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Norm of Gaussian integers in arithmetical progressions and narrow sectors

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Abstract. We proved the equidistribution of the Gaussian integer numbers in narrow sectors of the circle of radius $x^{\frac{1}{2}}$, $x \to \infty$, with the norms belonging to arithmetic progression $N(\alpha) \equiv \ell \pmod q$ with the common difference of an arithmetic progression q, $q \ll x^{\frac{2}{3}-\varepsilon}$.

Introduction

For the classical arithmetic functions $\tau(n)$ (the number of divisors for the positive integer n) and $r(n)$ (the number of representations for the positive integer n as sum of two squares of integers) there were obtained the asymptotic formulas of the sums

$$\sum_{\substack{n\equiv\ell \pmod q\\ n\leqslant x}} \tau(n) \quad \text{and} \quad \sum_{\substack{n\equiv\ell \pmod q\\ n\leqslant x}} r(n),$$

where q grows together with x and they are nontrivial for $q \ll x^{\frac{2}{3}-\varepsilon}$.

For the function $\tau(n)$ K. Liu, I. Shparlinskii and T. Zhang ([2]) obtained the extended region of non-triviality.

In the present paper we investigate the distribution of points from complex plane $\mathbb{C} = \{x + iy | x, y \in \mathbb{R}\}$, $\varphi_1 < \arg(x + iy) \leqslant \varphi_2$, $\varphi_2 - \varphi_1 <$

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$\frac{\pi}{2}$, $x^2+y^2 \equiv \ell \pmod q$, $x^2+y^2 \leqslant N$. Using the property of Hecke Z-function of the quadratic field $\mathbb{Q}(i)$ and the estimates of special exponential sums, we obtain a non-trivial asymptotic formula for the number of integer points under the circle's sectorial region in arithmetic progression with the growing difference progression.

Throughout this paper we use the following notations.

- p denotes a prime number in $\mathbb{Z}$;
- the Latin letters a, b, k, m, n, ℓ be the positive integers;
- $\Re z$ denotes the real part of z and $\Im z$ be the imaginary part of z;
- through $\mathbb{Z}$ we denote the ring of integers;
- $G=\mathbb{Z}[i]$ denotes the ring of Gaussian integers $a+bi$, $a,b\in\mathbb{Z}$, $i^2=-1$;
- G_γ (respectively, G^*_γ) be the ring of residue classes modulo γ (respectively, the multiplicative group of inversive element in G_γ);
- $N(\omega)$ is the norm of $\omega\in G$, $N(\omega)=|\omega|^2$;
- $Sp(\omega)$ is the trace of ω from $\mathbb{Q}(i)$ to $\mathbb{Q}$, $Sp(\omega)=2\Re\omega$;
- symbols "$\ll$" and "O" are equivalent;
- $s=\sigma+it\in\mathbb{C}$, $\Re s=\sigma$, $\Im s=t$;
- χ_q denotes the Dirichlet character modulo q over $\mathbb{Z}$
- $(a,q)=\gcd(a,q)$ in $\mathbb{Z}$;
- $(\alpha,\omega)=\gcd(\alpha,\omega)$ in G;

1. Auxiliary results

Let $\delta_1,\delta_2\in\mathbb{Q}(i)$ and $s=\sigma+it$. For the rational integer number m let us define the function sized by absolutely convergent series into semiplane $\Re s>1$:

$$Z_m(s,;\delta_1,\delta_2) := \sum_{\omega\in G}\frac{e^{4mi\arg\omega+\delta_1}}{N(\omega+\delta_1)^s}e^{\pi i Sp(\delta_2\cdot\omega)}.$$

It is obvious that with $m=0$ we get the Epstein zeta-function. With $\delta_1,\delta_2\in\mathbb{Q}_i$ we get the Hecke Z-function over the imaginary quadratic field $\mathbb{Q}(i)$.

Let $p>2$ be a prime rational number, $n\in\mathbb{N}$. Denote

$$E_{p^n} := \left\{\alpha\in G_{p^n}\middle| N(\alpha)\equiv\pm1 \pmod{p^n}\right\}. \tag{1}$$

It is also obvious that E_n is the subgroup of multiplicative group of residue classes modulo p^n over the ring G_{p^n}.

We call E_{p^n} the norm group in $G^*_{p^n}$.

Lemma 1. *Let $p \equiv 3 \pmod 4$ and E_n be the norm group in G_{p^n}. Then E_n is the cyclic group, $|E_n| = 2(p+1)p^{n-1}$, and let $u+iv$ be a generative element of E_n. Then exist $x_0, y_0 \in \mathbb{Z}^*_{p^n}$ such that*

$$(u+iv)^{2(p+1)} \equiv 1 + p^2 x_0 + ipy_0,$$
$$2x_0 + y_0^2 \equiv -2p^2 x_0^2 \pmod{p^3}.$$

Moreover, we have modulo p^n for any $t = 4, 5, \dots, p^{n-1}-1$,

$$\Re\Big((u+iv)^{2(p+1)t}\Big) = A_0 + A_1 t + A_2 t^2 + \cdots$$
$$\Im\Big((u+iv)^{2(p+1)t}\Big) = B_0 + B_1 t + B_2 t^2 + \cdots,$$

where

$$A_0 \equiv 1 \pmod{p^4}, \qquad B_0 \equiv 0 \pmod{p^4},$$
$$A_1 \equiv p^2 x_0 + \frac{1}{2}p^2 y_0^2 \equiv -\frac{5}{2}x_0^2 p^4 \pmod{p^5},$$
$$B_1 \equiv py_0(1 - p^2 x_0) \pmod{p^4},$$
$$A_2 \equiv -\frac{5}{2}x_0^2 p^2 \pmod{p^5}, \qquad B_2 \equiv \frac{5}{3}p^3 x_0 y_0 \pmod{p^4},$$
$$A_j \equiv B_j \equiv 0 \pmod{p^3}, \quad j = 3, 4, \dots$$

(In greater details see [3])

Denote

$$(u+iv)^k = u(k) + iv(k), \quad 0 \leqslant k \leqslant 2p+1,$$
$$(u+iv)^{2(p+1)t+k} \equiv \sum_{j=0}^{n-1} (A_j(k) + iB_j(k))t^k \pmod{p^n}.$$

It is obvious that

$$A_j(k) = A_j u(k) - B_j v(k), \quad B_j(k) = A_j v(k) + B_j u(k).$$

Thus from Lemma 1 we infer

Corollary. *For $k = 0, 1, \dots, 2p+1$ we have*

$$u(0) = 1, \quad v(0) = 0, \quad (u(p+1), p) = 1, \quad p||v(p+1);$$
$$(u(k), p) = (v(k), p) = 1 \quad for \quad k \not\equiv 0 \pmod{\frac{p+1}{2}};$$
$$u(k) \equiv 0 \pmod p, \quad (v(k), p) = 1 \quad if \quad k = \frac{p+1}{2} \quad or \quad \frac{3p+1}{2};$$
$$u(k) \equiv u(-k), \quad v(k) \equiv -v(-k).$$

Hence, for $k \not\equiv 0 \pmod{\frac{p+1}{2}}$ *we have*

$$\begin{gathered} A_0(k) \equiv u(k) \pmod{p}, \quad B_0(k) \equiv v(k) \pmod{p}, \\ A_1(k) \equiv -py_0v(k), \quad B_1(k) \equiv py_0u(k) \pmod{p^2} \\ A_2(k) \equiv -\frac{5}{2}x_0^2p^2u(k), \quad B_2(k) \equiv -\frac{5}{2}x_0^2p^2v(k) \pmod{p^4}. \end{gathered} \tag{2}$$

For $k = \frac{p+1}{2}$ *or* $\frac{3p+1}{2}$ *we obtain*

$$p||A_1(k), \quad p^2||B_1(k), \quad p^2||A_2(k), \quad B_2(k) \equiv 0 \pmod{p^3}. \tag{3}$$

Moreover,

$$\begin{gathered} A_1(0) \equiv -\frac{5}{2}x_0^2p^4 \pmod{p^5}, \quad B_1(0) \equiv 0 \pmod{p^4}, \\ A_2(0) \equiv -\frac{5}{2}x_0^2p^2 \pmod{p^5}, \quad B_2(0) \equiv 0 \pmod{p^3}, \quad p^2||A_1(p+1), \\ p||B_1(p+1), \qquad p^2||A_2(p+1), \quad B_2(p+1) \equiv 0 \pmod{p^3}. \end{gathered} \tag{4}$$

At last for all $k = 0, 1, \dots, 2p+1$

$$A_j(k) \equiv B_j(k) \equiv 0 \pmod{p^3}, \quad j = 3, 4, \dots.$$

Lemma 2. *Let* $q = p^\ell$ *with* $\ell \geqslant 1$*,* $g(y)$ *is the polynomial in form*

$$g(y) = A_1y + pA_2y^2 + p^{\lambda_3}A_3y^3 + \cdots + p^{\lambda_k}A_ky^k, \quad k \geqslant 3,$$

with $A_j \in \mathbb{Z}$*,* $(A_j, p) = 1$*,* $j = 3, \dots, k$*,* $2 \leqslant \lambda_3 \leqslant \lambda_4 \leqslant \cdots \leqslant \lambda_k$*. Then we have*

$$S_q := \sum_{y=1}^{q-1} e^{2\pi i \frac{g(y)}{p^\ell}} = p^{\left[\frac{\ell}{2}\right]} \sum_{\substack{y \in \mathbb{Z}_{p^{[\ell/2]}} \\ g'(y) \equiv 0 \pmod{p^{[\ell/2]}}}} B_q(y), \tag{5}$$

where

$$B_q(y) = \begin{cases} 0 & \textit{if} \quad (A_1, p) = 1, \\ 1 & \textit{if} \quad \ell \equiv 0 \pmod{2}, \\ & \qquad A_1 \equiv 0 \pmod{p}, \\ \sum_{z=0}^{p-1} e^{2\pi i \frac{\left(\left(\frac{A_1}{p} + 2A_2\right)z + 2z^2\right)}{p}} & \textit{if} \quad \ell \equiv 1 \pmod{2}, \\ & \qquad A_1 \equiv 0 \pmod{p}. \end{cases}$$

Proof. The proof of this assertion repeats the proofs of Lemmas 12.3 and 12.4 in [1]. □

For $p \equiv 1 \pmod 4$ or $p = 2$ the norm groups are not the cyclic groups. We shall use the description of the solutions $x^2 + y^2 \equiv 1 \pmod{p^n}$ for these cases.

Lemma 3. *Let (x, y) is a solution of the congruence $x^2+y^2 \equiv 1 \pmod{p^\ell}$, $p > 2$ is a prime number. Then all solutions with $(x_0, p) = 1$ are described in the following manner*

$$x = x(0)f(y_0, t), \quad y = y_0 + pt, \quad t = 0, 1, \ldots, p^{\ell-1} - 1, \tag{6}$$

where $x(0)$ runs all solutions of the congruence

$$x^2 \equiv 1 - y_0^2 \pmod{p^n},$$

y_0 runs all solutions of the congruence

$$x_0^2 + y_0^2 \equiv 1 \pmod p$$

with $x_0 \not\equiv 0 \pmod p$, and

$$f(y_0, t) = 1 + p\frac{y_0}{y_0^2 - 1}t + p^2\frac{1 - y_0}{y_0^2 - 1}t^2 + p^{\lambda_3}X_3(y_0)t^3 + \cdots + p^{\lambda_s}X_s(y_0)t^s,$$

under conditions $(X_j(y_0), p) = 1$, $\lambda_j \geqslant 3$, $s \leqslant \left[\ell\frac{p-1}{p-2}\right]$.

For the solutions of the congruence $x^2 + y^2 \equiv 1 \pmod{p^\ell}$ with $x_0 \equiv 0 \pmod p$ we have

$$x = pt, \quad y \equiv \pm\left(1 - \frac{1}{2}p^2t^2\right) \pmod{p^4}. \tag{7}$$

(Here, the multiplicative inverse for 2 and $y_0^2 - 1$ is considered modulo p^n).

Lemma 3′. *Let $s = \left[\frac{\ell-1}{2}\right]$. There exists the polynomial*

$$f(t) = 1 + 2^{\lambda_1}A - 1t^2 + \cdots + 2^{\lambda_s}A_s t^{2s},$$

with $A_j \equiv 1 \pmod 2$, $\lambda_j \geqslant 2j + 1$, $j = 1, \ldots, s$, such that all solutions of the congruence $x^2 + y^2 \equiv 1 \pmod{p^\ell}$ can be written as

$$\begin{aligned} &x = 4t, \quad y = \pm f(t) \quad \text{or} \quad x = 4t, \quad y = \pm(2^{\ell-1} - 1)f(t), \\ &t = 0, 1, \ldots, 2^{\ell-2} - 1. \end{aligned} \tag{8}$$

Lemma 4. *Let us $I(\ell, q)$ be the number of solutions of the congruence*

$$u^2 + v^2 \equiv a \pmod q, \quad (a, q) = \prod_{p|q} p^{t_0}.$$

Then we have

$$I(a, q) = c(a, q) q \prod_{p^t || q} \left(1 - \frac{\chi_4(p^{t_0+1})}{p} \left(1 - \chi_4(p^{t-t_0})\right) + \left(1 - \frac{1}{p}\right) \sum_{b=t-t_0}^{t-1} \chi_4(p^{t-b})\right),$$

where

$$c(a, q) = \begin{cases} 1 & \text{if } (q, 2) = 1, \\ 1 & \text{if } 2||q, \\ 1 & \text{if } q \equiv 0 \pmod 4, \ t_0 > t - 2, \\ 2 & \text{if } q \equiv 0 \pmod 4, \ t_0 < t - 2 \text{ and } \frac{a}{2^{t_0}} \equiv 1 \pmod 4, \\ 0 & \text{if } q \equiv 0 \pmod 4, \ t_0 \leqslant t - 2 \text{ and } \frac{a}{2^{t_0}} \equiv 3 \pmod 4 \end{cases}$$

This lemma follows from the equation

$$I(a, p^t) = \sum_{u,v \in \mathbb{Z}_{p^t}} \frac{1}{p^t} \sum_{z \in \mathbb{Z}^*_{p^t}} e^{2\pi i \frac{z(y^2+v^2-\ell)}{p^t}}$$

and the values of the Gaussian sums $\sum_{x \in \mathbb{Z}_{p^t}} e^{2\pi i \frac{zx^2}{p^t}}$.

Similarly, we obtain the description of the solutions of the congruence $x^2 + y^2 \equiv -1 \pmod{p^\ell}$, $p \equiv 1 \pmod 4$. Indeed, let c_0 be the solution of the congruence $x^2 \equiv -1 \pmod{p^\ell}$. Then

$$x = c_0 x(0) f_1(y_0, t), \quad y = y_0 + pt, \quad t = 0, 1, \ldots, p^{\ell-1} - 1,$$

where $f_1(y_0, t)$ is as $f(y_0, t)$.

2. The main results

We consider the generalized Hecke Z-function of quadratic field $\mathbb{Q}(i)$

$$Z_m(s; \delta_1, \delta_2) := \sum_{\substack{\omega \in G \\ \omega \neq \delta_1}} \frac{e^{4mi \arg(\omega+\delta_1)}}{N(\omega + \delta_1)} e^{\pi i Sp(\omega \delta_2)}, \quad (\Re s > 1),$$

where $\delta_1, \delta_2 \in \mathbb{Q}(i)$, $m \in \mathbb{Z}$. This function satisfies the functional equation

$$\begin{aligned}\pi^{-1}\Gamma(2|m|+s)Z_m(s;\delta_1,\delta_2)\\ = \pi^{-(1-s)}\Gamma(2|m|+1-s)Z_{-m}(1-s;-\delta_2,\delta_1)e^{\pi i Sp(\delta_1\delta_2)}.\end{aligned} \tag{9}$$

The function $Z_m(s;\delta_1,\delta_2)$ is an entire function except the case $m=0$ and the Gaussian integer δ_2 when $Z_m(s;\delta_1,\delta_2)$ is holomorphic for all complex s exclusive $s=1$ where it has a simple pole with residue π.

We define the multiplicative character modulo q over $G^*_{p^\ell}$ as

$$\chi(\omega) = \chi_{p^\ell}(N(\omega)),$$

where χ_{p^ℓ} is the character modulo p^ℓ in $\mathbb{Z}^*_{p^\ell}$.

Let $\Xi_m(\omega) := e^{4mi\arg\omega}\chi(\omega) = e^{4mi\arg\omega}\chi_{p^\ell}(N(\omega))$. Then from (9) we have for $Z(s;\Xi_m) := \sum_\omega \frac{\Xi_m(\omega)}{N(\omega)^s}$ the following functional equation

$$Z(s;\Xi_m) = \kappa(\Xi_m)\Psi(s,\Xi_m)Z(1-s,\overline{\Xi}_m), \tag{10}$$

where

$$\begin{aligned}\kappa(\Xi_m) &= (N(p^\ell))^{-\frac{1}{2}}\sum_{\tau\in G_{p^\ell}}\chi(N(\tau))e^{Sp\frac{\tau}{p^\ell}},\\ \Psi(s,\Xi_m) &= \left(\frac{1}{\pi}N(p^\ell)^{\frac{1}{2}}\right)^{1-2s}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}.\end{aligned} \tag{11}$$

Denote

$$r_m(n) = \sum_{\substack{u,v\in\mathbb{Z}\\ u^2+v^2=n}} e^{4mi\arg(u+iv)}.$$

From this we have

$$\sum_{n\leqslant x} r_m(n)\chi_{p^\ell}(n) = \sum_{\substack{u,v\in\mathbb{Z}\\ u^2+v^2=n\leqslant x}} e^{4mi\arg(u+iv)}\chi_{p^\ell}(n).$$

Therefore,

$$F_m(s) = \sum_{n=1}^{\infty}\frac{r_m(n)}{n^s} = \sum_{\chi_q}\overline{\chi}_q(a)\cdot Z(s;\Xi_m).$$

We get by the Perron's formula on an arithmetic progression with $c > 1$, $T > 1$, $(a, p^\ell) = 1$, $0 < \varepsilon < \frac{1}{2}$ the following equality

$$\sum_{\substack{n\equiv a \pmod{p^\ell}\\ n\leqslant x}} r_m(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} F_m(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{Tp^\ell(c-1)}\right) + O\left(x^\varepsilon\right)$$
$$= \operatorname*{res}_{s=0,1}\left(F_m(s)\frac{x^s}{s}\right) + \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} F_m(s)\frac{x^s}{s}ds + \max_{-\varepsilon\leqslant\Re s\leqslant c}\left|\frac{1}{s}F_m(s)x^s\right|$$
$$+ O\left(\frac{x^c}{Tp^\ell(c-1)}\right) + O\left(x^\varepsilon\right), \tag{12}$$

where ε is a positive arbitrary small number.

From the functional equation for $Z(s,\Xi)$, summing all over character χ_{p^ℓ}, we have for $\Re s < 0$

$$F_m(s) = \pi^{-1+2s}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)} \times \sum_{\substack{\omega\in G\\ (\omega,p^\ell)=1}} \frac{e^{-4mi\arg\omega}}{N(\omega)^{1-s}} \sum_{\substack{\tau\in G^*_{p^\ell}\\ N(\tau)\equiv aN(\omega) \pmod{p^\ell}}} e^{\frac{Sp(\tau)}{p^\ell}}.$$

Consider the sum

$$\sum\nolimits_0 := \sum_{\substack{\tau\in G^*_{p^\ell}\\ N(\tau)\equiv n \pmod{p^\ell}\\ (n,p^\ell)=1}} e^{\pi i Sp(\frac{\tau}{p^\ell})}.$$

For $p \equiv 3 \pmod 4$, we apply the representation of elements from the norm group E_{p^ℓ}. Lemma 1 and its Corollary give

$$\sum\nolimits_0 = \sum_{k=0}^{2p+1} e^{2\pi i\frac{A'_0(k)}{p^\ell}} \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i\frac{A'_1(k)t+A'_2(k)t^2+\cdots}{p^\ell}}$$
$$= e^{2\pi i\frac{A'_0(0)}{p^\ell}} \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i\frac{A'_1(0)t+A'_2(0)t^2+\cdots}{p^\ell}}$$
$$+ e^{2\pi i\frac{A'_0(p+1)}{p^\ell}} \sum_{t=0}^{p^{\ell-1}-1} e^{2\pi i\frac{A'_1(p+1)t+A'_2(p+1)t^2+\cdots}{p^\ell}},$$

where $A'_j(0)$ and $A'_j(p+1)$ differ from $A_j(j)$ and $A-j(p+1)$ only by the multiplier $N(\omega)a$.

Now Lemma 3 gives

$$E_0 = p^{\frac{\ell}{2}} \left(e^{2\pi i \frac{A_0'(0)}{p^\ell}} + e^{2\pi i \frac{A_0'(p+1)}{p^\ell}} \right) \times \begin{cases} 1 & if \quad \ell \equiv 0 \pmod 2, \\ e^{-\frac{2\pi i A_1'(2A_2')^{-1}}{p}} & if \quad \ell \equiv 1 \pmod 2. \end{cases} \tag{13}$$

If $p \equiv 1 \pmod 4$ or $p = 2$ we use Lemma 1 and then obtain $E_0 = O\left(p^{\ell \frac{1}{2}}\right)$ with an absolute constant in the symbol "O".

Now we able to prove the main theorems.

Let us denote through $A(x; \varphi_1, \varphi_2; a, p^\ell)$ the number of points (u, v) in the circle $(u^2 + v^2) \leqslant x$ under conditions

$$\begin{aligned} &u, v \in \mathbb{Z}, \quad \varphi_1 < \arg(u + iv) \leqslant \varphi_2, \\ &u^2 + v^2 \equiv a \pmod{p^\ell}, \quad (a, p^\ell) = 1. \end{aligned} \tag{14}$$

Theorem 1. *For $x \to \infty$ the following estimate*

$$\sum_{\substack{n \equiv a \pmod{p^\ell} \\ n \leqslant x}} r_m(n) = \varepsilon \frac{\pi x}{p^\ell} k_0 \left(1 - \frac{\chi_4(p)}{p}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{p^{\frac{\ell}{4}}} M^{1+\varepsilon}\right) + O\left(p^{\frac{\ell}{2}} M^{1+\varepsilon}\right), \tag{15}$$

holds, where $\varepsilon_m = 0$ if $m \neq 0$, $\varepsilon_0 = 1$, $k_0 = 1$ if $p > 2$, or $k = 2$ if $p = 2$, $\ell \geqslant 3$; $M = |m| + 3$, $\varepsilon > 0$ is an arbitrary small number; constants in the symbols can depend only on ε.

Proof. The function $F_m(s)$ has a pole in $s = 1$ only if $m = 0$:

$$\operatorname*{res}_{s=1} F_0(s) = \frac{\pi x}{p^\ell} k_0 \left(1 - \frac{\chi_4(p)}{p}\right).$$

The estimate for $F_m(0)$ is easy proving by the Phragmen-Lindelöff principle and the estimates of $Z_m(s)$ on the bounds of stripe $-\varepsilon \leqslant \Re s \leqslant 1 + \varepsilon$. Therefore, we have

$$\operatorname*{res}_{s=0} F_m(s) \ll p^{\frac{\ell}{2}} (|m| + 3) \log(|m| + 3).$$

Hence,

$$\sum_{\substack{n\equiv a \pmod{p^\ell}\\ n\leqslant x}} r_m(n) = \varepsilon_m \frac{\pi x}{p^\ell} \sum_{\substack{u,v\in\mathbb{Z}_{p^\ell}\\ u^2+v^2\equiv a \pmod{p^\ell}}} 1 + O\left(p^{\frac{\ell}{2}}(|m|+3)\log(|m|+3)\right) +$$
$$+ \frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} F_m(s)\frac{x^s}{s}ds + O\left(\frac{x^c}{Tp^\ell(c-1)} + x^\varepsilon\right). \tag{16}$$

Note that

$$\varepsilon_m \frac{\pi x}{p^\ell} \sum_{\substack{u,v\in\mathbb{Z}_{p^\ell}\\ u^2+v^2\equiv a \pmod{p^\ell}}} 1 = \varepsilon_m \frac{\pi x}{p^\ell} k_0 \left(1 - \frac{\chi_4(p)}{p}\right),$$

where

$$F_m(s) = \pi^{-1+2s}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)} \times \sum_{\substack{\omega\in G\\ (\omega,p^\ell)=1}} \frac{e^{-4mi\arg\omega}}{N(\omega)^{1-s}} \sum_{\substack{\tau\in G^*_{p^\ell}\\ N(\tau)\equiv aN(\omega) \pmod{p^\ell}}} e^{\pi i\frac{Sp(\tau)}{p^\ell}}.$$

Thus, using the estimate of the sum $\sum_0$ and the Stirling formula for the gamma-function $\Gamma(z)$, we at once obtain the estimate of the integral in (16)

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} F_m(s)\frac{x^s}{s}ds \ll T^{1+2\varepsilon}p^{\frac{\ell}{2}+\varepsilon}x^{-\varepsilon} \ll T^{1+2\varepsilon}p^{\frac{\ell}{2}}. \tag{17}$$

Choosing $c = 1 + (logx)^{-1}$, $T = \frac{x^{\frac{1}{2}}}{p^{\frac{3\ell}{4}}}$, we get assertion of Theorem 1. □

The following theorems stem from this result and Vinogradov's lemma (see, [4], Lemma 12, pp. 261-262).

Theorem 2. *In the sectorial region* $u^2+v^2 \leqslant x$, $u^2+v^2\equiv a \pmod{p^\ell}$, $\varphi_1 < \arg(u+iv) \leqslant \varphi_2$, $\varphi_2 - \varphi_1 \gg x$ *the following asymptotic formula holds:*

$$A(x;\varphi_1,\varphi_2;a,p^\ell) := \sum_{\substack{u,v\\ u^2+v^2\equiv a \pmod{p^\ell}\\ \varphi_1<\arg(u+iv)\leqslant\varphi_2\\ u^2+v^2\leqslant x}} 1 =$$
$$= \frac{\varphi_2-\varphi_1}{2}\cdot\frac{k_0 x}{p^\ell}\left(1-\frac{\chi_4(p)}{p}\right) + O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{p^{\frac{\ell}{4}}}\right)$$

Theorem 3. *Let p be a prime number, $\ell \geqslant 3$, and $p^{\frac{3\ell}{2-4\kappa}} \leqslant x \leqslant p^{2\ell}$, $0 < \kappa \leqslant \frac{1}{8} - \frac{1}{4\ell}$, $\varphi_2 - \varphi_1 \gg x^{-\kappa}$. Then we have*

$$A(x;\varphi_1,\varphi_2;a,p^\ell) = \frac{\varphi_2-\varphi_1}{2}\cdot\frac{x}{p^\ell}\left(1-\frac{\chi_4(p)}{p}\right) + O\left(\frac{x^{1-\kappa}}{p^\ell}\log x^\kappa\right).$$

Actually, in Vinogradov's lemma we take $\Omega = \frac{\pi}{2}$, $\delta = x^\kappa$, $\Delta = x^{-\alpha}$ and let $\Delta \leqslant \varphi_2 - \varphi_1 < \frac{\pi}{4} - 2\kappa$. Then $f(\varphi_1,\varphi_2)$ be the function from that lemma.

Consider the function

$$\Phi(\varphi_1,\varphi_2) = \frac{1}{4}\sum_{\substack{u^2+v^2\leqslant x\\ u^2+v^2\equiv a \pmod{p^\ell}}} f(\arg(u+iv)).$$

Then we have

$$\Phi(\varphi_1,\varphi_2) = \sum_{\substack{u^2+v^2\leqslant x\\ u^2+v^2\equiv a \pmod{p^\ell}}} \sum_{m=-\infty}^{\infty} a_m e^{4mi\arg(u+iv)} = $$
$$= \sum_{m=-\infty}^{\infty} a_m \sum_{\substack{n\equiv a \pmod{p^\ell}\\ n\leqslant x}} r_m(n),$$

(here a_m are the coefficients from the Vinogradov's lemma).

We take $r = 3$ (in the notation of the Vinogradov's lemma) and take into account that

$$a_0 = \frac{1}{\Omega}(\varphi_2-\varphi_1+\Delta)$$
$$|a_m| \leqslant \begin{cases} \frac{1}{\Omega}(\varphi_2-\varphi_1+\Delta) & \\ \frac{2}{\pi|m|} & \quad if \quad m\neq 0, \\ \frac{2}{\pi|m|}\left(\frac{r\Omega}{\pi|m|\Delta}\right)^r & \end{cases}$$

then after simple calculations we get Theorem 2 and Theorem 3.

Taking into account that Hecke characters and Gauss exponential sums have the multiplicative properties modulo q, we have the following assertion.

Theorem 4. *In the sectorial region* $u^2+v^2 \leqslant x$, $u^2+v^2 \equiv a \pmod q$, $\varphi_1 < \arg(u+iv) \leqslant \varphi_2$, $\varphi_2-\varphi_1 \gg x$ *the following asymptotic formula holds:*

$$A(x;\varphi_1,\varphi_2;a,q) := \sum_{\substack{u,v \\ u^2+v^2\equiv a \pmod q \\ \varphi_1<\arg(u+iv)\leqslant\varphi_2 \\ u^2+v^2\leqslant x}} 1 =$$

$$= \frac{\varphi_2-\varphi_1}{2}\cdot\frac{k_0 x}{q}\prod_{p|q}\left(1-\frac{\chi_4(p)}{p}\right)+O\left(\frac{x^{\frac{1}{2}+\varepsilon}}{q^{\frac{1}{4}}}\right).$$

Remark. The result of Theorem 1 can be improved in case $p \equiv 3 \pmod 4$ and $\ell \geqslant 3$ in view of the fact that we have the precise meaning of the sum E_0 (see (13)).

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