

Injective stabilization of additive functors, III. Asymptotic stabilization of the tensor product

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ABSTRACT. The injective stabilization of the tensor product is subjected to an iterative procedure that utilizes its bifunctor property. The limit of this procedure, called the asymptotic stabilization of the tensor product, provides a homological counterpart of Buchweitz’s asymptotic construction of stable cohomology. The resulting connected sequence of functors is isomorphic to Triulzi’s J -completion of the Tor functor. A comparison map from Vogel homology to the asymptotic stabilization of the tensor product is constructed and shown to be always epic. The category of finitely presented functors is shown to be complete and cocomplete. As a consequence, the inert injective stabilization of the tensor product with fixed variable a finitely generated module over an artin algebra is shown to be finitely presented. Its defect and consequently all right-derived functors are determined. New notions of asymptotic torsion and cotorsion are introduced and are related to each other.

1. Introduction

This is the third in a series of papers on applications of what should be called “homological algebra in degree zero”, this time to stable homological algebra. The specific application dealt with in this part is a

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new generalization of Tate homology (as opposed to Tate cohomology) to arbitrary modules over arbitrary rings. As an unexpected byproduct of this construct, we introduce new concepts of asymptotic torsion and asymptotic cotorsion, thus establishing a connection with the torsion and cotorsion introduced in the previous paper [13] of this series.

Our interest in torsion stems from its importance for the problem of recognizing syzygy modules, as was expounded by M. Auslander in [3]. The significance of this problem transcends the boundary of algebra. In mathematical systems theory it is related to the controllability of linear systems. In topology, a similar problem is referred to as “delooping”. Of special interest are infinite syzygy modules. Their topological counterparts are known as infinite loop spaces [1]. Over Gorenstein local rings these are precisely maximal Cohen-Macaulay modules. In an obvious sense these are “asymptotic” objects and they are closely related to Tate cohomology as was shown by Buchweitz in [6].

Even more surprisingly, the new results (and their proofs) show the advantages of focusing on the category of finitely presented (aka coherent) functors, due to its completeness and cocompleteness. This indicates the emergence of a “coherent homological algebra”, which is based on redefining classical homological constructions that involve colimits. Further applications of the new techniques will be given in a subsequent paper.

The original motivation for this paper (in fact, for the entire series) came from an obvious imbalance between generalized Tate cohomology and generalized Tate homology: the former admits (at least) three constructions whereas the latter – only two. More precisely, the available homological constructions are obvious analogues of their cohomological counterparts, leaving Buchweitz’s generalization of Tate cohomology without a homological analogue (see the next section for more details).

The main technical tool used in this paper is the notion of the injective stabilization of an additive functor from modules to abelian groups. This concept goes back to foundational works of M. Auslander in the 1960s ([2] and [4]). Recall that the injective stabilization of an additive functor F is defined as the kernel of the natural transformation from F to its zeroth right-derived functor. It is usually denoted by \overline{F} . For a given module B , $\overline{F}(B)$ can be easily computed: if $\iota : B \rightarrow I$ is monic with I injective, then $\overline{F}(B) \simeq \text{Ker } F(\iota)$. (See [12] for a detailed treatment of this construct.) In particular, one can take $F := A \otimes _$. Unlike the tensor product itself, its injective stabilization is not balanced. For this reason, the overline is replaced by a harpoon and $\overline{F}(B)$ becomes $A \overline{\otimes} B$, with the harpoon

pointing to the active variable, i.e., the variable being embedded in an injective. The resulting expression is in fact a bifunctor. Let Ω denote the syzygy operation in a projective resolution and Σ denote the cosyzygy operation in an injective resolution. While neither Ω nor Σ is well-defined (their values on a module depend on the chosen resolutions), $\Omega^i A \overline{\otimes} \Sigma^i B$ is well-defined and is again a bifunctor. Moreover, there is a sequence of canonically defined natural transformations

$$\dots \longrightarrow \Omega^2 A \overline{\otimes} \Sigma^2 B \xrightarrow{\Delta_2} \Omega A \overline{\otimes} \Sigma B \xrightarrow{\Delta_1} A \overline{\otimes} B,$$

and the limit $\overline{T}_0(A, B)$ of this sequence, called here the *asymptotic stabilization of the tensor product*, yields the desired generalization of Tate homology. The above construct works for arbitrary modules over arbitrary rings.

Now we give a brief outline of the paper. Section 2 describes various generalizations of Tate cohomology. Section 3 does the same for Tate homology. Section 4 deals with two lemmas, both related to connecting homomorphisms in spatial diagrams. In section 5 we give the first construction of stable homology, as described above via the asymptotic stabilization of the tensor product. In the same section we show that the result is a connected sequence of functors. Another construction of stable homology, based on the sequence $\text{Tor}_1(\Omega^i A, \Sigma^{i+1} B)$, is given in section 6. The result is again a connected sequence of functors, isomorphic to $\overline{T}_0(-, -)$. Its connecting homomorphism comes from a doubly infinite exact sequence obtained by splicing the familiar long sequence of the Tor functors in positive degrees and a long sequence of iterated injectives stabilizations of the tensor product. Yet another construction of stable homology is given in section 7. This time it is based on the sequence $S^1 \text{Tor}_1(\Omega^i A, -)$, where S^1 denotes the right satellite. The resulting connected sequence of functors is also isomorphic to $\overline{T}_0(A, -)$ and has the additional advantage of being obviously isomorphic to the J -completion of Tor, introduced and studied by M. Triulzi [17].¹ In section 8, we construct a comparison transformation from Vogel homology to $\overline{T}_0(-, -)$ and show that it is always epic. This evokes a similar situation in topology where one has an epic natural transformation from Steenrod-Sitnikov homology to Čech homology. Based on that analogy, the first author conjectured that the kernel of the comparison transformation from Vogel homology to the asymptotic stabilization of the

¹The authors are grateful to L. Avramov for pointing our attention to that reference.

tensor product should be given by a suitable first derived limit. A recent result of I. Emmanouil and P. Manousaki [8] shows that this is indeed the case. In section 9 we show that the category of finitely presented functors is (co)complete. This sets the stage for section 10, where we establish the coherence of the inert asymptotic stabilization when the fixed argument satisfies certain finiteness conditions. All finitely generated modules over an artin algebra satisfy this condition. In the same section we compute the defect of the inert asymptotic stabilization, which allows us to determine all of its right-derived functors. These results are then applied to the torsion functor \mathfrak{s} introduced in [13]. This leads naturally to the introduction of an asymptotic torsion and asymptotic cotorsion, and we show that the defect of the asymptotic torsion is isomorphic to the asymptotic cotorsion of the ring viewed as a module on the opposite side.

The surprising connection with topology mentioned above hints at further developments. Treating Buchweitz's cohomological construction as an analogue of stable homotopy groups (both are byproducts of universal inversion of a functor) one is forced to look for a homotopy-theoretic counterpart of the asymptotic stabilization. The proper context for doing this is an axiomatic homotopy theory, or, even more broadly, quite general categories with suitable cylinders. The relevant results will appear elsewhere.

We follow the terminology and notation established in [12]; the reader may benefit from reviewing that source. Some results contained in the present paper overlap with some results obtained by the second author in his PhD thesis [16].

2. Stable cohomology

Tate cohomology was invented around 1950. We assume that the reader is familiar with this notion, but if they need to refresh their memory, we recommend [5] and [7]. In 1977, F. T. Farrell [9] constructed a cohomology theory for groups of finite virtual cohomological dimension that, for finite groups, gave the same result as Tate cohomology. In the mid-1980s, R.-O. Buchweitz [6] constructed a generalization of Tate (and Farrell) cohomology that worked over arbitrary Gorenstein commutative rings.

2.1. Vogel cohomology

At about the same time as Buchweitz, Pierre Vogel [11] came up with his own generalization of Tate cohomology, and while he was interested

in arbitrary group rings, his approach actually worked over any ring. We now review that construction.

Let Λ be a (unital) ring and M and N (left) Λ -modules. Choose projective resolutions $(\mathbf{P}, \partial) \rightarrow M$ and $(\mathbf{Q}, \partial) \rightarrow N$. Forgetting the differentials, we have \mathbb{Z} -diagrams \mathbf{P} and \mathbf{Q} of left Λ -modules, together with a \mathbb{Z} -diagram (\mathbf{P}, \mathbf{Q}) of abelian groups. The latter has $\prod_i \text{Hom}(P_i, Q_{i+n})$ as its degree n component. It contains the subdiagram $(\mathbf{P}, \mathbf{Q})_b$ of bounded maps, whose degree n component is $\prod_i \text{Hom}(P_i, Q_{i+n})$. Passing to the quotient, we have a short exact sequence of diagrams

$$0 \rightarrow (\mathbf{P}, \mathbf{Q})_b \rightarrow (\mathbf{P}, \mathbf{Q}) \rightarrow \widehat{(\mathbf{P}, \mathbf{Q})} \rightarrow 0$$

The standard definition, $D(f) := \partial \circ f - (-1)^{\deg f} f \circ \partial$, yields a differential on the middle diagram, which clearly restricts to a differential on the subdiagram of bounded maps. Thus the inclusion map is actually an inclusion of complexes, and the corresponding quotient becomes the quotient complex. By construction, the maps in this short exact sequence are chain maps between the constructed complexes. The n th Vogel cohomology group of M with coefficients in N , where $n \in \mathbb{Z}$, is then defined as the n th cohomology group of the complex $\widehat{(\mathbf{P}, \mathbf{Q})}$. We denote it by $V^n(M, N)$.

2.2. Buchweitz cohomology

As we mentioned before, Buchweitz was interested in a generalized Tate cohomology over Gorenstein rings, but his construction (actually, one of two proposed) turned out to work for any ring. We now describe his approach. Again, let Λ be an arbitrary (unital) ring, M and N (left) Λ -modules, and $\Lambda\text{-Mod}$ the category of left Λ -modules and homomorphisms. First, we pass to the category $\Lambda\text{-Mod}$ of modules modulo projectives, which has the same objects as $\Lambda\text{-Mod}$, but whose morphisms $\underline{(M, N)}$ are defined as the quotient groups $(M, N)/P(M, N)$, where $P(M, N)$ is the subgroup of all maps that can be factored through a projective module. The composition of classes of homomorphisms is defined as the class of the composition of representatives. One of the advantages of this new category is that the syzygy operation Ω on $\Lambda\text{-Mod}$ becomes an additive endofunctor on $\Lambda\text{-Mod}$. In particular, for M and N we have a sequence of homomorphisms of abelian groups

$$\underline{(M, N)} \rightarrow \underline{(\Omega M, \Omega N)} \rightarrow \underline{(\Omega^2 M, \Omega^2 N)} \rightarrow \dots$$

The n th Buchweitz cohomology group $B^n(M, N)$, $n \in \mathbb{Z}$ is defined as

$$\varinjlim_{n+k, k \geq 0} \underline{(\Omega^{n+k} M, \Omega^k N)}.$$

2.3. Mislin's construction

Yet another generalization of Tate cohomology was given by G. Mislin [14] in 1994. It is a special case of a considerably more general construct. For a cohomological (or, more generally, connected) sequence of functors $\{F^i\}$, $i \in \mathbb{Z}$ Mislin uses a sequence of natural transformations

$$F^i \longrightarrow S_1(F^{i+1}) \longrightarrow S_2(F^{i+2}) \longrightarrow \dots,$$

where S_j denotes the j th left satellite, and defines what he calls the P -completion of $\{F^i\}$ as

$$\varinjlim_{k \geq 0} S_k(F^{i+k}) =: M^i F.$$

Evaluating the colimit on the group cohomology (viewed as a cohomological functor of the coefficients), he gets a new cohomological (or connected if the original sequence is connected but not necessarily cohomological) sequence of functors. He then proves that, for groups of finite virtual cohomological dimension, the new cohomology is isomorphic to Farrell cohomology. Moreover, he also establishes, for arbitrary groups, an isomorphism between his construction and Buchweitz's cohomology (called in the paper the Benson-Carlson cohomology, after the two authors, who independently found Buchweitz's cohomology in 1992). We remark that Mislin's construction is completely general and applies, in particular, to the Ext functors over any ring.

3. Stable homology

At this point, one may ask if there are *homological* analogues of the various cohomology theories discussed above. The answer to this question is less clear. First, there was no "Tate homology" in Tate's original work: only the Hom functor was used with complete resolutions. However, at the same time when P. Vogel constructed his cohomology, he also constructed a homology theory. We begin by reviewing his construction.

3.1. Vogel homology

Let Λ be a ring, M a left Λ -module and N a right Λ -module. Choose a projective resolution $(\mathbf{P}, \partial) \longrightarrow M$ and an injective resolution $N \longrightarrow (\mathbf{I}, \partial)$. Forgetting the differentials, we have \mathbb{Z} -diagrams \mathbf{P} and \mathbf{I} of left and, respectively, right Λ -modules, together with a \mathbb{Z} -diagram $\mathbf{P} \hat{\otimes} \mathbf{I}$ of

abelian groups. The latter has $\prod_i (P_i \otimes I^{i-n})$ as its degree n component. It contains the subdiagram $\mathbf{P} \otimes \mathbf{I}$, whose degree n component is $\prod_i (P_i \otimes I^{i-n})$. Passing to the quotient, we have a short exact sequence of diagrams

$$0 \longrightarrow \mathbf{P} \otimes \mathbf{I} \longrightarrow \mathbf{P} \widehat{\otimes} \mathbf{I} \longrightarrow \mathbf{P} \overset{\vee}{\otimes} \mathbf{I} \longrightarrow 0$$

The standard definition

$$\begin{aligned} D(a \otimes b) &:= \partial_P(a) \otimes b + (-1)^{\deg a} a \otimes \partial_I(b) \\ &= (\partial_P \otimes 1 + (-1)^{\deg_1(-)} 1 \otimes \partial_I)(a \otimes b), \end{aligned} \tag{3.1}$$

where a and b are homogeneous elements of \mathbf{P} and, respectively, \mathbf{I} , and $\deg_1(-)$ picks the degree of the first factor of a decomposable tensor, gives rise to a differential on $\mathbf{P} \otimes \mathbf{I}$. It is easy to check that it extends to a differential, denoted by D again, on $\mathbf{P} \widehat{\otimes} \mathbf{I}$. Indeed, if $s \in (\mathbf{P} \widehat{\otimes} \mathbf{I})_n$ is a degree n element, then $s = (s_i)_{i \in \mathbb{Z}}$, where each $s_i \in P_i \otimes I^{i-n}$ is just a finite sum of decomposable tensors. For each $k \in \mathbb{Z}$, define

$$D : \prod_i (P_i \otimes I^{i-n}) \longrightarrow (P_k \otimes I^{k+1-n}) : s \mapsto (\partial \otimes 1)(s_{k+1}) + (-1)^k (1 \otimes \partial)(s_k)$$

Now, we obtain the desired differential by the universal property of direct product.

As a consequence, the third term in the short exact sequence above becomes a complex, and Vogel homology is now defined by setting

$$V_n(M, N) := H_{n+1}(\mathbf{P} \overset{\vee}{\otimes} \mathbf{I}). \tag{3.2}$$

3.2. The J -completion

A homological analogue of Mislin’s cohomological P -completion, called the J -completion, was defined by M. Triulzi in his PhD thesis [17]. Like its cohomological prototype, it is defined on connected sequences of functors, but even if the original sequence is cohomological, the result doesn’t seem to be cohomological²; one can only claim that the resulting sequence is connected. For reference, we denote it by $M_i F$.

We summarize the existing constructions in the following table:

Cohomology	Homology
$V^i(M, N)$	$V_i(M, N)$
$B^i(M, N)$?
$M^i F$	$M_i F$

²This is related to the fact that the inverse limit is not an exact functor.

One of the goals of this paper is to replace the question mark by a homological analogue of Buchweitz's construction.

4. Two lemmas on connecting homomorphisms

In this section, we gather general observations on connecting homomorphisms.

4.1. Spatial diagrams: front, bottom, and right-hand faces

Lemma 4.1.1. *Suppose the 3D diagram in Figure 1 is commutative and:*

- 1) *any three-term sequence with arrows running in the same direction is exact (i.e., exact at the middle term);*
- 2) *each arrow preceded by an arrow in the same direction is epic;*
- 3) *the three middle three-term sequences $L''M''N''$, $M'_2M_2M''_2$, and $N'_2N_2N''_2$ on the front, the bottom and the right-hand faces of this cube are short-exact, i.e., each sequence is exact in the middle, the first map is monic, and the second map is epic.*

Then the image of the connecting homomorphism $\text{Ker } \alpha \rightarrow L''_2$ (in the front face) is in $\text{Ker } \beta$, and the composition of the connecting homomorphisms $\text{Ker } \alpha \rightarrow L''_2$ and $\text{Ker } \beta \rightarrow N'_2$ (in the bottom face) equals the negative of the connecting homomorphism $\text{Ker } \alpha \rightarrow N'_2$ (in the right-hand face).

Proof. Diagram chase. □

4.2. Spatial diagrams: top, back, and left-hand faces

Now we look at the composition of connecting homomorphisms in the three remaining planes of the cube.

Lemma 4.2.1. *Suppose the 3D diagram in Figure 2 is commutative and:*

- 1) *any three-term sequence with arrows running in the same direction is exact;*
- 2) *each arrow preceded by an arrow in the same direction is epic;*
- 3) *the three middle three-term sequences $M'_1M_1M''_1$, $L'M'_1N'$, and $L'LL''$ on the top, the back, and the left-hand faces of this cube are short-exact, i.e., each sequence is exact in the middle, the first map is monic, and the second map is epic. Moreover, the two horizontal sequences LMN and $M'MM''$ passing through the center of the cube are also short-exact.*

Then the image of $\text{Ker } \alpha \cap \text{Ker } \gamma$ under the connecting homomorphism $\text{Ker } \alpha \rightarrow N'_1$ (in the top face) is in $\text{Ker } \beta$, and on $\text{Ker } \alpha \cap \text{Ker } \gamma$

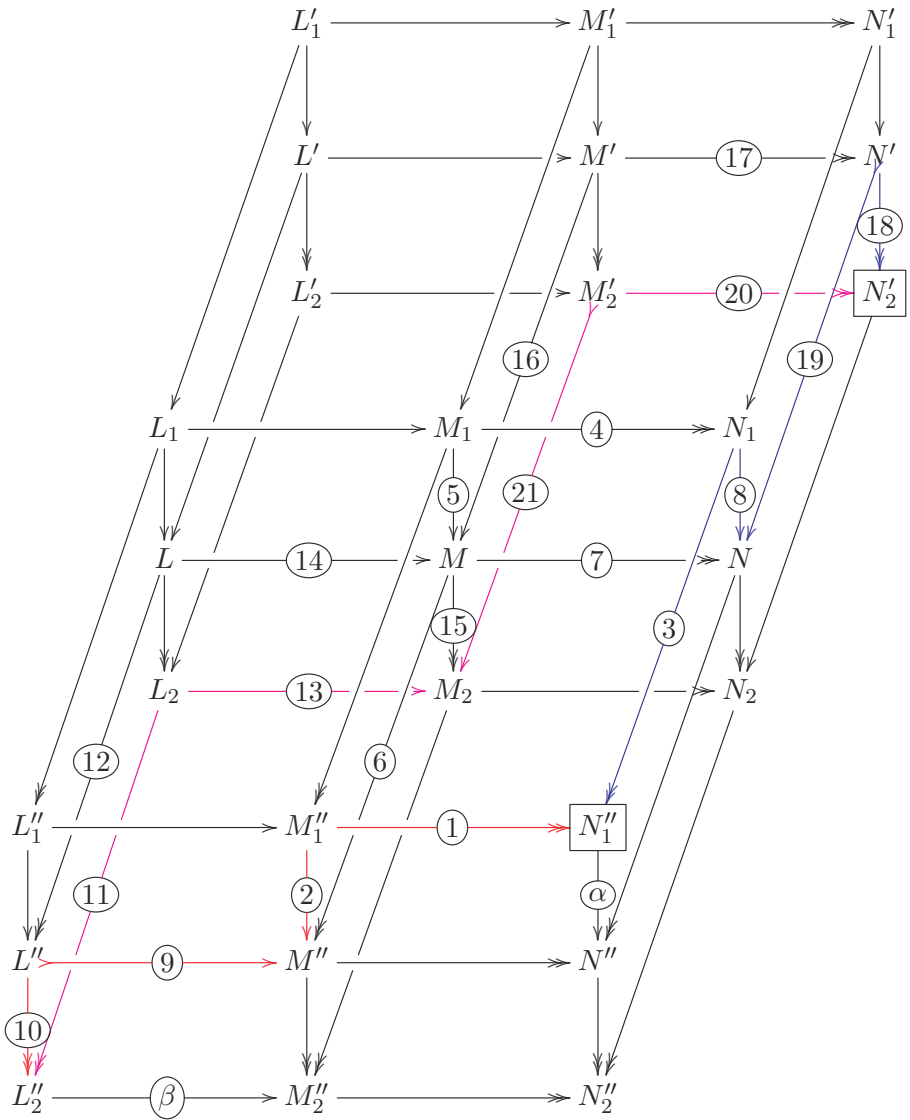


FIGURE 1.

the composition of the connecting homomorphisms $\text{Ker } \alpha \rightarrow N'_1$ and $\text{Ker } \beta \rightarrow L'_2$ (in the back face) coincides with the connecting homomorphism $\text{Ker } \gamma \rightarrow L'_2$ (in the left-hand face).

Proof. Diagram chase. □

5. The asymptotic stabilization: the first construction

5.1. The construction

Our next goal is to introduce what we shall call the *asymptotic stabilization* of the tensor product, which is a limit of a sequence of maps between injective stabilizations of tensor products of iterated syzygy and cosyzygy modules. This can be done in three equivalent ways, the first one being dealt with in this section.

We begin by constructing a homomorphism $\Omega A \overline{\otimes} \Sigma B \rightarrow A \overline{\otimes} B$ of abelian groups, where A is a right Λ -module and B is a left Λ -module. Choosing a projective resolution $P^\bullet \rightarrow A$ and an injective resolution $B \rightarrow I^\bullet$ and tensoring the short exact sequences

$$0 \rightarrow \Omega A \rightarrow P_0 \rightarrow A \rightarrow 0$$

and

$$0 \rightarrow B \rightarrow I^0 \rightarrow \Sigma B \rightarrow 0,$$

we have a commutative diagram of solid arrows whose rows, columns, and diagonal are exact:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \text{Tor}_1(A, \Sigma B) & & \\
 & & & \swarrow & \downarrow & \searrow & \\
 & & \Omega A \overline{\otimes} \Sigma B & \xrightarrow{\quad} & \Omega A \otimes \Sigma B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 \Omega A \otimes B & \longrightarrow & \Omega A \otimes I^0 & \longrightarrow & \Omega A \otimes \Sigma B & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow & & \\
 0 \longrightarrow & P_0 \otimes B & \longrightarrow & P_0 \otimes I^0 & \longrightarrow & P_0 \otimes \Sigma B & \longrightarrow & \Omega A \otimes I^1 \\
 \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Tor}_1(A, \Sigma B) \xrightarrow{\alpha} & A \otimes B & \longrightarrow & A \otimes I^0 & \longrightarrow & A \otimes \Sigma B & \longrightarrow & P_0 \otimes I^1 \\
 \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & & 0 & & 0 & &
 \end{array} \tag{5.1}$$

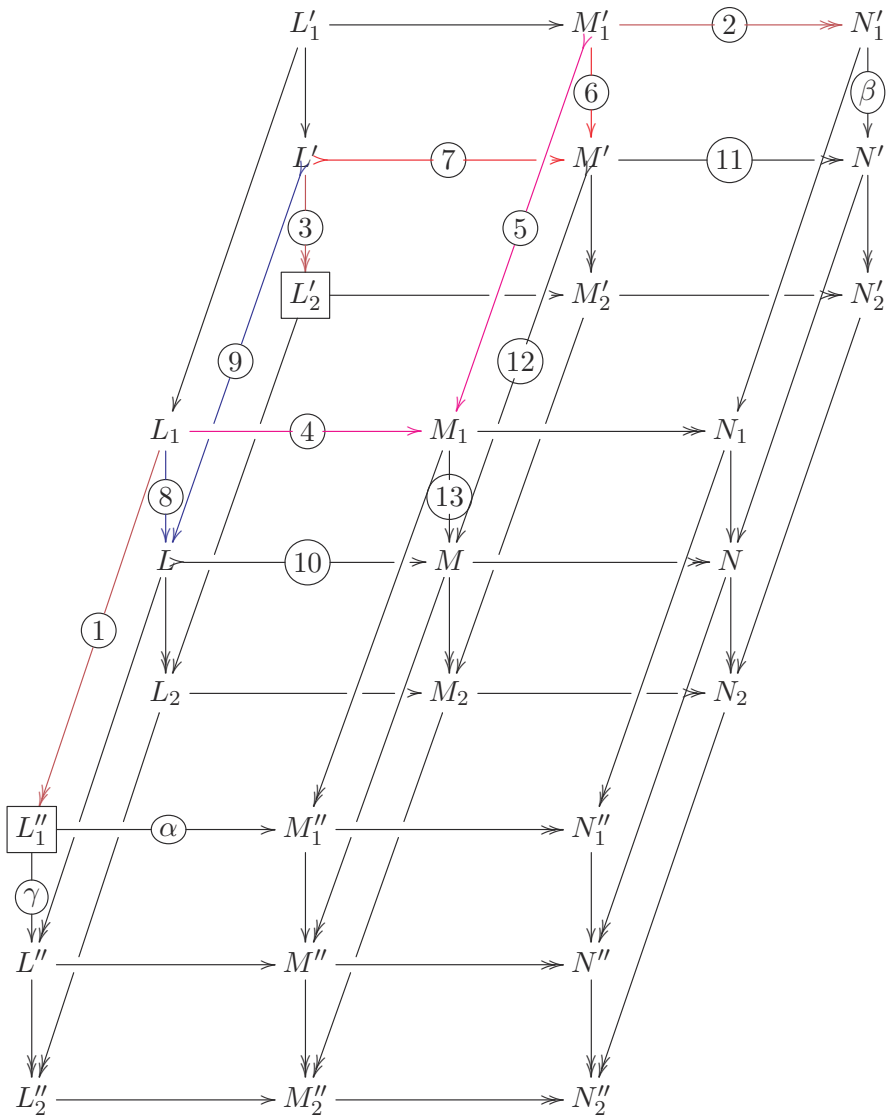


FIGURE 2.

As $P_0 \otimes -$ is an exact functor, the bottom southeast map is monic. Since the composition of this map with $\Omega A \overline{\otimes} \Sigma B \rightarrow \Omega A \otimes \Sigma B \rightarrow P_0 \otimes \Sigma B$ is zero, by the universal property of kernels, we have the dotted map, making the top triangle commute. Notice that this map is monic. Restricting the connecting homomorphism in the snake lemma, we have a map $\Omega A \overline{\otimes} \Sigma B \rightarrow A \overline{\otimes} B$. Iteration of this process yields a sequence

$$\dots \longrightarrow \Omega^2 A \overline{\otimes} \Sigma^2 B \xrightarrow{\Delta_2} \Omega A \overline{\otimes} \Sigma B \xrightarrow{\Delta_1} A \overline{\otimes} B. \tag{5.2}$$

Proposition 5.1.1. *The homomorphism $\Delta_1 : \Omega A \overline{\otimes} \Sigma B \rightarrow A \overline{\otimes} B$, and hence any Δ_i , is functorial in both A and B .*

Proof. Standard diagram chase together with the functoriality of $-\overline{\otimes}-$ in each argument [12, Lemmas 9.1 and 9.2]. \square

For any integer n (including negative values), the process of constructing the sequence (5.2) may be repeated with $\Omega^{k+n} A$ in place of $\Omega^k A$, yielding sequences

$$M_n(A, B) := \{\Omega^{k+n} A \overline{\otimes} \Sigma^k B, \Delta_{k+1}\}_{k, k+n \geq 0} \tag{5.3}$$

Definition 5.1.2. *The asymptotic stabilization $\overline{T}_n(A, -)$ of the left tensor product in degree n with coefficients in the right Λ -module A is*

$$\overline{T}_n(A, -)(B) := \overline{T}_n(A, B) := \varprojlim_{k, k+n \geq 0} \Omega^{k+n} A \overline{\otimes} \Sigma^k B = \varprojlim M_n(A, B)$$

It is easy to see that each \overline{T}_n is a bifunctor additive in each variable. We shall say that $\overline{T}_n(A, -)$ is the *active* asymptotic stabilization. Clearly, it is injectively stable, i.e., vanishes on injectives. We shall say that $\overline{T}_n(-, B)$ is the *inert* asymptotic stabilization. Clearly, it is projectively stable.

Next we observe that the \overline{T}_n allow infinite dimension shifts in both directions if one utilizes both arguments.

Lemma 5.1.3. *For all integers n and all nonnegative integers k , there are isomorphisms of functors*

$$\overline{T}_n(A, \Sigma^k -) \simeq \overline{T}_{n-k}(A, -) \quad \text{and} \quad \overline{T}_n(\Omega^k A, -) \simeq \overline{T}_{n+k}(A, -)$$

Proof. The sequences (including the structure maps) for the components of the former (respectively, latter) pair of functors at any right Λ -module can be obviously chosen to be shifts of each other. \square

Now we want to discuss the vanishing of the functors $\overline{T}_\bullet(A, -)$. The first result is an immediate consequence of the definitions.

Proposition 5.1.4. *If the right global dimension of Λ is finite then $\overline{T}_n(A, -) = 0$ for all integers n .* \square

Proposition 5.1.5. *If the flat dimension of A is finite, then $\overline{T}_n(A, -) = 0$ for all integers n .*

Proof. As the diagram (5.1) shows, we have an injection $\Omega A \overline{\otimes} \Sigma B \rightarrow \text{Tor}_1(A, \Sigma B)$. In particular, $\Omega^{n+k} A \overline{\otimes} \Sigma^k B$, $n+k, k \geq 1$ embeds in $\text{Tor}_1(\Omega^{n+k-1} A, \Sigma^k B)$. But the latter vanishes for $n+k-1 \geq \text{fl. dim } A$. \square

It is known that the vanishing of stable cohomology in one degree implies its vanishing in all degrees. We do not know if a similar statement is true for $\overline{T}_\bullet(A, -)$. A partial result is provided by

Proposition 5.1.6. *If $\overline{T}_n(A, -) = 0$ for some integer n , then $\overline{T}_m(A, -) = 0$ for all $m < n$. If, in addition, Λ is quasi-Frobenius, then $\overline{T}_m(A, -) = 0$ for all $m \in \mathbb{Z}$.*

Proof. The first assertion is an immediate consequence of the first isomorphism of Lemma 5.1.3. Suppose now that Λ is quasi-Frobenius. Since projective modules are injective, for any positive integer k , any right Λ -module B is a k th cosyzygy module in an injective resolution of $\Omega^k B$, i.e., $B \simeq \Sigma^k \Omega^k B$. Therefore,

$$\overline{T}_{n+k}(A, B) \cong \overline{T}_{n+k}(A, \Sigma^k \Omega^k B) \cong \overline{T}_n(A, \Omega^k B) = 0. \quad \square$$

5.2. The connectedness property

Now we want to define, for each short exact sequence $0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0$ of left Λ modules, homomorphisms $\omega_n : \overline{T}_n(A, B'') \rightarrow \overline{T}_{n-1}(A, B')$ making $(\overline{T}_\bullet(A, -), \omega_\bullet)$ a connected sequence of functors.³ By Lemma 5.1.3, it suffices to define $\omega_1 : \overline{T}_1(A, B'') \rightarrow \overline{T}_0(A, B')$.

³Any sequence of additive functors can be made connected by choosing the zero map as the connecting homomorphism. Our choice will be nonzero.

The snake lemma yields a map $\kappa_1^1 : \Omega A \overset{\rightharpoonup}{\otimes} B'' \longrightarrow \Omega A \overset{\rightharpoonup}{\otimes} \Sigma B'$. Similarly, we have maps $\kappa_1^i : \Omega^i A \overset{\rightharpoonup}{\otimes} \Sigma^{i-1} B'' \longrightarrow \Omega^i A \overset{\rightharpoonup}{\otimes} \Sigma^i B'$ for each natural i . The next step, as one would expect, is to show that the maps κ_1^i are compatible with the structure maps Δ . Actually, this is not true since the corresponding squares anticommute rather than commute. This motivates

Definition 5.2.1. *For each integer i , set $\omega_1^i := (-1)^i \kappa_1^i$.*

Notice that both the Δ and the κ are connecting homomorphisms in suitable diagrams. We now have

Theorem 5.2.2. *The pair $(\overset{\rightharpoonup}{T}_\bullet(A, -), \omega_\bullet)$, is a connected sequence of functors.*

Proof. We already remarked that $\overset{\rightharpoonup}{T}$ is an additive functor. Therefore, given an exact sequence of left Λ -modules $0 \longrightarrow B' \xrightarrow{\alpha} B \xrightarrow{\beta} B'' \longrightarrow 0$, the composition

$$\overset{\rightharpoonup}{T}_n(A, B') \xrightarrow{\overset{\rightharpoonup}{T}_n(A, \alpha)} \overset{\rightharpoonup}{T}_n(A, B) \xrightarrow{\overset{\rightharpoonup}{T}_n(A, \beta)} \overset{\rightharpoonup}{T}_n(A, B'')$$

of the induced maps is zero. The fact that $\overset{\rightharpoonup}{T}_{n-1}(A, \alpha) \circ \omega_n = 0$ follows from the snake lemma. For the same reason, $\omega_n \circ \overset{\rightharpoonup}{T}_n(A, \beta) = 0$. Thus it remains to show that the ω_n are functorial. But this follows from the functoriality of the connecting homomorphism in the snake lemma and Lemmas 4.1.1 and 4.2.1. \square

6. The asymptotic stabilization: the second construction

Next we want to show that the asymptotic stabilization $\overset{\rightharpoonup}{T}_\bullet(A, B)$ can be computed via the Tor functors, completely bypassing the need to use the injective stabilization of the tensor product. This approach uses non-functorial tools but has the advantage of being formulated in terms of familiar operations. It will also offer an intuitive perspective on Proposition 5.1.5.

6.1. The construction

We start with the definition of the Tor functor as the first left satellite of the tensor product

$$0 \longrightarrow \text{Tor}_1(A, -) \longrightarrow \Omega A \otimes - \longrightarrow P_0 \otimes - \longrightarrow A \otimes - \longrightarrow 0 \quad (6.1)$$

which forms the rightmost column of the diagram (5.1). Repeatedly shifting the first argument of the Tor functors by Ω and the second by Σ and gluing those diagrams by the connecting homomorphisms we end up with the commutative diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \text{Tor}_1(\Omega A, \Sigma^2 B) & \xrightarrow{\Gamma_2} & \text{Tor}_1(A, \Sigma B) & \xrightarrow{\Gamma_1} & A \otimes B \\
 & & \nearrow & & \nearrow & & \nearrow \\
 & & & \Delta_2 & & \Delta_1 & \\
 & \xrightarrow{\cdot} & \Omega A \otimes \overline{\Sigma B} & \xrightarrow{\Delta_1} & A \otimes \overline{B} & & \\
 & & & & & & \\
 & & & & & &
 \end{array} \tag{6.2}$$

As a result, we have

Theorem 6.1.1. *The sequence of the Tor functors in the above diagram is functorial in the first argument. For any integer n , the two families of parallel arrows in the (suitably shifted) above diagram induce mutually inverse isomorphisms⁴*

$$\overline{\text{T}}_n(A, -)(B) = \varprojlim_{k, k+n \geq 0} \Omega^{k+n} A \otimes \overline{\Sigma^k B} \simeq \varprojlim_{k, k+n \geq 0} \text{Tor}_1(\Omega^{k+n} A, \Sigma^{k+1} B)$$

All southeast maps are epic, and all northeast maps are monic.

Proof. The first assertion follows from the functoriality of the connecting homomorphism. The second assertion is immediate. The fact that all southeast maps are epic follows from the snake lemma, and the fact that all northeast maps are monic is already seen in the diagram (5.1). \square

Remark 6.1.2. Since all southeast maps are epic, all northeast maps are monic, and since an epi-mono factorization of a morphism in an abelian category is determined uniquely up to an isomorphism, the lower sequence is determined uniquely up to an isomorphism by the maps in the upper sequence. In particular, this yields new equivalent definitions of both the injective stabilization and the asymptotic stabilization of the tensor product.

Remark 6.1.3. While the top sequence results in a bifunctorial construction, the terms of that sequence are not functorial in the second variable. This is due to the fact that the symbol ΣB is only defined up to (injective) stable equivalence, but the Tor functor is not injectively stable and thus fails in general to preserve composition when combined with Σ .

⁴The reader has probably noticed that, as promised, this theorem implies Proposition 5.1.5.

Remark 6.1.4. A cosyzygy sequence for A gives rise to an exact sequence

$$\mathrm{Tor}_1(A, B) \longrightarrow \mathrm{Tor}_1(A, I^0) \longrightarrow \mathrm{Tor}_1(A, \Sigma B) \xrightarrow{\delta} A \overline{\otimes} B \longrightarrow 0,$$

where δ is (the corestriction of) the connecting homomorphism from the snake lemma. Thus if the injective I^0 is flat, then δ is an isomorphism.

Example 6.1.5. Suppose that Λ is quasi-Frobenius or, more generally, left IF (i.e., each injective left module is flat). Then, by Remark 6.1.4, the southeast maps are all isomorphisms, making the two systems isomorphic. The next example shows that the two systems may be isomorphic over other types of rings.

Example 6.1.6. Let $\Lambda := \mathbb{Z}$, $A := \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, and $B := \mathbb{Z}$. Then, taking $\Sigma B \simeq \mathbb{Q}/\mathbb{Z}$, we have, since the injective stabilization vanishes on injectives ([12, Lemma 4.5]), $\Omega A \overline{\otimes} \Sigma B = 0$.⁵ To compute $A \overline{\otimes} B$, we apply the functor $\mathbb{Z}/p\mathbb{Z} \otimes -$ to the injective envelope $\mathbb{Z} \rightarrow \mathbb{Q}$ of \mathbb{Z} . The kernel of the resulting map $\mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} \otimes \mathbb{Q}$ is $\mathbb{Z}/p\mathbb{Z}$. Finally, we compute $\mathrm{Tor}_1(A, \Sigma B)$ by using a projective resolution of $A = \mathbb{Z}/p\mathbb{Z}$. The result is the kernel of the map $\mathbb{Q}/\mathbb{Z} \xrightarrow{p} \mathbb{Q}/\mathbb{Z}$, which is the subgroup $\mathbb{Z}/p\mathbb{Z} = \{0, 1/p, \dots, (p-1)/p\}$. Moreover, the diagram (5.1) shows that, in this case, the map $\mathrm{Tor}_1(A, \Sigma B) \rightarrow A \overline{\otimes} B$ is an isomorphism. Since all the remaining terms in the two directed systems vanish, we have that the southeast maps in (6.2) make the two systems isomorphic.

6.2. The connectedness property

The second construction of the asymptotic stabilization also produces a connected sequence of functors. This is certainly true since the first and the second constructions produce, isomorphic results. But it can be shown that the second construction has a connecting homomorphism of its own and that it is compatible with the one from the first construction. The proof of this fact is based on the functorial doubly infinite exact sequence [12, (9.1)], Lemmas 4.1.1 and 4.2.1, and a standard diagram chase.

⁵Alternatively, since ΩA is projective, one can use [12, Lemma 4.8]).

7. The asymptotic stabilization: the third construction

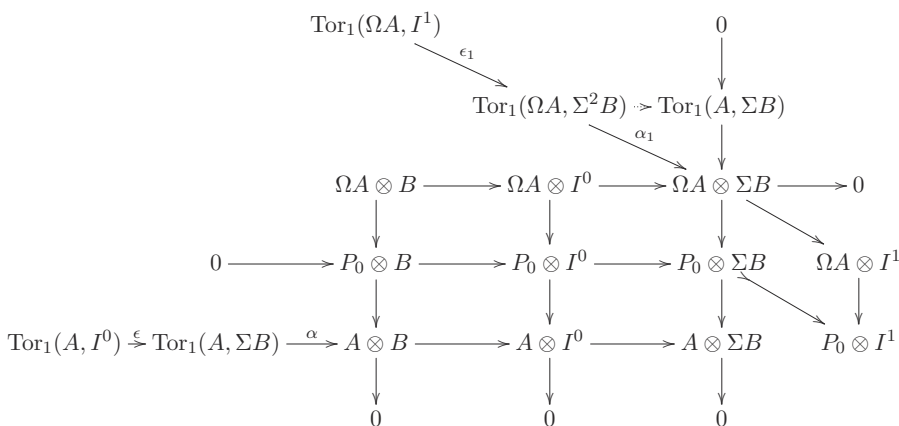
The goal of this section is to establish an isomorphism between the asymptotic stabilization of the tensor product and the J -completion of the univariate Tor .

7.1. Stabilization via satellites

The isomorphism $S^1\text{Tor}_1(A, -) \cong A \overline{\otimes} -$ [12, Proposition 9.3] (S^1 denotes the right satellite) suggests a stabilization sequence

$$\Delta_i : S^1\text{Tor}_1(\Omega^{i+1}A, -) \circ \Sigma^{i+1} \longrightarrow S^1\text{Tor}_1(\Omega^i A, -) \circ \Sigma^i.$$

Clearly, it suffices to do this for $i = 0$, and we shall again use the diagram (5.1). This yields a commutative diagram of solid arrows



with exact rows and columns. Moreover, the diagonal is a fragment of a long homology exact sequence and, at the same time, the bottom row of (5.1) with the fixed argument specialized to ΩA and with ΣB replaced by $\Sigma^2 B$ and I^0 replaced by I^1 . The diagram shows that α_1 factors through $\text{Tor}_1(A, \Sigma B)$, giving rise to a unique dotted map making a commutative triangle. As $\text{Tor}_1(A, \Sigma B) \rightarrow \Omega A \otimes \Sigma B$ is monic, the dotted map composed with ϵ_1 is zero, and therefore gives rise to a unique map

$$\text{Coker } \epsilon_1 = S^1\text{Tor}_1(\Omega A, -)(\Sigma B) \longrightarrow \text{Tor}_1(A, \Sigma B)$$

Composing it with the canonical epimorphism

$$\text{Tor}_1(A, \Sigma B) \longrightarrow \text{Coker } \epsilon = S^1\text{Tor}_1(A, -)(B),$$

we declare the resulting composition to be the structure map

$$\Delta_1 : S^1\mathrm{Tor}_1(\Omega A, -)(\Sigma B) \longrightarrow S^1\mathrm{Tor}_1(A, -)(B).$$

Similar arguments yield maps Δ_i for all natural i and the compatibility with the connecting homomorphisms. We now have

Theorem 7.1.1. *The connecting homomorphism in the diagram (5.1) induces an isomorphism of connected sequences of functors*

$$(S^1\mathrm{Tor}_1(\Omega^i A, -) \circ \Sigma^i, \Delta_i) \simeq (\Omega^i A \otimes \overline{\Sigma^i -}, \Delta_i)$$

natural in A . □

Passing to the componentwise limits in the foregoing isomorphisms, we have, in summary, that the three constructions of the asymptotic stabilization of the tensor product yield isomorphic connected sequences of functors.

Corollary 7.1.2. *The asymptotic stabilization $\overline{\mathrm{T}}(A, -)$ and the J -completion of the connected sequence $\mathrm{Tor}_*(A, -)$ are isomorphic as connected sequences of functors.*

Proof. This follows from the fact that the directed system involved in the construction of the J -completion [17] and the directed system used in the construction of the asymptotic stabilization are isomorphic. The isomorphism is precisely the one appearing in Theorem 7.1.1. □

Remark 7.1.3. For a more precise statement of the foregoing corollary see Proposition 8.2.1 below.

8. The comparison homomorphisms

At the moment we have three constructions of stable homology: Vogel homology, the J -completion of the univariate Tor functors, and the asymptotic stabilization of the tensor product. Our next goal is to compare them.

8.1. From Vogel homology to the asymptotic stabilization of the tensor product

First, we want to construct a natural transformation from Vogel homology to the asymptotic stabilization of the tensor product. This will

be done in degree zero; all other degrees are treated similarly. Let A be a right Λ -module with a projective resolution $(P, \partial_P) \rightarrow A$, and B be a left Λ -module with an injective resolution $B \rightarrow (I, \partial_I)$. Recall that the differential on $V_\bullet(A, B)$ is induced by $\partial_P \otimes 1 + (-1)^{\deg_1(-)} 1 \otimes \partial_I$ (see (3.1)). To simplify notation, we set

$$d_P := \partial_P \otimes 1 \quad \text{and} \quad d_I := 1 \otimes \partial_I.$$

A homology class in $V_0(A, B)$ can be represented by an infinite sequence

$$s = (s_i)_{i=1}^\infty \in (P_1 \otimes I^0) \times (P_2 \otimes I^1) \times \dots$$

which vanishes under the differential of $V_\bullet(A, B)$. This means that

$$D(s) = (d_P(s_1), -d_I(s_1) + d_P(s_2), d_I(s_2) + d_P(s_3), -d_I(s_3) + d_P(s_4), \dots)$$

represents the zero class in $V_{-1}(A, B)$ and therefore has only finitely many nonzero components. Let k be the smallest index such that for all $i \geq 0$

$$d_I(s_{k+i}) = (-1)^i d_P(s_{k+i+1})$$

Observe that since $s_{k+1} \in P_{k+1} \otimes I^k$, $d_P(s_{k+1}) \in P_k \otimes I^k$. Denote $d_P(s_{k+1})$ by \bullet in the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 \Omega^{k+1}A \otimes \Sigma^k B & \longrightarrow & \Omega^{k+1}A \otimes I^k & \longrightarrow & \Omega^{k+1}A \otimes \Sigma^{k+1} B & \longrightarrow & 0 \\
 \downarrow & & \downarrow \circ & & \downarrow & & \\
 0 \longrightarrow & P_k \otimes \Sigma^k B & \longrightarrow & P_k \otimes I^k & \longrightarrow & P_k \otimes \Sigma^{k+1} B & \longrightarrow 0 \\
 & \square & & \bullet & & \downarrow & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \Omega^k A \otimes \Sigma^k B & \longrightarrow & \Omega^k A \otimes I^k & \longrightarrow & \Omega^k A \otimes \Sigma^{k+1} B & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

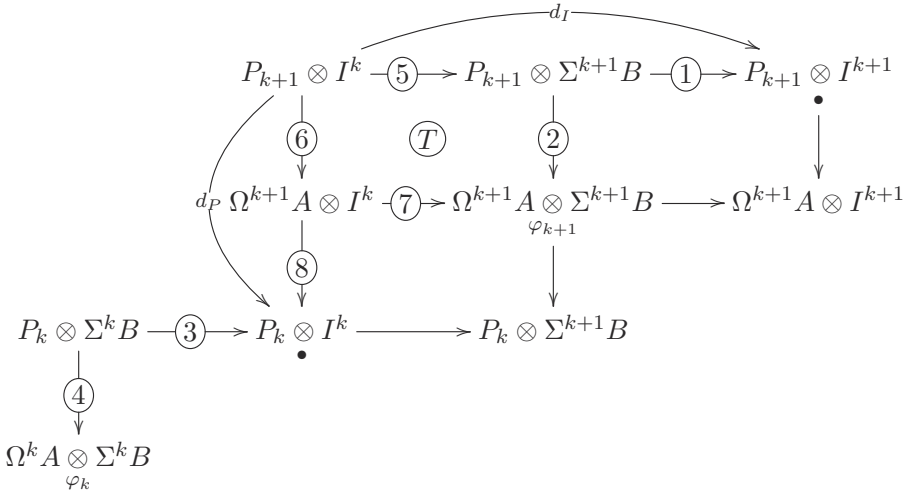
Since $\bullet = d_I(s_k)$, it pulls back to some element \square . Pushing it down, we produce $\omega_k \in \Omega^k A \otimes \Sigma^k B$. Since $\bullet = d_P(s_{k+1})$, the element \bullet is the image of some element \circ in $\Omega^{k+1}A \otimes I^k$. By the commutativity of the diagram, the image of ω_k in $\Omega^k A \otimes I^k$ is zero, i.e., $\omega_k \in \Omega^k A \otimes \overline{\Sigma^k B}$, and we set $\varphi_k := \omega_k$.

This process is well-defined up to choice of sign. To see this, notice that the element \bullet goes to 0 when applying the horizontal map. Hence,

by commutativity of the diagram, s_{k+1} also goes to 0 using the vertical top right map and hence is in the kernel of this map. Now one can apply the map from the snake lemma to produce the exact same element ω_k . Since we may also take the negative of this connecting homomorphism, we even have the freedom to choose $\pm\omega_k$. Once this choice is fixed, ω_k will be well-defined. Iterating this process, for any $i \geq 0$, we set

$$\varphi_{k+i} := \begin{cases} \omega_{k+i} & \text{if } i \equiv 0, 3 \pmod{4} \\ -\omega_{k+i} & \text{if } i \equiv 1, 2 \pmod{4}. \end{cases}$$

We claim that the sequence $(\varphi_k, \varphi_{k+1}, \dots)$ is coherent, i.e., in the notation of (5.2), $\Delta_n(\varphi_n) = \varphi_{n-1}$ for any $n \geq k + 1$. It suffices to check this claim for $n = k + 1$; the remaining cases are similar. To this end, we examine the commutative diagram



Here $(1) \circ (5) = d_I$, $(8) \circ (6) = d_P$, and the bullets denote

$$d_P(s_{k+2}) = -d_I(s_{k+1}) = -(1) \circ (5)(s_{k+1}) \quad \text{and} \quad d_P(s_{k+1}).$$

The element φ_{k+1} is obtained from the upper bullet by applying $(2) \circ (1)^{-1}$ and φ_k is obtained from the lower bullet by applying $(4) \circ (3)^{-1}$. Since the square T commutes,

$$\varphi_{k+1} = -(2) \circ (1)^{-1}(d_P(s_{k+2})) = -(7) \circ (6) \circ (5)^{-1} \circ (1)^{-1}(d_P(s_{k+2})).$$

Recalling the construction of Δ_{k+1} (this is just the restriction of the connecting homomorphism in the diagram (5.1)), one easily checks that

$\Delta_{k+1}(\varphi_{k+1}) = \varphi_k$. Thus, the sequence $(\varphi_k, \varphi_{k+1}, \dots)$ is coherent. It uniquely extends to a coherent sequence $(\varphi_i)_{i=0}^\infty$, and we set $\kappa_0(s) := (\varphi_i)_{i=0}^\infty$. A similar argument yields $\kappa_l : V_l(A, -) \rightarrow \overline{T}_l(A, -)$ for each integer l .

Theorem 8.1.1. *Let A be a right Λ -module. For each $l \in \mathbb{Z}$,*

$$\kappa_l : V_l(A, -) \rightarrow \overline{T}_l(A, -)$$

is an epic natural transformation.

Proof. The naturality follows from that of the connecting homomorphism. The epic part is primarily a diagram chase and is left to the reader. \square

8.2. From the asymptotic stabilization of the tensor product to the J -completion of the univariate Tor

Let U be a connected sequence of functors and $M_\bullet(U)$ its J -completion (see 3.2). In [17, Proposition 6.1.2], Triulzi shows that there is a morphism of connected sequences of functors $\tau : M_\bullet(U) \rightarrow U$ satisfying the following universal property. Given any morphism $\beta : V \rightarrow U$, where V is a connected sequence of functors that is injectively stable in all degrees, there exists a unique morphism $\phi : V \rightarrow M_\bullet(U)$ such that $\phi\tau = \beta$. From this, we can now establish a commutative diagram of comparison maps between Vogel homology, the asymptotic stabilization of the tensor product, and the J -completion of the univariate Tor functor.

Proposition 8.2.1. *For any module A , there is a commutative diagram of connected sequences of functors*

$$\begin{array}{ccc} V_\bullet(A, -) & \xrightarrow{\kappa} & \overline{T}_\bullet(A, -) \\ \theta \downarrow & \swarrow \begin{smallmatrix} \cong \\ \varepsilon \end{smallmatrix} & \downarrow \lambda \\ M_\bullet(\text{Tor}(A, -)) & \xrightarrow{\tau} & \text{Tor}(A, -) \end{array}$$

where τ is the J -completion of $\text{Tor}(A, -)$. Moreover, $\lambda\kappa$ is the canonical natural transformation from Vogel homology to Tor.

Proof. For λ , take the natural transformation from the limit to the first term. The diagonal isomorphism ε is taken from Corollary 7.1.2. Under that isomorphism, λ is identified with τ , i.e., the lower triangle commutes. Now set $\theta := \varepsilon\kappa$, thus making the whole square commute. The last assertion is verified by a direction calculation. \square

Since the connected sequence of functors $V_\bullet(A, -)$ is J -complete, the universal property of the J -completion yields

Corollary 8.2.2. *θ is the unique lifting of the canonical natural transformation $\lambda\kappa : V_\bullet(A, -) \rightarrow \text{Tor}(A, -)$ against τ . \square*

As an immediate application of Proposition 8.2.1 and Corollary 8.2.2, we have (see also [17, Corollary 6.2.10] and [16, Theorem 69])

Proposition 8.2.3. *The comparison map from Vogel homology to the J -completion of Tor is epic in each degree. \square*

In view of the foregoing result, it is natural to try and identify the kernel of $\kappa : V_\bullet(A, -) \rightarrow \overline{T}_\bullet(A, -)$ or, equivalently, of $\theta : V_\bullet(A, -) \rightarrow M_\bullet(\text{Tor}(A, -))$. Driven by a formal analogy between κ and the natural transformation from Steenrod-Sitnikov homology to Čech homology, the first author conjectured in 2014 that the kernel of κ should be given by a derived limit. The following recent result of I. Emmanouil and P. Manousaki shows that this is indeed the case.

Theorem 8.2.4 ([8], Theorem 2.2). *There is an exact sequence*

$$0 \longrightarrow \varprojlim_i^1 \text{Tor}_{\bullet+i+1}(A, \Sigma^i -) \longrightarrow V_\bullet(A, -) \longrightarrow M_\bullet(\text{Tor}(A, -)) \longrightarrow 0. \quad \square$$

9. (Co)completeness of finitely presented functors

Our goal in this section is to show that the category $fp(\Lambda\text{-Mod}, \text{Ab})$ of finitely presented functors is complete and cocomplete. It appears that these results, stated in the different language of left-exact sequences, were first established by Ron Gentle [10, Remark 1.3. (b)]. The proofs presented here are different as we work directly in the functor categories.

9.1. The category of finitely presented covariant functors is complete

Yoneda's lemma classifies natural transformations from a representable functor to an arbitrary functor. It is also possible to describe natural transformations going in the opposite direction when the arbitrary functor is replaced by a finitely presented one. Thus, let $(X, -) \rightarrow (Y, -) \rightarrow F \rightarrow 0$ be exact, where the first map is of the form $(f, -)$ for some $f : Y \rightarrow X$. $\text{Ker } f$ is called the defect of F , denoted by $w(F)$. Given a

representable functor $(Z, -)$ we take natural transformations into it from the above presentation of F , which results in an exact sequence

$$0 \longrightarrow (F, (Z, -)) \longrightarrow (Z, Y) \longrightarrow (Z, X).$$

On the other hand, mapping Z into the exact sequence $0 \longrightarrow w(F) \longrightarrow Y \longrightarrow X$, we have an exact sequence

$$0 \longrightarrow (Z, w(F)) \longrightarrow (Z, Y) \longrightarrow (Z, X).$$

Comparing the two sequences, we have

Lemma 9.1.1 (The co-Yoneda lemma). *There is a binatural isomorphism*

$$(F, (Z, -)) \cong (Z, w(F)).$$

*In other words, the contravariant Yoneda embedding $\Upsilon : \Lambda\text{-Mod} \longrightarrow fp(\Lambda\text{-Mod}, \text{Ab})$ and the defect are adjoint to each other on the right.*⁶ \square

For later use, we also recall

Lemma 9.1.2. *The defect and the zeroth left-derived functor of the contravariant Yoneda's embedding are adjoint to each other on the left, i.e., for any finitely presented covariant functor F and any module A there is a binatural isomorphism*

$$(L^0\Upsilon(A), F) \simeq (w(F), A).$$

Proof. Follows from the definition of the zeroth left-derived functor, Yoneda's lemma, and the fact that $(w(F), -) \simeq R^0F$. \square

Combining the previous two lemmas, we have

Proposition 9.1.3. *The defect is a (contravariant) biadjoint. In particular, it interchanges limits and colimits.* \square

Proposition 9.1.4. *The category of finitely presented covariant functors on Λ -modules is complete and limits can be computed componentwise.*

Proof. The category of finitely presented covariant functors is abelian with kernels and cokernels defined componentwise. Thus it suffices to show that this category has products and that products can be computed componentwise. First, we show this for products of representable functors. More precisely, we claim that the desired product $\prod(X_i, -)$ is just $(\prod X_i, -)$ ⁷ with structure maps induced by the canonical injections

⁶It is not difficult to show that this isomorphism is canonical, i.e., independent of the chosen presentation of F . This is the reason for using the \cong sign rather than \simeq .

⁷The reader is cautioned against making a claim that this is obvious. This is not obvious and requires a proof because the product on the left should be taken in a functor category.

$\iota_i : X_i \rightarrow \coprod X_i$. To see that, let F be a finitely presented functor and suppose we have a family of natural transformations $\alpha_i : F \rightarrow (X_i, -)$. As we just saw, each α_i is uniquely determined by an element of $(X_i, w(F))$, which we denote again by α_i . These elements give rise to a unique $\beta \in (\coprod X_i, w(F)) \cong (F, (\coprod X_i, -))$ such that $\beta \iota_i = \alpha_i$ for each i . Switching back to functors and natural transformations, we have a unique β such that $(\iota_i, -)\beta = \alpha_i$ for each i . This establishes the claim for representable functors.⁸ Moreover, as the contravariant Hom converts coproducts in each component to a product in abelian groups, the products of representables can be computed componentwise. Since AB4^* holds in the category of abelian groups, it now follows that products of finitely presented functors exist and are computed componentwise. \square

9.2. The category of finitely presented covariant functors is cocomplete

Lemma 9.2.1. *Finitely presented covariant functors commute with products.*

Proof. This follows from the facts that covariant representable functors have this property and products preserve epimorphisms in abelian groups. \square

Theorem 9.2.2. *The category $fp(\Lambda\text{-Mod}, \text{Ab})$ of finitely presented functors is cocomplete.*

Proof. It is convenient to introduce the following notation. Given a natural transformation $\alpha : (A, -) \rightarrow F$, set $\mathbf{b}_\alpha := \alpha_A(1_A) \in F(A)$. By Yoneda’s lemma, this is the element that uniquely determines α .

First we show that $fp(\Lambda\text{-Mod}, \text{Ab})$ has coproducts and we begin with coproducts of representables. Let $\{X_i\}_{i \in I}$ be an arbitrary family of modules. Associated with it is the family $\{(X_i, -)\}_{i \in I}$ of projectives in $fp(\Lambda\text{-Mod}, \text{Ab})$. Let $\pi_j : \coprod X_i \rightarrow X_j$ be the canonical projections. We now claim that the family

$$(\pi_j, -) : (X_j, -) \rightarrow (\coprod X_i, -)$$

is a coproduct of $\{(X_i, -)\}_{i \in I}$. To show this, for any functor F and any family of natural transformations $\beta_j : (X_j, -) \rightarrow F$, we need to find a

⁸Using a more conceptual language, we have just shown that the contravariant Yoneda embedding converts coproducts into products.

natural transformation $\alpha : (\prod X_i, -) \rightarrow F$ making each diagram

$$\begin{array}{ccc} (X_j, -) & \xrightarrow{(\pi_j, -)} & (\prod X_i, -) \\ & \searrow \beta_j & \swarrow \alpha \\ & F & \end{array}$$

commute. Each β_j is determined by $\mathbf{b}_{\beta_j} \in F(X_j)$. By Lemma 9.2.1, the canonical map $f : F(\prod X_i) \rightarrow \prod F(X_i)$ in the commutative diagram

$$\begin{array}{ccc} F(\prod X_i) & \xrightarrow{F(\pi_j)} & F(X_j) \\ f \downarrow \cong & \nearrow p_j & \\ \prod F(X_i) & & \end{array}$$

is an isomorphism. We can now define α by setting $\mathbf{b}_\alpha := f^{-1}(\prod \mathbf{b}_{\beta_i})$. By Yoneda's lemma, it suffices to check that $F(\pi_j)(\mathbf{b}_\alpha) = \mathbf{b}_{\beta_j}$ for each j , i.e.,

$$F(\pi_j)(f^{-1}(\prod \mathbf{b}_{\beta_i})) = \mathbf{b}_{\beta_j},$$

which is immediate from the commutative diagram above. We have thus shown that the family $(\pi_i, -) : (X_i, -) \rightarrow (\prod X_i, -)$ is a coproduct of the $(X_i, -)$ and, in particular, the coproduct is represented by $\prod X_i$.⁹

Now we can move on to the case of arbitrary finitely presented functors. Let $\{F_i\}_{i \in I}$ be a family of finitely presented functors with presentations

$$(Y_i, -) \rightarrow (X_i, -) \rightarrow F_i \rightarrow 0$$

We claim that the cokernel of the induced natural transformation

$$\prod (Y_i, -) \rightarrow \prod (X_i, -)$$

is the desired coproduct of the F_i . This follows from a general fact: if in an abelian category there is a coproduct of a family of morphisms, then the cokernel of this coproduct is the coproduct of the corresponding cokernels. In summary, we have a defining exact sequence

$$(\prod Y_i, -) \rightarrow (\prod X_i, -) \rightarrow \prod F_i \rightarrow 0 \quad (9.1)$$

As a result, we have that the category of finitely presented functors has coproducts. On the other hand, analogous to equalizers, coequalizers exist in this category. It now follows that $fp(\Lambda\text{-Mod}, \text{Ab})$ is cocomplete. \square

⁹Using a more conceptual language, we have just shown that the contravariant Yoneda embedding converts products into coproducts.

Remark 9.2.3. Because the direct product cannot be taken out of the contravariant argument of the Hom functor (i.e., the contravariant Hom functor does not convert direct products into direct sums), the coproduct, and therefore colimits, of finitely presented functors are not computed componentwise.

10. Coherence and defect of the inert asymptotic stabilization

10.1. Coherence of the inert asymptotic stabilization

We now turn attention to the inert univariate functor determined by \overline{T} and give a sufficient condition for it to be finitely presented. We begin by recalling some known (at least to the experts) preliminary results. Recall that a functor is said to be finitely presented if it is a cokernel of a natural transformation between representable functors.

Lemma 10.1.1. [2, Lemma 6.1] *If the left Λ -module B is finitely presented, then so is the functor $-\otimes B$.*¹⁰

Proof. Let $P_1 \rightarrow P_0 \rightarrow B \rightarrow 0$ be a finite presentation. By the right-exactness of the tensor product, the sequence $-\otimes P_1 \rightarrow -\otimes P_0 \rightarrow -\otimes B \rightarrow 0$ is exact. The duality for finitely generated projective modules yields a finite presentation $(P_1^*, -) \rightarrow (P_0^*, -) \rightarrow -\otimes B \rightarrow 0$. \square

Lemma 10.1.2. *If B is FP_2 , then $\text{Tor}_1(-, B)$ is finitely presented. More generally, if B is FP_{n+1} , $n \geq 1$, then $\text{Tor}_n(-, B)$ is finitely presented.*

Proof. By assumption, we have a syzygy sequence $0 \rightarrow \Omega B \rightarrow P \rightarrow B \rightarrow 0$, where all modules are finitely presented. The corresponding long exact sequence

$$0 \rightarrow \text{Tor}_1(-, B) \rightarrow -\otimes \Omega B \rightarrow -\otimes P \rightarrow -\otimes B \rightarrow 0$$

and Lemma 10.1.1 show that $\text{Tor}_1(-, B)$, being the kernel of a natural transformation between finitely presented functors, is finitely presented. The general case can now be treated by dimension shift. \square

Proposition 10.1.3. *Suppose that a left Λ -module B is FP_∞ and has an injective resolution (I^i, d^i) such that all I^i are also FP_∞ . Then, for all nonnegative integers l and i , the functors $\Omega^l - \overline{\otimes} \Sigma^i B$ are finitely presented.*

¹⁰The converse is also true, [ibid.].

Proof. Since injective stabilization vanishes on injectives and cosyzygy modules are defined up to (injectively) stable equivalence, we may assume that the $\Sigma^i B$ are computed using the resolution (I^i, d^i) . The cosyzygy sequences $0 \rightarrow \Sigma^i B \rightarrow I^i \rightarrow \Sigma^{i+1} B \rightarrow 0$ show that all cosyzygy modules of B are FP_∞ . Each such sequence gives rise to a long exact sequence of Tor functors, which yields a presentation

$$\text{Tor}_1(\Omega^l -, I^i) \rightarrow \text{Tor}_1(\Omega^l -, \Sigma^{i+1} B) \rightarrow \Omega^l - \overline{\otimes} \Sigma^i B \rightarrow 0.$$

Rewriting it as

$$\text{Tor}_1(-, \Omega^l I^i) \rightarrow \text{Tor}_1(-, \Omega^l \Sigma^{i+1} B) \rightarrow \Omega^l - \overline{\otimes} \Sigma^i B \rightarrow 0$$

and using Lemma 10.1.2, we have the desired result. □

Theorem 10.1.4. *Suppose that a left Λ -module B is FP_∞ and that it has an injective resolution all of whose terms are also FP_∞ . Then the inert asymptotic stabilizations $\overline{T}_n(-, B)$, $n \in \mathbb{Z}$, are finitely presented.*

Proof. By dimension shift, it suffices to assume that $n = 0$. Since \overline{T}_0 is a bifunctor, for any right Λ -module A , we have $\overline{T}_0(-, B)(A) \simeq \overline{T}_0(A, -)(B)$, which is $\varprojlim_{k \geq 0} (\Omega^k A \overline{\otimes} \Sigma^k B)$. By Proposition 9.1.4, the latter is just $\varprojlim_{k \geq 0} (\Omega^k - \overline{\otimes} \Sigma^k B)(A)$. The constructed isomorphism is functorial in A and therefore we have a functor isomorphism

$$\overline{T}_0(-, B) \simeq \varprojlim_{k \geq 0} (\Omega^k - \overline{\otimes} \Sigma^k B).$$

Propositions 10.1.3 and 9.1.4 now show that $\overline{T}_0(-, B)$ is finitely presented. □

As an immediate consequence of the just proved result, we have

Theorem 10.1.5. *If B is a finitely generated module over an artin algebra, then the functors $\overline{T}_n(-, B)$, $n \in \mathbb{Z}$ are finitely presented.* □

10.2. The defect of the inert asymptotic stabilization

In view of the foregoing theorems, it is natural to try and describe the defect of the inert stabilization when it is finitely presented. To this end, we first establish an auxiliary result. Given a right module A , choose a syzygy sequence $0 \rightarrow \Omega A \rightarrow P \rightarrow A \rightarrow 0$. Given a left module B , choose a cosyzygy sequence $0 \rightarrow B \rightarrow I \rightarrow \Sigma B \rightarrow 0$ and a syzygy sequence $0 \rightarrow \Omega \Sigma B \rightarrow Q \rightarrow \Sigma B \rightarrow 0$. Lifting the identity map on ΣB , we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Omega \Sigma B & \longrightarrow & Q & \longrightarrow & \Sigma B \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & B & \longrightarrow & I & \longrightarrow & \Sigma B \longrightarrow 0
 \end{array} \tag{10.1}$$

In the leftmost vertical map we replace B with ΣB to obtain a map $\Omega \Sigma^2 B \rightarrow \Sigma B$. Applying Ω , we have a map $\Omega^2 \Sigma^2 B \rightarrow \Omega \Sigma B$. Iterating this process and applying the functor $\text{Tor}_1(A, -)$ we have a sequence

$$\dots \longrightarrow \text{Tor}_1(A, \Omega^2 \Sigma^2 B) \longrightarrow \text{Tor}_1(A, \Omega \Sigma B) \longrightarrow \text{Tor}_1(A, B).$$

Finally, replacing B with ΣB , we have the sequence

$$\dots \longrightarrow \text{Tor}_1(A, \Omega^2 \Sigma^3 B) \longrightarrow \text{Tor}_1(A, \Omega \Sigma^2 B) \longrightarrow \text{Tor}_1(A, \Sigma B). \tag{10.2}$$

Now we recall the second construction of the asymptotic stabilization and the intertwining diagram (6.2).

Lemma 10.2.1. *The sequence (10.2) is isomorphic to the top row of (6.2).*

Proof. The proof can be accomplished by tensoring the diagram (10.1) with the syzygy sequence $0 \rightarrow \Omega A \rightarrow P \rightarrow A \rightarrow 0$, doing a diagram chase, and using the balance of the bifunctor Tor . The tedious but more or less straightforward details are left to the reader. \square

Before we can describe the defect of the asymptotic stabilization, we need to compute the defect of the univariate Tor functor.

Lemma 10.2.2. *Suppose $B \in \text{mod-}\Lambda$ is FP_2 . Then $w(\text{Tor}_1(-, B)) \simeq \text{Ext}^1(B, \Lambda)$, where the isomorphism is functorial in B .*

Proof. Since B is FP_2 , we can find a syzygy sequence $0 \rightarrow \Omega B \rightarrow Q \rightarrow B \rightarrow 0$ all of whose terms are finitely presented. Dualizing it into Λ we have an exact sequence $0 \rightarrow B^* \rightarrow Q^* \rightarrow (\Omega B)^* \rightarrow \text{Ext}^1(B, \Lambda) \rightarrow 0$. On

the other hand, the same syzygy sequence gives rise to an exact sequence of functors

$$0 \longrightarrow \mathrm{Tor}_1(-, B) \longrightarrow - \otimes \Omega B \longrightarrow - \otimes Q \longrightarrow - \otimes B \longrightarrow 0.$$

By Lemmas 9.1.1 and 9.1.2, w is a contravariant biadjoint (see also [13, the diagram after Theorem 4.2]) and hence exact. Applying it to the sequence above and using [12, Example 3.14], we have an exact sequence $0 \rightarrow B^* \rightarrow Q^* \rightarrow (\Omega B)^* \rightarrow w(\mathrm{Tor}_1(-, B)) \rightarrow 0$. The isomorphism claim now follows. The functoriality claim is clear. \square

We are now ready to describe the defect of the asymptotic stabilization.

Theorem 10.2.3. *Suppose that a left Λ -module B is FP_∞ and that it has an injective resolution all of whose terms are also FP_∞ . Then*

$$w(\overline{\mathrm{T}}_0(-, B)) \simeq \varinjlim \mathrm{Ext}^1(\Omega^i \Sigma^{i+1} B, \Lambda)$$

where the colimit is taken over the sequence of iterations of the map from (10.1). In particular, this formula applies to an arbitrary finitely generated module over an arbitrary artin algebra.

Proof.

$$\begin{aligned} w(\overline{\mathrm{T}}_0(-, B)) &\simeq w\left(\varprojlim \mathrm{Tor}_1(\Omega^i -, \Sigma^{i+1} B)\right) && \text{(by Theorem 6.1.1)} \\ &\simeq w\left(\varprojlim \mathrm{Tor}_1(-, \Omega^i \Sigma^{i+1} B)\right) && \text{(by Lemma 10.2.1)} \\ &\simeq \varinjlim w(\mathrm{Tor}_1(-, \Omega^i \Sigma^{i+1} B)) && \text{(since } w \text{ is a biadjoint)} \\ &\simeq \varinjlim \mathrm{Ext}^1(\Omega^i \Sigma^{i+1} B, \Lambda) && \text{(by Lemma 10.2.2)} \end{aligned}$$

\square

Remark 10.2.4. The defect in question can also be written as

$$w(\overline{\mathrm{T}}_0(-, B)) \simeq \varinjlim \mathrm{Ext}^1(\Sigma^{i+1} B, \Sigma^i \Lambda).$$

Replacing the syzygy endofunctor on the projectively stable category by the cosyzygy endofunctor on the injectively stable category, and universally inverting Σ , we have what we may call Buchweitz cohomology W^\bullet based on injectives. (See [15] for more details on this construct.) Arguments similar to the ones preceding Theorem 6.1.1 show that the universal inversion of

Σ can be replaced by a stabilization of Ext^1 (which replaces Hom modulo injectives). This leads to a surprising description of the defect:

$$w(\overline{T}_0(-, B)) \simeq W^0(B, \Lambda) = \varinjlim (\overline{\Sigma^i B}, \overline{\Sigma^i \Lambda}).$$

The reader is invited to compare this formula with that for the defect of the tensor product ([12, Example 3.14]): $w(- \otimes B) \simeq (B, \Lambda) = B^*$.

The just proved theorem immediately leads to a description of the right-derived functors of $(\overline{T}_0(-, B))$. To see this, recall ([2, top of page 210]) that, for a finitely presented functor F , the natural transformation $R^0 F \rightarrow (w(F), -)$ is an isomorphism. Since, for each n , the natural transformation $R^n F \rightarrow R^n R^0 F$ is always an isomorphism, we have

Corollary 10.2.5. *Under the assumptions of the theorem, the natural transformation*

$$\begin{aligned} R^n(\overline{T}_0(-, B)) &\longrightarrow \text{Ext}^n(\varinjlim \text{Ext}^1(\Omega^i \Sigma^{i+1} B, \Lambda), -) \\ &\simeq \text{Ext}^n(W^0(B, \Lambda), -) \end{aligned}$$

is an isomorphism for all n . □

In view of Remark 10.2.4, we have the following.

Corollary 10.2.6. *Under the assumption of the theorem, if Λ is of finite injective dimension as a left module over itself, then all derived functors of $\overline{T}_0(-, B)$ are zero.* □

Now recall the torsion functor $\mathfrak{s} = - \overline{\otimes} \Lambda$ introduced in [13]. The foregoing discussion motivates

Definition 10.2.7. *The functor*

$$\mathfrak{s}_\infty := \overline{T}_0(-, \Lambda) \simeq \varprojlim_{k \geq 0} (\Omega^k - \overline{\otimes} \Sigma^k \Lambda)$$

is called the asymptotic torsion functor.¹¹

For the next result, recall the cotorsion functor $\mathfrak{q} = (\overline{\Lambda}, -)$ introduced in [13] and defined on left modules. It is now natural to introduce

¹¹This is a functor on right modules. A similar definition applies to left modules.

Definition 10.2.8. *The functor*

$$\mathfrak{q}^\infty := W^0(\Lambda, -) = \varinjlim (\overline{\Sigma^i \Lambda}, \overline{\Sigma^i -})$$

may therefore be called the asymptotic cotorsion functor.¹²

Theorem 10.2.9. *Suppose that Λ , viewed as a left module over itself, has an injective resolution all of whose terms are FP_∞ . Then*

$$w(\mathfrak{s}_\infty) \simeq \mathfrak{q}^\infty({}_\Lambda \Lambda),$$

i.e., the defect of the asymptotic torsion on right modules is isomorphic to the asymptotic cotorsion of Λ viewed as a left module over itself. \square

Remark 10.2.10. The reader should compare this formula with [13, Corollary 5.4] showing that

$$w(\mathfrak{s}) \simeq \mathfrak{q}({}_\Lambda \Lambda)$$

when the injective envelope of ${}_\Lambda \Lambda$ is finitely presented.

Specializing Corollary 10.2.5 to $B := {}_\Lambda \Lambda$, we have

Corollary 10.2.11. *Under the assumptions of the theorem, the canonical natural transformation*

$$R^n \mathfrak{s}_\infty \longrightarrow \text{Ext}_\Lambda^n(\mathfrak{q}^\infty({}_\Lambda \Lambda), -)$$

is an isomorphism for all n . \square

References

- [1] J. F. Adams. *Infinite loop spaces*, volume 90 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.
- [2] M. Auslander. Coherent functors. In *Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965)*, pages 189–231. Springer, New York, 1966.
- [3] M. Auslander. Anneaux de Gorenstein, et torsion en algèbre commutative. *Séminaire d'Algèbre Commutative dirigé par Pierre Samuel (Paris, 1967)*
- [4] M. Auslander and M. Bridger. *Stable module theory*. Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969.
- [5] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

¹²A similar definition applies to right modules.

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- [6] R.-O. Buchweitz. *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*. 1986. Available at: <http://hdl.handle.net/1807/16682>
- [7] H. Cartan and S. Eilenberg. *Homological algebra*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. With an appendix by David A. Buchsbaum, Reprint of the 1956 original.
- [8] I. Emmanouil and P. Manousaki. On the stable homology of modules. *J. Pure Appl. Algebra*, 221(9):2198–2219, 2017.
- [9] F. T. Farrell. An extension of Tate cohomology to a class of infinite groups. *J. Pure Appl. Algebra*, 10(2):153–161, 1977/78.
- [10] R. Gentle. T.T.F. theories for left and right exact sequences. *J. Pure Appl. Algebra*, 75(3):237–258, 1991.
- [11] F. Goichot. Homologie de Tate-Vogel équivariante. *J. Pure Appl. Algebra*, 82(1):39–64, 1992.
- [12] A. Martsinkovsky and J. Russell. Injective stabilization of additive functors. I. Preliminaries. *J. Algebra* 530:429-469, 2019.
- [13] A. Martsinkovsky and J. Russell. Injective stabilization of additive functors. II. (Co)torsion and the Auslander-Gruson-Jensen functor. *J. Algebra*, 548:53–95, 2020.
- [14] G. Mislin. Tate cohomology for arbitrary groups via satellites. *Topology Appl.*, 56(3):293–300, 1994.
- [15] B. E. A. Nucinkis. Complete cohomology for arbitrary rings using injectives. *J. Pure Appl. Algebra*, 131(3):297–318, 1998.
- [16] J. Russell. *A functorial approach to linkage and the asymptotic stabilization of the tensor product*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)—Northeastern University.
- [17] M. E. Triulzi. Completion constructions in homological algebra and finiteness conditions. ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Dr.sc.math.)—Eidgenössische Technische Hochschule Zuerich (Switzerland).

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