

Computing bounds for the general sum-connectivity index of some graph operations*

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ABSTRACT. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Denote by $d_G(u)$ the degree of a vertex $u \in V(G)$. The general sum-connectivity index of G is defined as $\chi_\alpha(G) = \sum_{u_1 u_2 \in E(G)} (d_G(u_1) + d_G(u_2))^\alpha$, where α is a real number. In this paper, we compute the bounds for general sum-connectivity index of several graph operations. These operations include corona product, cartesian product, strong product, composition, join, disjunction and symmetric difference of graphs. We apply the obtained results to find the bounds for the general sum-connectivity index of some graphs of general interest.

Introduction

Let G be an undirected, simple, finite and connected graph whose vertex set is $V(G)$ and edge set is $E(G)$. The distance between two vertices u_1 and u_2 in G , denoted by $d_G(u_1, u_2)$, is the length of a shortest path between them. For a vertex $u \in V(G)$, $d_G(u)$ denotes the degree of u . An edge between two vertices u_1 and u_2 of G is denoted by $u_1 u_2$. The order and size of G is denoted by n_G and m_G , respectively. The maximum vertex degree of G is denoted by Δ_G and the minimum vertex degree of G is denoted by δ_G . A complete graph of order n is denoted by K_n .

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A topological index is designed on the ground of transformation of a molecular graph into a number which characterizes the topology of that graph. In chemical graph theory, the degree based topological indices are very important. A topological index, denoted by $Top(G)$, of a graph G is equal to the topological index $Top(H)$ of a graph H if and only if G and H are isomorphic.

In the theoretical point of view and applications, the *Wiener index* is the first and most studied topological index. Initially, Wiener index was known as path index but after some time it was renamed as Wiener index. In 1947, the Wiener index was introduced by the chemist Harold Wiener [18] and is defined as follows:

$$W(G) = \frac{1}{2} \sum_{u_1 \in V(G)} \sum_{u_2 \in V(G)} d_G(u_1, u_2).$$

The *hyper Wiener index* is the extension of Wiener index. The *hyper Wiener index* $WW(G)$ of a graph G is defined as

$$WW(G) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} (d_G(u, v) + d_G(u, v)^2).$$

The *Randić index* (or product connectivity index) was introduced by Randić [14] in 1975. It is the one of the most used molecular descriptors in structure property and structure activity relationship studies and is defined as follows:

$$R(G) = \sum_{u_1 u_2 \in E(G)} \frac{1}{\sqrt{d_G(u_1) d_G(u_2)}}.$$

The *general Randić connectivity index* (or general product-connectivity index) R_α was defined by Kier and Hall [12] in 1976 and is defined as follows:

$$R_\alpha(G) = \sum_{u_1 u_2 \in E(G)} (d_G(u_1) d_G(u_2))^\alpha,$$

where α is a real number. Then $R_{-1/2}$ is the Randić connectivity index. Another variant of the Randić index of G is the harmonic index, denoted by $H(G)$ and is defined as follows:

$$H(G) = \sum_{ab \in E(G)} \frac{2}{d_G(a) + d_G(b)} = 2\chi_{-1}(G).$$

In 2009, the *sum-connectivity index* $\chi(G)$ was introduced by Zhou and Trinajstić [19] and is defined as follows:

$$\chi(G) = \sum_{u_1 u_2 \in E(G)} \frac{1}{\sqrt{d_G(u_1) + d_G(u_2)}}.$$

A generalized form of the sum-connectivity index was introduced by Zhou and Trinajstić [20] in 2010. It is called *general sum-connectivity index* $\chi_\alpha(G)$ and is defined as:

$$\chi_\alpha(G) = \sum_{u_1 u_2 \in E(G)} (d_G(u_1) + d_G(u_2))^\alpha, \quad (1)$$

where α is a real number. Then $\chi_{-1/2}$ is the sum-connectivity index. The first general Zagreb index of G is introduced by Li and Gutman [13] and is defined as:

$$M_1^\alpha(G) = \sum_{u_1 \in V(G)} d_G(u_1)^\alpha = \sum_{u_1 u_2 \in E(G)} (d_G(u_1)^{(\alpha-1)} + d_G(u_2)^{(\alpha-1)}),$$

where α is a real number with $\alpha \neq 0$ and $\alpha \neq 1$. If $\alpha = 2$ then $M_1^2(G)$ becomes first Zagreb index and if $\alpha = 3$ then $M_1^3(G)$ is called F -index. Some chemical graphs that are very interesting in chemical graph theory, can be obtained by the use of different graph operations (graph products). It is important to understand that how the topological indices and topological invariants of such graph operations are related to the topological indices and topological invariants of components of these graph products.

Recently, Khalifeh et al. [10, 11] gave the exact expressions for hyper Wiener index and, first and second Zagreb indices of several graph operations. In 2015, Shetty et al. [15] derived formulae for harmonic index of some graph operations. For a detailed study on topological indices of graph operations, we refer to [2–8, 16]. Very recently, Wang et al. [17] computed the lower and upper bounds for different indices of tricyclic graphs. Gao et al. [9] gave some exact expressions for the hyper-Zagreb index of some graph operations. Akhter et al. [1] gave exact formulae of general sum-connectivity index for some graph operations.

In this paper, we give some bounds for the general sum-connectivity index of several graph operations when $\alpha < 0$. These graph operations include corona product, cartesian product, strong product, join, composition, disjunction and symmetric difference of graphs. We apply these results to find the bounds for the general sum-connectivity index of some graphs of general interest.

1. Bounds for the general sum-connectivity index

In this section, we derive some bounds for general sum-connectivity index of several graph operations. Let G and H be two simple connected graphs whose vertex sets are disjoint. For each $u \in V(G)$ and $v \in V(H)$, we have

$$\Delta_G \geq d_G(u), \quad \delta_G \leq d_G(u), \quad (2)$$

$$\Delta_H \geq d_H(v), \quad \delta_H \leq d_H(v). \quad (3)$$

The equality holds if and only if G and H are regular graphs.

1.1. The corona product

The corona product of G and H , denoted by $G \odot H$, is a graph obtained by taking one copy of G and n_G copies of H and joining the vertex u that is on i -th position in G to every vertex in i -th copy of H . The order and size of $G \odot H$ are $n_G(1 + n_H)$ and $m_G + n_G m_H + n_G n_H$, respectively. The degree of a vertex $u \in V(G \odot H)$ is given by

$$d_{G \odot H}(u) = \begin{cases} d_G(u) + n_H & \text{if } u \in V(G), \\ d_H(u) + 1 & \text{if } u \in V(H). \end{cases} \quad (4)$$

In the following theorem, the bounds on the general sum-connectivity index of corona product of two graphs are computed.

Theorem 1. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G \odot H) \leq \beta_2$, where*

$$\begin{aligned} \beta_1 &= 2^\alpha m_G (\Delta_G + n_H)^\alpha + 2^\alpha n_G m_H (\Delta_H + 1)^\alpha \\ &\quad + n_G n_H (\Delta_G + \Delta_H + n_H + 1)^\alpha, \\ \beta_2 &= 2^\alpha m_G (\delta_G + n_H)^\alpha + 2^\alpha n_G m_H (\delta_H + 1)^\alpha \\ &\quad + n_G n_H (\delta_G + \delta_H + n_H + 1)^\alpha. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (4) in equation (1), we obtain

$$\begin{aligned}
 \chi_\alpha(G \odot H) &= \sum_{uv \in E(G)} (d_G(u) + d_G(v) + 2n_H)^\alpha \\
 &\quad + n_G \sum_{uv \in E(H)} (d_H(u) + d_H(v) + 2)^\alpha \\
 &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} (d_G(u) + n_H + d_H(v) + 1)^\alpha \\
 &\geq \sum_{uv \in E(G)} (2\Delta_G + 2n_H)^\alpha + n_G \sum_{uv \in E(H)} (2\Delta_H + 2)^\alpha \\
 &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} (\Delta_G + n_H + \Delta_H + 1)^\alpha \\
 &= 2^\alpha m_G (\Delta_G + n_H)^\alpha + 2^\alpha n_G m_H (\Delta_H + 1)^\alpha \\
 &\quad + n_G n_H (\Delta_G + \Delta_H + n_H + 1)^\alpha.
 \end{aligned} \tag{5}$$

Similarly, we can compute

$$\begin{aligned}
 \chi_\alpha(G \odot H) &\leq 2^\alpha m_G (\delta_G + n_H)^\alpha + 2^\alpha n_G m_H (\delta_H + 1)^\alpha \\
 &\quad + n_G n_H (\delta_G + \delta_H + n_H + 1)^\alpha.
 \end{aligned} \tag{6}$$

The equality in (5) and (6) holds if and only if G and H are regular graphs. \square

Let $t \geq 1$ and \overline{K}_t be the complement of K_t . Then t -thorny graph of G is the corona product of G and \overline{K}_t . The following corollary is an easy consequence of Theorem 1.

Corollary 1. *For $\alpha < 0$, the following holds:*

$$\begin{aligned}
 2^\alpha m_G (\Delta_G + t) + n_G t (\Delta_G + t + 1) &\leq \chi_\alpha(G \odot \overline{K}_t) \\
 &\leq 2^\alpha m_G (\delta_G + t) + n_G t (\delta_G + t + 1).
 \end{aligned}$$

1.2. The cartesian product

The cartesian product of G and H , denoted by $G \square H$, is a graph whose vertex set is $V(G \square H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \square H$ whenever $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$ or $[u_1$ and u_2 are adjacent in G and $v_1 = v_2]$. The order of the cartesian product of two graphs is the product of number of vertices of G and H , and the size is $m_G n_H + m_H n_G$. If G and H are regular graphs

then $G \square H$ is also regular graph. The degree of a vertex $(u, v) \in V(G \square H)$ is

$$d_{G \square H}((u, v)) = d_G(u) + d_H(v). \tag{7}$$

In the following theorem, we compute bounds on the general sum-connectivity index of $G \square H$.

Theorem 2. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G \square H) \leq \beta_2$, where*

$$\beta_1 = 2^\alpha m_{G \square H}(\Delta_G + \Delta_H)^\alpha, \beta_2 = 2^\alpha m_{G \square H}(\delta_G + \delta_H)^\alpha.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (7) in equation (1), we obtain

$$\begin{aligned} \chi_\alpha(G \square H) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} (2d_G(u_1) + d_H(v_1) + d_H(v_2))^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} (d_G(u_1) + d_G(u_2) + 2d_H(v_1))^\alpha \\ &\geq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (\Delta_G + \Delta_H)^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} 2^\alpha (\Delta_G + \Delta_H)^\alpha \\ &= 2^\alpha (n_G m_H + n_H m_G) (\Delta_G + \Delta_H)^\alpha \\ &= 2^\alpha m_{G \square H} (\Delta_G + \Delta_H)^\alpha. \end{aligned} \tag{8}$$

One can analogously compute the following for $\alpha < 0$:

$$\chi_\alpha(G \square H) \leq 2^\alpha m_{G \square H} (\delta_G + \delta_H)^\alpha. \tag{9}$$

The equality in (8) and (9) obviously holds if and only if G and H are regular graphs. □

1.3. The strong product

The strong product of G and H , denoted by $G \boxtimes H$, is a graph whose vertex set is $V(G \boxtimes H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \boxtimes H$ whenever $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$ or $[u_1$ and u_2 are adjacent in G and $v_1 = v_2]$ or $[u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)]$. The order of $G \boxtimes H$ is the product of number of vertices

of G and H , and the size is $n_G m_H + n_H m_G + 2m_G m_H$. The degree of a vertex $(u, v) \in V(G \boxtimes H)$ is

$$d_{G \boxtimes H}((u, v)) = d_G(u) + d_H(v) + d_G(u)d_H(v). \tag{10}$$

We compute bounds on the general sum-connectivity index of $G \boxtimes H$ in the following theorem.

Theorem 3. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G \boxtimes H) \leq \beta_2$, where*

$$\begin{aligned} \beta_1 &= 2^\alpha m_{G \boxtimes H} (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha, \\ \beta_2 &= 2^\alpha m_{G \boxtimes H} (\delta_G + \delta_H + \delta_G \delta_H)^\alpha. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (10) in equation (1), we obtain

$$\begin{aligned} \chi_\alpha(G \boxtimes H) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} (2d_G(u_1) + (d_H(v_1) + d_H(v_2)) \\ &\quad + d_G(u_1)(d_H(v_1) + d_H(v_2)))^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} ((d_G(u_1) + d_G(u_2)) + 2d_H(v_1) \\ &\quad + d_H(v_1)(d_G(u_1) + d_G(u_2)))^\alpha \\ &\quad + 2 \sum_{u_1 u_2 \in E(G)} \sum_{v_1 v_2 \in E(H)} ((d_G(u_1) + d_G(u_2)) + (d_H(v_1) \\ &\quad + d_H(v_2)) + d_G(u_1)d_H(v_1) + d_G(u_2)d_H(v_2)))^\alpha \\ &\geq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{u_1 u_2 \in E(G)} 2^\alpha (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha \\ &\quad + 2 \sum_{u_1 u_2 \in E(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha \\ &= 2^\alpha (n_G m_H + n_H m_G + 2m_G m_H) (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha \\ &= 2^\alpha m_{G \boxtimes H} (\Delta_G + \Delta_H + \Delta_G \Delta_H)^\alpha. \end{aligned} \tag{11}$$

Analogously, one can compute the following:

$$\chi_\alpha(G \boxtimes H) \leq 2^\alpha m_{G \boxtimes H} (\delta_G + \delta_H + \delta_G \delta_H)^\alpha. \tag{12}$$

If G and H are regular graphs then the equality in (11) and (12) holds. \square

1.4. The join

The join of G and H , denoted by $G + H$, is a union of graphs G and H together with all the edges joining the sets of vertices of G and H . The order and size of $G + H$ are $n_G n_H$ and $m_G + m_H + n_G n_H$, respectively. The degree of a vertex u in $G + H$ is given by

$$d_{G+H}(u) = \begin{cases} d_G(u) + n_H & \text{if } u \in V(G), \\ d_H(u) + n_G & \text{if } u \in V(H). \end{cases} \tag{13}$$

We compute bounds on the general sum-connectivity index for join of two graphs in the following theorem.

Theorem 4. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G + H) \leq \beta_2$, where*

$$\begin{aligned} \beta_1 &= 2^\alpha m_G(\Delta_G + n_H)^\alpha + 2^\alpha m_H(\Delta_H + n_G)^\alpha \\ &\quad + n_G n_H(\Delta_G + \Delta_H + n_G + n_H)^\alpha, \\ \beta_2 &= 2^\alpha m_G(\delta_G + n_H)^\alpha + 2^\alpha m_H(\delta_H + n_G)^\alpha \\ &\quad + n_G n_H(\delta_G + \delta_H + n_G + n_H)^\alpha. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (13) in equation (1), we get

$$\begin{aligned} \chi_\alpha(G + H) &= \sum_{uv \in E(G)} (d_G(u) + d_G(v) + 2n_H)^\alpha \\ &\quad + \sum_{uv \in E(H)} (d_H(u) + d_H(v) + 2n_G)^\alpha \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} (d_G(u) + n_H + d_H(v) + n_G)^\alpha \\ &\geq \sum_{uv \in E(G)} 2^\alpha (\Delta_G + n_H)^\alpha + \sum_{ab \in E(H)} 2^\alpha (\Delta_H + n_G)^\alpha \\ &\quad + \sum_{u \in V(G)} \sum_{v \in V(H)} (\Delta_G + \Delta_H + n_G + n_H)^\alpha \\ &= 2^\alpha m_G(\Delta_G + n_H)^\alpha + 2^\alpha m_H(\Delta_H + n_G)^\alpha \\ &\quad + n_G n_H(\Delta_G + \Delta_H + n_G + n_H)^\alpha. \end{aligned} \tag{14}$$

Similarly, we can show that

$$\begin{aligned} \chi_\alpha(G + H) &\leq 2^\alpha m_G(\delta_G + n_H)^\alpha + 2^\alpha m_H(\delta_H + n_G)^\alpha \\ &\quad + n_G n_H(\delta_G + \delta_H + n_G + n_H)^\alpha. \end{aligned} \tag{15}$$

If G and H are regular graphs then we obtain the equality in (14) and (15). □

1.5. The composition

The composition or lexicographic product of G and H , denoted by $G[H]$, is the graph whose vertex set is $V(G[H]) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G[H]$ whenever $[u_1 u_2 \in E(G)]$ or $[v_1$ and v_2 are adjacent in H and $u_1 = u_2]$. The order of $G[H]$ is the product of number of vertices of G and H , and size is $m_G n_H^2 + n_G m_H$. The degree of a vertex $(u, v) \in V(G[H])$ is

$$d_{G[H]}((u, v)) = n_H d_G(u) + d_H(v). \quad (16)$$

In the following theorem, we calculate bounds on the general sum-connectivity index for composition of two graphs.

Theorem 5. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G[H]) \leq \beta_2$, where*

$$\beta_1 = 2^\alpha m_{G[H]}(n_H \Delta_G + \Delta_H)^\alpha, \beta_2 = 2^\alpha m_{G[H]}(n_H \delta_G + \delta_H)^\alpha.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (16) in equation (1), we obtain

$$\begin{aligned} \chi_\alpha(G[H]) &= \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} (2n_H d_G(u_1) + d_H(v_1) + d_H(v_2))^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} (n_H(d_G(u_1) + d_G(u_2)) \\ &\quad + d_H(v_1) + d_H(v_2))^\alpha \\ &\geq \sum_{u_1 \in V(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (n_H \Delta_G + \Delta_H)^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} 2^\alpha (n_H \Delta_G + \Delta_H)^\alpha \\ &= 2^\alpha (n_G m_H + n_H^2 m_G) (n_H \Delta_G + \Delta_H)^\alpha \\ &= 2^\alpha m_{G[H]} (n_H \Delta_G + \Delta_H)^\alpha. \end{aligned} \quad (17)$$

Analogously, one can compute the upper bound

$$\chi_\alpha(G[H]) \leq 2^\alpha m_{G[H]} (n_H \delta_G + \delta_H)^\alpha. \quad (18)$$

The equality in (17) and (18) obviously holds if and only if G and H are regular graphs. \square

1.6. The disjunction

The disjunction of G and H , denoted by $G \vee H$, is a graph whose vertex set is $V(G \vee H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \vee H$ whenever $[v_1v_2 \in E(H)]$ or $[u_1u_2 \in E(G)]$. The order of $G \vee H$ is the product of number of vertices of graphs G and H , and size is $m_G n_H^2 + m_H n_G^2 - 2m_G m_H$. The degree of a vertex $(u, v) \in G \vee H$ is

$$d_{G \vee H}((u, v)) = n_H d_G(u) + n_G d_H(v) - d_G(u) d_H(v). \tag{19}$$

In the following theorem, we compute lower and upper bounds on the general sum-connectivity index for $G \vee H$.

Theorem 6. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G \vee H) \leq \beta_2$, where*

$$\beta_1 = 2^\alpha m_{G \vee H} (n_H \Delta_G + n_G \Delta_H)^\alpha, \beta_2 = 2^\alpha m_{G \vee H} (n_H \delta_G + n_G \delta_H)^\alpha.$$

The equality holds if and only if G and H are regular graphs.

Proof. Using (2) and (19) in equation (1), we obtain

$$\begin{aligned} \chi_\alpha(G \vee H) &= \sum_{u_1 \in V_1} \sum_{u_2 \in V(G)} \sum_{v_1 v_2 \in E(H)} (n_H(d_G(u_1) + d_G(u_2)) + n_G(d_H(v_1) \\ &\quad + d_H(v_2)) - (d_G(u_1)d_H(v_1) + d_G(u_2)d_H(v_2)))^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} (n_H(d_G(u_1) + d_G(u_2)) + n_G(d_H(v_1) \\ &\quad + d_H(v_2)) - (d_G(u_1)d_H(v_1) + d_G(u_2)d_H(v_2)))^\alpha \\ &\quad - 2 \sum_{u_1 u_2 \in E(G)} \sum_{v_1 v_2 \in E(H)} (n_H(d_G(u_1) + d_G(u_2)) + n_G(d_H(v_1) \\ &\quad + d_H(v_2)) - (d_G(u_1)d_H(v_1) + d_G(u_2)d_H(v_2)))^\alpha \\ &\geq \sum_{u_1 \in V(G)} \sum_{u_2 \in V(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (n_H \Delta_G + n_G \Delta_H - \Delta_G \Delta_H)^\alpha \\ &\quad + \sum_{v_1 \in V(H)} \sum_{v_2 \in V(H)} \sum_{u_1 u_2 \in E(G)} 2^\alpha (n_H \Delta_G + n_G \Delta_H - \Delta_G \Delta_H)^\alpha \\ &\quad - 2 \sum_{u_1 u_2 \in E(G)} \sum_{v_1 v_2 \in E(H)} 2^\alpha (n_H \Delta_G + n_G \Delta_H - \Delta_G \Delta_H)^\alpha \\ &= 2^\alpha (n_G m_H + n_H m_G - 2m_G m_H) (n_H \Delta_G + n_G \Delta_H - \Delta_G \Delta_H)^\alpha \\ &= 2^\alpha m_{G \vee H} (n_H \Delta_G + n_G \Delta_H)^\alpha. \end{aligned} \tag{20}$$

Similarly, we can compute,

$$\chi_\alpha(G \vee H) \leq 2^\alpha m_{G \vee H} (n_H \delta_G + n_G \delta_H)^\alpha. \tag{21}$$

The equality in (20) and (21) obviously holds if and only if G and H are regular. \square

1.7. The symmetric difference

The symmetric difference of G and H , denoted by $G \oplus H$, is a graph whose vertex set is $V(G \oplus H) = V(G) \times V(H)$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \oplus H$ whenever $[u_1 u_2 \in E(G)]$ or $[v_1 v_2 \in E(H)]$ but not both. The order of $G \oplus H$ is the product of number of vertices graphs of G and H , and size is $m_G n_H^2 + m_H n_G^2 - 4m_G m_H$. The degree of a vertex $(u, v) \in V(G \oplus H)$ is $d_{G \oplus H}((u, v)) = n_H d_G(u) + n_G d_H(v) - 2d_G(u)d_H(v)$. In the following theorem, we compute bounds for the general sum-connectivity index for symmetric difference of two graphs. The proof is similar to the proof of Theorem 6, hence omitted.

Theorem 7. *Let $\alpha < 0$. Then $\beta_1 \leq \chi_\alpha(G \oplus H) \leq \beta_2$, where*

$$\begin{aligned} \beta_1 &= 2^\alpha m_{G \oplus H} (n_H \Delta_G + n_G \Delta_H - 2 \Delta_G \Delta_H)^\alpha, \\ \beta_2 &= 2^\alpha m_{G \oplus H} (n_H \delta_G + n_G \delta_H - 2\delta_G \delta_H)^\alpha. \end{aligned}$$

The equality holds if and only if G and H are regular graphs.

Remark. In Theorems 1–7, we assumed that $\alpha < 0$. However, if $\alpha > 0$ then all inequalities in Theorems 1–7 will be reversed.

2. Few examples

A path P_n of length $n - 1$ is a graph with vertex set $\{u_i | i = 1, \dots, n\}$ and edge set $\{u_i u_{i+1} | i = 1, \dots, n - 1\}$. A cycle C_n of length n is a graph with vertex set $\{u_i | i = 1, \dots, n\}$ and edge set $\{u_i u_{i+1} | i = 1, \dots, n - 1\} \cup \{u_n u_1\}$. In this section, we apply the results of Section 1 to find bounds of some particular graphs.

The bottleneck graph, denoted by B , of a graph G is the hydrogen suppressed molecular graph. It is defined as $B = K_2 \odot G$. Let G_1, G_2, \dots, G_n be the pairwise disjoint graphs. The bridge graph with respect to the vertices $v_j \in G_j$ where $j = 1, 2, \dots, n$, denoted by $B(G_1, G_2, \dots, G_n, v_1, v_2, \dots, v_n)$ is the graph obtained by connecting the vertices $v_j \in V(G_j)$ and $v_{j+1} \in V(G_{j+1})$ by an edge $v_j v_{j+1}$, where $j = 1, 2, \dots, n$. Taking n copies of

the graph G , $n \geq 2$, and a vertex $v \in V(G)$, a special case of the bridge graph is defined by $G_n(G, v) = B(G, G, \dots, G, v, v, v, \dots, v)$. Note that $G_1(G, v) = G$ for any vertex $v \in V(G)$. Let \bar{v} be a vertex of P_3 of degree 2. Then it is known that $G_n(P_3, v) \cong P_n \odot \overline{K_2}$. Also, for any $u \in V(C_k)$, we know that $G_n(C_k, u) \cong P_n \odot K_2$. Using Theorem 1, the upper and lower bounds on the general sum-connectivity index of B , $P_n \odot \overline{K_2}$ and $P_n \odot K_2$ for $n \geq 2$ are as follows:

$$\begin{aligned} 2^\alpha(n_G + 1)^\alpha + 2^{\alpha+1}m_G(\Delta_G + 1)^\alpha + 2n_G(\Delta_G + n + 2)^\alpha \\ \leq \chi_\alpha(B) \leq 2^\alpha(n_G + 1)^\alpha + 2^{\alpha+1}m_G(\delta_G + 1)^\alpha + 2n_G(\delta_G + n + 2)^\alpha, \\ 2^\alpha n(4^\alpha + 2 \times 5^\alpha) - 8^\alpha \leq \chi_\alpha(P_n \odot \overline{K_2}) \leq 2^\alpha n(3^\alpha + 2^{\alpha+1}) - 6^\alpha, \\ 2^\alpha n(4^\alpha + 2^\alpha + 2 \times 3^\alpha) - 8^\alpha \leq \chi_\alpha(P_n \odot K_2) \leq \\ n(6^\alpha + 4^\alpha + 2 \times 5^\alpha) - 6^\alpha. \end{aligned}$$

The graph $G = P_n \square C_m$ and $S = C_n \square C_m$ denote a nanotube and nanotrous, respectively, where $m \geq 3$ and $n \geq 2$. Using Theorem 2, we can compute bounds on the general sum-connectivity index of nanotube G as follows:

$$m(2n - 1)8^\alpha \leq \chi_\alpha(G) \leq m(2n - 1)6^\alpha.$$

The nanotrous S is a regular graph. Using Theorem 2, we can compute the exact formula for the general sum-connectivity index of S as follows:

$$\chi_\alpha(S) = 2^{3\alpha+1}mn.$$

The suspension of a graph G , denoted by $K_1 + G$, is the join of K_1 and G . The complete bipartite graph K_{n_1, n_2} is defined as $\overline{K_{n_1}} + \overline{K_{n_2}}$. The wheel graph is defined by $W_n = K_1 + C_n$ and cone graph is defined by $C_{n_1, n_2} = C_{n_1} + \overline{K_{n_2}}$. Using Theorem 4, we can compute the bounds on the general sum-connectivity index of $K_1 + G$ as follows:

$$\begin{aligned} 2^\alpha m_G(\Delta_G + 1)^\alpha + n_G(\Delta_G + n_G + 1)^\alpha \\ \leq \chi_\alpha(K_1 + G) \leq 2^\alpha m_G(\delta_G + 1)^\alpha + n_G(\delta_G + n_G + 1)^\alpha. \end{aligned}$$

Using Theorem 4, we can compute the exact formulae for the general sum-connectivity index of K_{n_1, n_2} , W_n and C_{n_1, n_2} as follows:

$$\begin{aligned} \chi_\alpha(K_{n_1, n_2}) &= n_1 n_2 (n_1 + n_2)^\alpha, \\ \chi_\alpha(W_n) &= 6^\alpha n + n(n + 3)^\alpha, \\ \chi_\alpha(C_{n_1, n_2}) &= 2^\alpha n_1 (n_2 + 2)^\alpha + n_1 n_2 (n_1 + n_2 + 2)^\alpha. \end{aligned}$$

The fence graph, denoted by $P_n[P_2]$, is the composition of P_n and P_2 . Using Theorem 5, we can compute the bounds on the general sum-connectivity index of $P_n[P_m]$.

$$10^\alpha(5n - 4) \leq \chi_\alpha(P_n[P_2]) \leq 6^\alpha(5n - 4).$$

The closed fence graph, denoted by $C_n[P_2]$, is the composition of C_n and P_2 . The graph $C_n[P_2]$ is a regular graph. Using Theorem 5, we can compute the exact formula for the general sum-connectivity index of $C_n[P_2]$ as follows:

$$\chi_\alpha(C_n[P_2]) = 5 \times 10^\alpha n.$$

By using Theorems 3, 6 and 7, the bounds on the general sum-connectivity index of $P_n \boxtimes P_m$, $P_n \vee P_m$ and $P_n \oplus P_m$ are given below:

- 1) $4^{2\alpha}(4mn - 3m - 3n + 2) \leq \chi_\alpha(P_n \boxtimes P_m) \leq 6^\alpha(4mn - 3m - 3n + 2)$,
- 2) $4^\alpha[mn(m + n - 2) + m(2 - m) + n(2 - n) - 2](m + n - 2)^\alpha \leq \chi_\alpha(P_n \vee P_m) \leq 4^\alpha[mn(m + n - 2) + m(2 - m) + n(2 - n) - 2](m + n - 2)^\alpha$,
- 3) $4^\alpha[mn(m + n - 4) + m(4 - m) + n(4 - n) - 4](m + n - 4)^\alpha \leq \chi_\alpha(P_n \oplus P_m) \leq 2^\alpha[mn(m + n - 4) + m(4 - m) + n(4 - n) - 4](m + n - 2)^\alpha$.

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