© Algebra and Discrete Mathematics
Volume **29** (2020). Number 2, pp. 173–179
DOI:10.12958/adm1545 RESEARCH ARTICLE

Morita equivalent unital locally matrix algebras*

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Communicated by A. P. Petravchuk

Abstract. We describe Morita equivalence of unital locally matrix algebras in terms of their Steinitz parametrization. Two countable-dimensional unital locally matrix algebras are Morita equivalent if and only if their Steinitz numbers are rationally connected. For an arbitrary uncountable dimension α and an arbitrary not locally finite Steinitz number s there exist unital locally matrix algebras A, B such that $\dim_F A = \dim_F B = \alpha$, $\mathbf{st}(A) = \mathbf{st}(B) = s$, however, the algebras A, B are not Morita equivalent.

Introduction

Let F be a ground field. Throughout the paper we consider unital associative F–algebras. An algebra A with a unit 1_A is called a *unital locally matrix algebra* if an arbitrary finite collection of elements $a_1, \ldots, a_s \in A$ lies in a subalgebra B, $1_A \in B \subset A$, that is isomorphic to a matrix algebra $M_n(F)$, $n \geqslant 1$.

The idea of parametrization of unital locally matrix algebras with Steinitz numbers was introduced by J. G. Glimm [1]. Diagonal locally simple Lie algebras of countable dimension were parametrized with Steinitz numbers by A. A. Baranov and A. G. Zhilinskii in [2,3]. The extension of these results to regular relation structures was done in [4].

In this paper we apply Steinitz parametrisation to Morita equivalence classes of unital locally matrix algebras. We show that two countable-dimensional unital locally matrix algebras are Morita equivalent if and

*The second author was partially supported by the grant for scientific researchers of the "Povir u sebe" Ukranian Foundation.

2010 MSC: 03C05, 03C60.

Key words and phrases: locally matrix algebra, Steinitz number, Morita equivalence.

only if their Steinitz numbers are rationally connected. This result does not extend to the uncountable case. Moreover, for an arbitrary uncountable dimension α and an arbitrary not locally finite Steinitz number s there exist unital locally matrix algebras A, B such that $\dim_F A = \dim_F B = \alpha$, $\mathbf{st}(A) = \mathbf{st}(B) = s$, however, the algebras A, B are not Morita equivalent.

1. Preliminaries

Let $\mathbb{P}$ be the set of all primes and $\mathbb{N}$ be the set of all positive integers. A *Steinitz* number (see [5]) is an infinite formal product of the form

$$\prod_{p\in\mathbb{P}} p^{r_p}, \tag{1}$$

where $r_p \in \mathbb{N} \cup \{0, \infty\}$ for all $p \in \mathbb{P}$. The product of two Steinitz numbers

$$\prod_{p\in\mathbb{P}} p^{r_p} \quad \text{and} \quad \prod_{p\in\mathbb{P}} p^{k_p}$$

is a Steinitz number

$$\prod_{p\in\mathbb{P}} p^{r_p+k_p},$$

where we assume, that $t + \infty = \infty + t = \infty + \infty = \infty$ for all non-negative integers t.

Denote by $\mathbb{SN}$ the set of all Steinitz numbers. Note, that the set $\mathbb{N}$ is a subset of $\mathbb{SN}$.

A Steinitz number (1) is called *locally finite* if $r_p \neq \infty$ for any $p \in \mathbb{P}$. The numbers $\mathbb{SN} \setminus \mathbb{N}$ are called *infinite* Steinitz numbers.

J. G. Glimm [1] parametrised countable-dimensional locally matrix algebras with Steinitz numbers. In [6] we studied Steinitz numbers of unital locally matrix algebras of arbitrary dimensions.

Let A be an infinite-dimensional locally matrix algebra with a unit 1_A over a field F and let $D(A)$ be the set of all positive integers n such that there is a subalgebra A', $1_A \in A' \subseteq A$, $A' \cong M_n(F)$.

Definition 1. The least common multiple of the set $D(A)$ is called the Steinitz number $\mathbf{st}(A)$ of the algebra A.

Given two unital locally matrix algebras A and B their tensor product $A \otimes_F B$ is a unital locally matrix algebra and $\mathbf{st}(A \otimes_F B) = \mathbf{st}(A) \cdot \mathbf{st}(B)$ (see [7]). In particular, a matrix algebra $M_k(A)$ is a unital locally matrix algebra and $\mathbf{st}(M_k(A)) = k \cdot \mathbf{st}(A)$.

Theorem 1 ([1], see also [4])**.** *If A and B are unital locally matrix algebras of countable dimension then A and B are isomorphic if and only if* $\mathbf{st}(A) = \mathbf{st}(B)$.

Let A be an algebraic system. The universal elementary theory $UTh(A)$ consists of universal closed formulas (see [8]) that are valid on A. The systems A and B of the same signature are universally equivalent if $UTh(A) = UTh(B)$.

In [6] we showed that for unital locally matrix algebras A, B of dimension $> \aleph_0$ the equality $\mathbf{st}(A) = \mathbf{st}(B)$ does not necessarily imply that A and B are isomorphic. However, $\mathbf{st}(A) = \mathbf{st}(B)$ is equivalent to A, B being universally equivalent.

2. Morita equivalence

Definition 2. Two unital algebras A, B are called Morita equivalent if categories of their left modules are equivalent.

Let $e \in A$ be an idempotent. We refer to the subalgebra eAe as a *corner* of the algebra A. An idempotent $e \in A$ is said to be *full* if $AeA = A$. K.Morita [9] (see also [10, 11]) proved that the algebras A, B are Morita equivalent if and only if there exists $n \geqslant 1$ and a full idempotent e in the matrix algebra $M_n(A)$ such that $B \cong eM_n(A)e$. Thus B is isomorphic to a corner of the algebra $M_n(A)$.

We say that a property P is *Morita invariant* if any two Morita equivalent algebras do satisfy or do not satisfy P simultaneously.

An F-algebra A is a tensor product of finite-dimensional matrix algebras if

$$A \cong \otimes_{i \in I} A_i, \quad A_i \cong M_{n_i}(F), \ n_i \geqslant 1.$$

Every tensor product (see [11]) of finite-dimensional matrix algebras is a locally matrix algebra. G. Köthe [12] showed that the reverse is true for countable-dimensional algebras. A.G.Kurosh [13] (see also [7, 14]) constructed examples of locally matrix algebras that do not decompose into a tensor product of finite-dimensional matrix algebras.

Lemma 1. (1) *Being a locally matrix algebra is a Morita invariant property.*

(2) *Being a tensor product of finite-dimensional matrix algebras is a Morita invariant property.*

Proof. (1) Let algebras A, B be Morita equivalent. Then there exists $n \geqslant 1$ and a full idempotent $e \in M_n(A)$ such that $B \cong eM_n(A)e$. If the algebra A is locally matrix then so is the matrix algebra $M_n(A)$. J.Dixmier

[15] showed that a corner of a locally matrix algebra is a locally matrix algebra. Hence B is a locally matrix algebra.

(2) Now suppose that $A \cong \otimes_{i\in I} A_i$, $A_i \cong M_{n_i}(F)$, $n_i \geqslant 1$. Then

$$M_n(A) \cong M_n(F) \otimes_F A \cong M_n(F) \otimes_F (\otimes_{i\in I} A_i).$$

There exists a finite subset $I_0 \subset I$, $\mid I_0 \mid < \infty$, such that $e \in M_n(F) \otimes_F (\otimes_{i\in I_0} A_i)$. As above, the corner $e(M_n(F) \otimes_F (\otimes_{i\in I_0} A_i))e$ is a matrix algebra. Hence

$$B \cong eM_n(A)e \cong e(M_n(F) \otimes_F (\otimes_{i\in I_0} A_i))e \otimes_F (\otimes_{i\in I\setminus I_0} A_i),$$

which completes the proof of the lemma. □

Definition 3. We say that nonzero Steinitz numbers s_1, s_2 are rationally connected if there exists a rational number $q \in \mathbb{Q}$ such that $s_2 = q \cdot s_1$.

Theorem 2. 1) *If unital locally matrix algebras A, B are Morita equivalent then their Steinitz numbers $\mathbf{st}(A)$, $\mathbf{st}(B)$ are rationally connected.*

2) *If unital locally matrix algebras A, B are countable-dimensional then they are Morita equivalent if and only if $\mathbf{st}(A)$, $\mathbf{st}(B)$ are rationally connected.*

3) *For an arbitrary not locally finite Steinitz number s there exist not Morita equivalent unital locally matrix algebras A, B of arbitrary uncountable dimensions such that $\mathbf{st}(A) = \mathbf{st}(B) = s$.*

4) *For a countable-dimensional unital locally matrix algebra A the Morita equivalence class of A is countable up to isomorphism. For a unital locally matrix algebra of an arbitrary dimension the Morita equivalence class is countable up to universal equivalence.*

Remark 1. Countability of Morita equivalence classes of finitely presented algebras was discussed in [16–18].

Let A be a locally matrix algebra, let $a \in A$. There exists a subalgebra $1_A \in A_1 < A$, $a \in A_1$, such that $A_1 \cong M_n(F)$, $n \geqslant 1$. Let r be the range of the matrix a in A_1. Let

$$r(a) = \frac{r}{n}, \quad 0 \leqslant r(a) \leqslant 1.$$

V.M.Kurochkin [14] noticed that the number $r(a)$ does not depend on a choice of the subalgebra A_1. We will call $r(a)$ the *relative range* of the element a.

Lemma 2. *Let e be an idempotent of a locally matrix algebra A. Then* $\mathbf{st}(eAe) = r(e) \cdot \mathbf{st}(A)$.

Proof. Consider the family of all matrix subalgebras $1_A \in A_i < A$, $A_i \cong M_{n_i}(F)$, $i \in I$, such that $e \in A_i$. Then $\mathbf{st}(A) = \mathrm{lcm}(n_i, i \in I)$. The range of the matrix e in A_i is equal to $r(e) \cdot n_i$. Hence

$$eA_ie \cong M_{r(e)\cdot n_i}(F) \quad \text{and} \quad \mathbf{st}(eAe) = \mathrm{lcm}(r(e)\cdot n_i, i \in I) = r(e)\cdot \mathbf{st}(A).$$

□

Proof of Theorem 2. 1) Let A, B be locally matrix algebras that are Morita equivalent. Hence there exists $k \geqslant 1$ and an idempotent $e \in M_k(A)$ such that $B \cong eM_k(A)e$. Let $r(e)$ be the relative range of the idempotent e in the locally matrix algebra $M_k(A)$. By Lemma 2

$$\mathbf{st}(B) = r(e)\cdot \mathbf{st}(M_k(A)) = r(e)\cdot k \cdot \mathbf{st}(A).$$

Since the number $r(e)\cdot k$ is rational it follows that the Steinitz numbers $\mathbf{st}(A)$, $\mathbf{st}(B)$ are rationally connected.

2) Let A, B be countable-dimensional locally matrix algebras. Suppose that their Steinitz numbers $\mathbf{st}(A)$, $\mathbf{st}(B)$ are rationally connected. Our aim is to prove that the algebras A, B are Morita equivalent. There exist integers k, $l \geqslant 1$ such that $k \cdot \mathbf{st}(A) = l \cdot \mathbf{st}(B)$. Consider the matrix algebras $M_k(A)$ and $M_l(B)$. We have

$$\mathbf{st}(M_k(A)) = k\cdot \mathbf{st}(A) = l \cdot \mathbf{st}(B) = \mathbf{st}(M_l(B)).$$

By Glimm's Theorem [1] the algebras $M_k(A)$ and $M_l(B)$ are isomorphic. Hence the algebras A, B are Morita equivalent.

3) Let S be a not locally finite Steinitz number. In [7] (see also [6] and [13]) we showed that there exists a locally matrix algebra A of an arbitrary uncountable dimension α such that $\mathbf{st}(A) = s$ and A is not isomorphic to a tensor product of finite dimensional matrix algebras. It is easy to see that there exists a locally matrix algebra B of dimension α such that $\mathbf{st}(B) = s$ and B is isomorphic to a tensor product of finite-dimensional matrix algebras. By Lemma 1 (2) the algebras A, B are not Morita equivalent.

4) For a countable-dimensional locally simple algebra A all algebras in its Morita equivalence class have Steinitz numbers $q \cdot \mathbf{st}(A)$, where q is a positive rational number, and are uniquely determined by their Steinitz numbers up to isomorphism. This implies that the Morita equivalence class of A is countable.

If the algebra A is not necessarily countable-dimensional then Steinitz numbers $q \cdot \mathbf{st}(A)$ determine universal elementary theories of algebras in this class (see [6]). Hence the Morita equivalence class of A is countable up to universal equivalence. This completes the proof of Theorem 2. □

If nonzero Steinitz numbers s_1, s_2 are rationally connected then it makes sense to talk about their ratio $q = \frac{s_2}{s_1}$ which is a rational number.

For a countable-dimensional locally matrix algebra A its Morita equivalence class *is ordered*: for algebras A_1, A_2 in this class we say that $A_1 < A_2$ if

$$\frac{\mathbf{st}(A_1)}{\mathbf{st}(A_2)} < 1.$$

Proposition 1. *Let A_1, A_2 be countable-dimensional Morita equivalent locally matrix algebras. Then*

$$\frac{\mathbf{st}(A_1)}{\mathbf{st}(A_2)} < 1 \text{ if and only if } A_1 \text{ is isomorphic to a proper corner of } A_2.$$

Proof. If $A_1 \cong eA_2e$, where e is a proper idempotent of the algebra A_2, then $\mathbf{st}(A_1) = r(e)\mathbf{st}(A_2)$ by Lemma 2. Hence

$$\frac{\mathbf{st}(A_1)}{\mathbf{st}(A_2)} = r(e) < 1.$$

Now let

$$\frac{\mathbf{st}(A_1)}{\mathbf{st}(A_2)} = \frac{m}{n} < 1,$$

where m, n are relatively prime integers. Then n is a divisor of $\mathbf{st}(A_2)$. Hence the algebra A_2 contains a subalgebra $1 \in A_2^{'} < A_2$, $A_2^{'} \cong M_n(F)$. Hence (see [13])

$$A_2 \cong A_2^{'} \otimes_F C \cong M_n(C),$$

where C is the centralizer of the subalgebra $A_2^{'}$ in A_2. Consider the idempotent $e = \mathrm{diag}(\underbrace{1, 1, \ldots, 1}_{m}, 0, \ldots, 0) \in M_n(C)$. By Lemma 2

$$\mathbf{st}(eM_n(C)e) = \frac{m}{n}\,\mathbf{st}(A_2) = \mathbf{st}(A_1).$$

By Glimm's Theorem A_1 is isomorphic to a corner of $M_n(C)$, hence to a corner of A_2. $\square$

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Received by the editors: 09.02.2020.