vdash, \perp)$ *is a trioid.*

The obtained trioid is denoted by S^f. It is known [21] that f is an averaging operator on S^f.

b) Let S be a semigroup, and let f be an averaging operator on S. Define operations $\dashv$, $\vdash$, and $\perp$ on S by

$$x \dashv y = x(yf), \qquad x \vdash y = (xf)y, \qquad x \perp y = xy$$

for all $x, y \in S$.

Proposition 2.2 ([21], Proposition 3.2)**.** $(S, \dashv, \vdash, \perp)$ *is a trioid.*

c) A semigroup S is called *rectangular* [21] if $xyz = xz$ for all x, y, $z \in S$.

Let S be a rectangular semigroup, let M be an arbitrary semigroup, and let $\pi : M \to S$ be a homomorphism. Define operations $\dashv$, $\vdash$, and $\perp$ on $S \times M$ by

$$(s,t) \dashv (p,g) = (s, tg), \qquad (s,t) \vdash (p,g) = ((t\pi)p, tg),$$
$$(s,t) \perp (p,g) = (sp, tg)$$

for all $(s,t), (p,g) \in S \times M$.

Proposition 2.3 ([18], Proposition 8). *$(S \times M, \dashv, \vdash, \perp)$ is a trioid.*

d) Let X be an arbitrary nonempty set, and let X^* be the set of all finite nonempty words over X. If $w \in X^*$, then denote the first (the last) letter of a word w by $w^{(0)}$ ($w^{(1)}$, respectively). Define operations $\dashv$, $\vdash$, and $\perp$ on X^* by

$$w \dashv u = w^{(0)} w^{(1)}, \qquad w \vdash u = u^{(0)} u^{(1)}, \qquad w \perp u = w^{(0)} u^{(1)}$$

for all $w, u \in X^*$.

Proposition 2.4 ([21], Proposition 3.4). *$(X^*, \dashv, \vdash, \perp)$ is a trioid.*

Other numerous examples of trioids can be found in [6, 18, 21, 27–30].

3. Free trialgebras

The material of this section is based on [6]. Here we present the constructions of the free trialgebra of an arbitrary rank and the free trialgebra (trioid) of rank 1.

Let $[n-1] = \{0, \ldots, n-1\}$ be a set with n elements. The set of nonempty subsets of $[n-1]$ is denoted by P_n. Observe that P_n is graded by cardinality of its members. The subset of P_n whose members have cardinality k is denoted by $P_{n,k}$. So, $P_n = P_{n,1} \cup \ldots \cup P_{n,n}$.

By definition, the *free trialgebra over the vector space* V is a trialgebra $\mathrm{Trias}(V)$ equipped with a map $V \to \mathrm{Trias}(V)$ which satisfies the following universal property. For any map $V \to A$, where A is a trialgebra, there is a unique extension $\mathrm{Trias}(V) \to A$ which is a morphism of trialgebras.

The tensor product of vector spaces over a field K is denoted by $\otimes$. Since the operations have no symmetry and since the relations let the variables in the same order, $\mathrm{Trias}(V)$ is completely determined by the free trialgebra on one generator (i.e., $V = K$). The latter is a graded vector space of the form

$$\mathrm{Trias}(K) = \bigoplus_{n \geqslant 1} \mathrm{Trias}(n).$$

From the motivation of defining the trialgebra type it is clear that for $n \in \{1, 2, 3\}$, a basis of $\mathrm{Trias}(n)$ is given by the elements of P_1, P_2 and P_3, respectively.

Consider the bijection

$$\mathrm{bij} : [i_1 - 1] \cup \ldots \cup [i_n - 1] \to [i_1 + \ldots + i_n - 1]$$

which sends $k \in [i_j - 1]$ to $i_1 + \ldots + i_{j-1} + k \in [i_1 + \ldots + i_n - 1]$.

Theorem 3.1 ([6], Theorem 1.7). *The free trialgebra* Trias(K) *on one generator is* $\bigoplus_{n\geqslant 1} K[P_n]$ *as a vector space. The binary operations* $\dashv$, $\perp$, *and* $\vdash$ *from* $K[P_p]\otimes K[P_q]$ *to* $K[P_{p+q}]$ *are given by*

$$X\dashv Y=\mathrm{bij}(X),\qquad X\perp Y=\mathrm{bij}(X\cup Y),\qquad X\vdash Y=\mathrm{bij}(Y),$$

where $X\in P_p$, $Y\in P_q$ *and* $\mathrm{bij}:[p-1]\times[q-1]\to[p+q-1]$.

Corollary 3.2 ([6], Corollary 1.8). *The free trialgebra* Trias(V) *on the vector space* V *is*

$$\mathrm{Trias}(V)=\bigoplus_{n\geqslant 1} K[P_n]\otimes V^{\otimes n},$$

and the operations are induced by the operations on Trias(K) *and concatenation.*

Proposition 3.3 ([6], Proposition 1.9). *The free trioid* $\mathfrak{T}$ *on one generator* x *is isomorphic to the trioid* $P=\bigcup_{n\geqslant 1} P_n$ *equipped with the operations described in Theorem 3.1 above.*

Since $\mathfrak{T}$ is the free trioid generated by x, there exists a unique trioid morphism $\phi:\mathfrak{T}\to P$.

Lemma 3.4 ([6], Lemma 1.10). *Any complete parenthesizing of*

$$\underbrace{(x\vdash\ldots\vdash x)}_{a_0}\vdash\underbrace{(x\dashv\ldots\dashv x)}_{a_1}\perp\underbrace{(x\dashv\ldots\dashv x)}_{a_2}\perp\ldots\perp\underbrace{(x\dashv\ldots\dashv x)}_{a_k},$$

where $a_0\geqslant 0$, $a_i\geqslant 1$ *for* $i=1,\ldots,k$, *gives the same element, denoted* ω, *in* $\mathfrak{T}$. *It is called the normal form of* ω. *Its image under* ϕ *in* P *is*

$$\underbrace{x\ldots x}_{a_0}\underbrace{\overline{x}\ldots x}_{a_1}\underbrace{\overline{x}\ldots x}_{a_2}\ldots x\underbrace{\overline{x}\ldots x}_{a_k}.$$

4. Free trioids

Loday and Ronco constructed the free trioid of rank 1 [6] (see also section 3). In this section, we describe the free trioid of an arbitrary rank and for this trioid give an isomorphic construction. We also consider an alternative construction for free trioids of rank 1.

Let X be an arbitrary nonempty set, $\overline{X}=\{\overline{x}\mid x\in X\}$, and let $\mathrm{F}[X]$ be the free semigroup on X. Let further $P\subset \mathrm{F}[X\cup\overline{X}]$ be a subsemigroup which contains words w with the element $\overline{x}$ $(x\in X)$ occuring in w at least one time. For every $w \in P$ denote by $\widetilde{w}$ the word obtained from w by change of all letters $\overline{x}$ ($x \in X$) by x. For instance, if $w = x\overline{x}\,\overline{y}x\overline{z}$, then $\widetilde{w} = xxyxz$. Obviously, $\widetilde{w} \in \mathrm{F}[X \cup \overline{X}] \setminus P$.

Define operations $\dashv$, $\vdash$, and $\perp$ on P by

$$w \dashv u = w\widetilde{u}, \qquad w \vdash u = \widetilde{w}u, \qquad w \perp u = wu$$

for all $w, u \in P$. The algebra $(P, \dashv, \vdash, \perp)$ is denoted by $\mathrm{Frt}(X)$.

The proof of the following statement is the same as the proof of Proposition 1.9 from [6] (see also Proposition 3.3) obtained for the free trioid of rank 1.

Proposition 4.1 ([28], Proposition 1)**.** $\mathrm{Frt}(X)$ *is the free trioid.*

If $X = \{x\}$, then $\mathrm{Frt}(X)$ is the free trioid of rank 1 presented by Loday and Ronco in [6]. In the latter paper it was shown that the free trialgebra over a vector space is completely determined by the free trialgebra on one generator and the description of the latter trialgebra is reduced to the description of the free trioid of rank 1 (see section 3).

Now we give another representation of the free trioid of an arbitrary rank.

As usual, $\mathbb{N}$ denotes the set of all positive integers. For every word ω over X the length of ω is denoted by ℓ_ω. For any $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \ldots, n\}$, $L \neq \varnothing$, we let $L + k = \{m + k \,|\, m \in L\}$. Define operations $\dashv'$, $\vdash'$, and $\perp'$ on the set

$$F = \{(w, L) \mid w \in \mathrm{F}[X],\ L \subseteq \{1, 2, \ldots, \ell_w\},\ L \neq \varnothing\}$$

by

$$(w, L) \dashv' (u, R) = (wu, L), \qquad (w, L) \vdash' (u, R) = (wu, R + \ell_w),$$
$$(w, L) \perp' (u, R) = (wu, L \cup (R + \ell_w))$$

for all $(w, L), (u, R) \in F$. The algebra $(F, \dashv', \vdash', \perp')$ is denoted by $\mathrm{FT}(X)$.

Theorem 4.2 ([21], Theorem 7.1)**.** *The free trioid* $\mathrm{Frt}(X)$ *is isomorphic to the trioid* $\mathrm{FT}(X)$*.*

Define operations $\prec$, $\succ$, and $\uparrow$ on the set

$$F' = \{(n, L) \mid n \in \mathbb{N},\ L \subseteq \{1, 2, \ldots, n\}, L \neq \varnothing\}$$

by

$$(n, L) \prec (m, R) = (n + m, L), \qquad (n, L) \succ (m, R) = (n + m, R + n),$$
$$(n, L) \uparrow (m, R) = (n + m, L \cup (R + n))$$

for all $(n, L), (m, R) \in F'$.

From Theorem 4.2 we obtain

Corollary 4.3 ([21], Corollary 7.1)**.** *Let* $|X| = 1$*. The free trioid* $\mathrm{Frt}(X)$ *of rank* 1 *is isomorphic to the trioid* $(F', \prec, \succ, \uparrow)$*.*

The trioid $(F', \prec, \succ, \uparrow)$ was also considered in [35]. Another representation of the free trioid of rank 1 can be found in [16].

For free trioids, characterizations of the least left (right) zero congruence, the least rectangular band congruence and the least n-nilpotent congruence were given in [28] and [29], respectively. The description of the least rectangular triband congruence on the free trioid follows from Theorem 3.1 (i) in [28]. The problem of the characterization of the least dimonoid congruences and the least semigroup congruence on the free trioid was solved in [27]. The least commutative congruence, the least commutative dimonoid congruences, and the least commutative semigroup congruence on the free trioid were presented in [11]. Decompositions of free trioids into tribands of subtrioids and bands of subtrioids were given in [28]. In [34], it was proved that endomorphism semigroups of free trioids are isomorphic if and only if the corresponding free trioids are isomorphic. The endomorphism monoid of the free trioid of rank 1 was studied in [35].

5. Free commutative trioids

In this section, we construct the free commutative trioid of rank 1 and show that the free commutative trioid of rank $n > 1$ is a subdirect product of a free commutative semigroup of rank n and the free commutative trioid of rank 1 [11]. We will use notations of section 4.

A trioid $(T, \dashv, \vdash, \perp)$ is called *commutative* if semigroups $(T, \dashv)$, $(T, \vdash)$, and $(T, \perp)$ are commutative.

Let Ω be the free monoid on the 3-element set $\{a, b, c\}$, and let θ denote the identity of Ω, that is, the empty word. By definition, the length ℓ_θ of θ is equal to 0 and $u^0 = \theta$ for any $u \in \Omega \setminus \{\theta\}$. For all $u_1, u_2 \in \Omega$ let

$$f_{\dashv}(u_1, u_2) = a, \qquad f_{\vdash}(u_1, u_2) = \begin{cases} b & \text{if } u_1 = u_2 = \theta, \\ a & \text{otherwise,} \end{cases}$$

$$f_{\perp}(u_1, u_2) = \begin{cases} c & \text{if } u_1 = c^k,\ u_2 = c^p,\ k, p \in \mathbb{N} \cup \{0\}, \\ a & \text{otherwise.} \end{cases}$$

The subset

$$\{y^k \mid y \in \{a, c\},\ k \in \mathbb{N} \cup \{0\}\} \cup \{b\}$$

of Ω is denoted by $\overline{\Omega}$. Define operations $\dashv$, $\vdash$, and $\perp$ on $\overline{\Omega}$ by

$$u_1 * u_2 = f_*(u_1, u_2)^{\ell_{u_1}+\ell_{u_2}+1}$$

for all $u_1, u_2 \in \overline{\Omega}$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(\overline{\Omega}, \dashv, \vdash, \perp)$ is denoted by FCT_1.

Theorem 5.1 ([11], Theorem 3.1). FCT_1 *is the free commutative trioid of rank* 1.

Now we construct the free commutative trioid of an arbitrary rank.

Let X be an arbitrary nonempty set, and let $\mathrm{F}^\star[X]$ be the free commutative semigroup on X. Define operations $\dashv$, $\vdash$, and $\perp$ on the set

$$A = \{(w, u) \in \mathrm{F}^\star[X] \times \mathrm{FCT}_1 \mid \ell_w - \ell_u = 1\}$$

by

$$(w_1, u_1) * (w_2, u_2) = (w_1 w_2, f_*(u_1, u_2)^{\ell_{u_1}+\ell_{u_2}+1})$$

for all $(w_1, u_1), (w_2, u_2) \in A$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(A, \dashv, \vdash, \perp)$ is denoted by $\mathrm{FCT}(X)$.

Theorem 5.2 ([11], Theorem 3.8). $\mathrm{FCT}(X)$ *is the free commutative trioid.*

A *subdirect product of two algebras* A_1 and A_2 is a subalgebra U of the direct product $A_1 \times A_2$ such that the projection maps $U \to A_1$ and $U \to A_2$ are surjections.

Corollary 5.3 ([11], Corollary 3.9). *The free commutative trioid of rank* $n > 1$ *is a subdirect product of a free commutative semigroup of rank* n *and the free commutative trioid of rank* 1.

Remark 5.4. *From the construction of* $\mathrm{FCT}(X)$ *it follows that* $\mathrm{FCT}(X)$ *is determined uniquely up to isomorphism by cardinality of the set* X. *Hence the automorphism group of* $\mathrm{FCT}(X)$ *is isomorphic to the symmetric group on* X.

The least dimonoid congruences and the least semigroup congruence on the free commutative trioid were described in [27].

6. Free n-nilpotent trioids

In this section, we construct free n-nilpotent trioids of an arbitrary rank and consider separately free n-nilpotent trioids of rank 1 [20, 29]. We will use notations of section 4.

An element 0 of a trioid $(T, \dashv, \vdash, \perp)$ is called *zero* [21] if

$$x * 0 = 0 * x = 0 * 0 = 0$$

for all $x \in T$ and $* \in \{\dashv, \vdash, \perp\}$. A trioid $(T, \dashv, \vdash, \perp)$ with zero is called *nilpotent* if for some $n \in \mathbb{N}$ and any $x_i \in T$, $1 \leqslant i \leqslant n+1$, and $*_j \in \{\dashv, \vdash, \perp\}$, $1 \leqslant j \leqslant n$, any parenthesizing of

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$$

gives $0 \in T$. The least such n is called the *nilpotency index* of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$ a nilpotent trioid of nilpotency index $\leqslant k$ is said to be *k-nilpotent*.

Fix $n \in \mathbb{N}$. Let $P_n \subset P$ be a set which contains words w with $\ell_w \leqslant n$. Define operations $\prec, \succ$, and $\uparrow$ on the set $P_n \cup \{0\}$ by

$$w \prec u = \begin{cases} w\widetilde{u}, & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases} \qquad w \succ u = \begin{cases} \widetilde{w}u, & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases}$$

$$w \uparrow u = \begin{cases} wu, & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases} \qquad w * 0 = 0 * w = 0 * 0 = 0$$

for all $w, u \in P_n$ and $* \in \{\prec, \succ, \uparrow\}$. The algebra $(P_n \cup \{0\}, \prec, \succ, \uparrow)$ is denoted by $P_n^0(X)$.

Theorem 6.1 ([29], Theorem 1)**.** *$P_n^0(X)$ is the free n-nilpotent trioid.*

Now we construct a trioid which is isomorphic to $P_n^0(X)$.

Define operations $\dashv$, $\vdash$, and $\perp$ on

$$\mathrm{FNT}_n = \{(w, L) \mid w \in \mathrm{F}[X],\ \ell_w \leqslant n,\ L \subseteq \{1, 2, \ldots, \ell_w\},\ L \neq \varnothing\} \cup \{0\}$$

by

$$(w, L) \dashv (u, R) = \begin{cases} (wu, L), & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases}$$

$$(w, L) \vdash (u, R) = \begin{cases} (wu, R + \ell_w), & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases}$$

$$(w, L) \perp (u, R) = \begin{cases} (wu, L \cup (R + \ell_w)), & \ell_{wu} \leqslant n, \\ 0, & \ell_{wu} > n, \end{cases}$$
$$(w, L) * 0 = 0 * (w, L) = 0 * 0 = 0$$

for all $(w, L), (u, R) \in \mathrm{FNT}_n \setminus \{0\}$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(\mathrm{FNT}_n, \dashv, \vdash, \perp)$ is denoted by $\mathrm{FNT}_n(X)$.

Lemma 6.2 ([20], Lemma 1). $\mathrm{FNT}_n(X)$ *is a trioid.*

Theorem 6.3 ([20], Theorem 3). *The free n-nilpotent trioid $P_n^0(X)$ is isomorphic to the trioid* $\mathrm{FNT}_n(X)$.

Consider separately free n-nilpotent trioids of rank 1.

Define operations $\dashv$, $\vdash$, and $\perp$ on the set

$$\mathrm{FNT}'_n = \{(m, L) \mid m \in \mathbb{N},\, m \leqslant n,\, L \subseteq \{1, 2, \ldots, m\},\, L \neq \varnothing\} \cup \{0\}$$

by

$$(m, L) \dashv (k, R) = \begin{cases} (m + k, L), & m + k \leqslant n, \\ 0, & m + k > n, \end{cases}$$
$$(m, L) \vdash (k, R) = \begin{cases} (m + k, R + m), & m + k \leqslant n, \\ 0, & m + k > n, \end{cases}$$
$$(m, L) \perp (k, R) = \begin{cases} (m + k, L \cup (R + m)), & m + k \leqslant n, \\ 0, & m + k > n, \end{cases}$$
$$(m, L) * 0 = 0 * (m, L) = 0 * 0 = 0$$

for all $(m, L), (k, R) \in \mathrm{FNT}'_n \setminus \{0\}$ and $* \in \{\dashv, \vdash, \perp\}$. The algebra $(\mathrm{FNT}'_n, \dashv, \vdash, \perp)$ is denoted by FNT^1_n.

Theorem 6.3 implies

Corollary 6.4 ([20], Corollary 1). *If $|X| = 1$, then* $\mathrm{FNT}_n(X) \cong \mathrm{FNT}^1_n$.

Examples of nilpotent trioids of nilpotency index 2 can be found in [29]. Decompositions of free n-nilpotent trioids into 0-bands of subtrioids and 0-tribands of subtrioids were given in [29]. The least dimonoid congruences and the least semigroup congruence on the free n-nilpotent trioid were presented in [20].

7. Free left (right) n-trinilpotent trioids

In this section, we construct the free left n-trinilpotent trioid and consider free left n-trinilpotent trioids of rank 1 [23] separately. We will use notations of section 4.

By Λ denote the signature of a trioid. Let $a_1, \ldots, a_n$ be individual variables. By $P(a_1, \ldots, a_n)$ we denote the set of all terms of algebras of the signature Λ having the form $a_1 \circ_1 \ldots \circ_{n-1} a_n$ with parenthesizing, where $\circ_1, \ldots, \circ_{n-1} \in \Lambda$. A trioid $(T, \dashv, \vdash, \perp)$ is called *left trinilpotent* if for some $n \in \mathbb{N}$, any $a \in T$ and any $p(a_1, \ldots, a_n) \in P(a_1, \ldots, a_n)$ the following identities hold:

$$p(a_1, \ldots, a_n) * a = p(a_1, \ldots, a_n), \tag{7.1}$$

$$p(a_1, \ldots, a_n) \vdash a = a_1 \vdash \ldots \vdash a_n, \tag{7.2}$$

where $* \in \{\dashv, \perp\}$. The least such n is called the *left trinilpotency index* of $(T, \dashv, \vdash, \perp)$. For $k \in \mathbb{N}$ a left trinilpotent trioid of left trinilpotency index $\leqslant k$ is said to be *left k-trinilpotent*. Obviously, in any trioid $(T, \dashv, \vdash, \perp)$, by axioms (T3), (T8) and associativity of the operation $\vdash$, we have

$$p(a_1, \ldots, a_n) \vdash a = a_1 \vdash \ldots \vdash a_n \vdash a.$$

Hence, if $(T, \vdash)$ is a left nilpotent semigroup of rank n [8], we get the identity (7.2). This explains how we obtain the third identity in the definition of a left trinilpotent trioid. Right k-trinilpotent trioids are defined dually.

Let $n, k \in \mathbb{N}$ and $L \subseteq \{1, 2, \ldots, n\}$. We regard

$$L + k = \{m + k \mid m \in L\}.$$

It is clear that $\varnothing + k = \varnothing$. For $L \neq \varnothing$, we let $L^{k,n} = \{m \in L \mid k + m \leqslant n\}$, and denote the least number of L by $L_{\min}$. Obviously, $L^{k,n} = \varnothing$ if $k + m > n$ for all $m \in L$.

Fix $n \in \mathbb{N}$. Let $w \in \mathrm{F}[X]$. If $\ell_w \geqslant n$, let $\overset{n}{\overrightarrow{w}}$ denote the initial subword with the length n of w, and if $\ell_w < n$, let $\overset{n}{\overrightarrow{w}} = w$. Define operations $\dashv, \vdash$, and $\perp$ on

$$V_n = \{(w, L) \mid w \in \mathrm{F}[X],\ \ell_w \leqslant n,\ L \subseteq \{1, 2, \ldots, \ell_w\},\ L \neq \varnothing\}$$

by

$$(w, L) \dashv (u, R) = (\overset{n}{\overrightarrow{wu}}, L),$$

$$(w, L) \vdash (u, R) = \begin{cases} (\overset{n}{\overrightarrow{wu}}, \{n\}), & n < \ell_w + R_{\min}, \\ (\overset{n}{\overrightarrow{wu}}, R^{\ell_w, n} + \ell_w) & \text{otherwise}, \end{cases}$$

$$(w, L) \perp (u, R) = (\overset{n}{\overrightarrow{wu}}, L \cup (R^{\ell_w, n} + \ell_w))$$

for all $(w, L), (u, R) \in V_n$. The algebra $(V_n, \dashv, \vdash, \perp)$ is denoted by $\mathrm{FT}_n^l(X)$.

Theorem 7.1 ([23], Theorem 3.1)**.** $\mathrm{FT}_n^l(X)$ *is the free left n-trinilpotent trioid.*

At the end of this section we construct a trioid which is isomorphic to the free left n-trinilpotent trioid of rank 1.

Fix $n \in \mathbb{N}$. For any $m \in \mathbb{N}$ let

$$\overset{n}{\overrightarrow{m}} = \begin{cases} m, & m \leqslant n, \\ n, & m > n. \end{cases}$$

Define operations $\dashv$, $\vdash$, and $\perp$ on

$$M_n = \{(k, L) \mid k \in \mathbb{N},\, k \leqslant n,\, L \subseteq \{1, 2, \ldots, k\},\, L \neq \varnothing\}$$

by

$$(k_1, L) \dashv (k_2, R) = (\overset{n}{\overrightarrow{k_1 + k_2}}, L),$$

$$(k_1, L) \vdash (k_2, R) = \begin{cases} (n, \{n\}), & n < k_1 + R_{\min}, \\ (\overset{n}{\overrightarrow{k_1 + k_2}}, R^{k_1, n} + k_1) & \text{otherwise}, \end{cases}$$

$$(k_1, L) \perp (k_2, R) = (\overset{n}{\overrightarrow{k_1 + k_2}}, L \cup (R^{k_1, n} + k_1))$$

for all (k_1, L), $(k_2, R) \in M_n$. The algebra $(M_n, \dashv, \vdash, \perp)$ is denoted by $\mathrm{F_1T}_n^l$.

Theorem 7.1 implies the following statement.

Corollary 7.2 ([23], Corollary 3.11)**.** *If* $|X| = 1$*, then* $\mathrm{FT}_n^l(X) \cong \mathrm{F_1T}_n^l$.

Remark 7.3. *In order to construct free right n-trinilpotent trioids we use the duality principle.*

It is known that the automorphism group of the free left (right) n-trinilpotent trioid is isomorphic to the symmetric group [23]. The problem of the description of the least left (right) n-trinilpotent congruence on the free trioid was first announced in [24].

8. Free rectangular trioids

In this section, we construct the free rectangular trioid [30].

A semigroup is called a *left (right) zero semigroup* provided that it satisfies the identity $xy = x$ $(xy = y)$. A semigroup S is a *rectangular band* if $xyx = x$ for all $x, y \in S$. Equivalently, a semigroup S is a *rectangular band* if $x^2 = x$, $xyz = xz$ for all $x, y, z \in S$. It is well-known that every rectangular band is isomorphic to the Cartesian product of the left zero semigroup and of the right zero semigroup. A trioid $(T, \dashv, \vdash, \perp)$ is called a *rectangular trioid* or a *rectangular triband* if $(T, \dashv)$, $(T, \vdash)$, and $(T, \perp)$ are rectangular bands.

Let X be an arbitrary nonempty set and $X^4 = X \times X \times X \times X$. Define operations $\dashv$, $\vdash$, and $\perp$ on X^4 by

$$(x_1, x_2, x_3, x_4) \dashv (y_1, y_2, y_3, y_4) = (x_1, x_2, x_3, y_4),$$
$$(x_1, x_2, x_3, x_4) \vdash (y_1, y_2, y_3, y_4) = (x_1, y_2, y_3, y_4),$$
$$(x_1, x_2, x_3, x_4) \perp (y_1, y_2, y_3, y_4) = (x_1, x_2, y_3, y_4)$$

for all $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in X^4$. The algebra $(X^4, \dashv, \vdash, \perp)$ is denoted by $\mathrm{FRT}(X)$.

Theorem 8.1 ([30], Theorem 1)**.** $\mathrm{FRT}(X)$ *is the free rectangular triband.*

Examples of rectangular tribands can be found in [30]. Decompositions of free rectangular tribands into bands of subtrioids, tribands of subsemigroups and tribands of subtrioids were given in [30]. It is known that the automorphism group of $\mathrm{FRT}(X)$ is isomorphic to the symmetric group on X, and any rectangular triband is semilattice indecomposable [30]. The least left (right) zero congruence and the least rectangular band congruence on the free rectangular trioid were described in [30]. The least dimonoid congruences and the least semigroup congruence on the free rectangular trioid were presented in [27].

Note that the main results of sections 5–8 can be applied to constructing the corresponding relatively free trialgebras.

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