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The sparing number of the powers of certain Mycielski graphs

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Abstract. In this paper, we discuss the sparing number of the power graphs of the Mycielski graphs of certain graph classes.

1. Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [1, 3, 14]. In this paper, by a graph we mean a simple, connected, finite and non-trivial graph $G = (V(G); E(G))$ with the set of vertices $V(G)$ and the set of edges $E(G)$. Given an integer $m \geqslant 2$, we denote by P_m the path on m vertices and by C_m the cycle on m vertices.

If r is a positive integer, the r-th power of G, denoted by G^r, is a graph with the same vertex set such that two vertices are adjacent in G^r if only if the distance between them is at most r.

The following theorem on graph powers is an important and a very useful result in our present study.

Theorem 1.1. If d is the diameter of a graph G, then G^d is a complete graph.

An *independent set* of a graph G is a subset I of the vertex set $V(G)$, such that no two elements (vertices) in I are adjacent. An independence set I of G is said to have *maximum incidence* in G if the number of edges in G having one of their end vertices in I is maximum when compared to the other independent sets of G.

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1.1. Mycielski graph of a graph

Consider a graph G with $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$. Apply the following steps to the graph G.

(i) Take the set of new vertices $U = \{u_1, u_2, u_3, \dots, u_n\}$ and add edges from each vertex u_i of U to the vertices v_j if the corresponding vertex v_i is adjacent to v_j in G,

(ii) Take another new vertex u and add edges to all elements in U.

The new graph thus obtained is called the *Mycielski graph* of G and is denoted by $\mu(G)$ (see [4]). For the ease of the notation in context of graph powers, we denote the Mycielski graph of a graph G by $\hat{G}$.

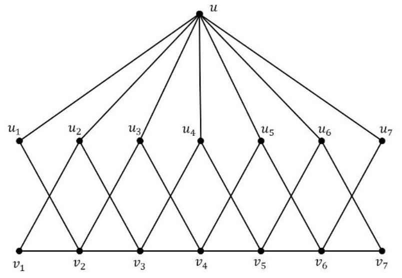
The following figures illustrate the Mycielski graphs of a path and a cycle.

1.2. Sparing number of a graph

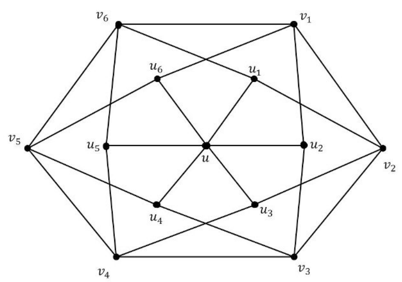
The *sumset* of two sets A and B of integers, denoted by $A + B$, is defined as $A + B = \{a + b : a \in A, b \in B\}$ (see [5]). If A or B is countably infinite, then their sumset $A + B$ will also be countably infinite. Hence, all sets we consider here are finite sets of non-negative integers.

Let X be a non-empty finite set of non-negative integers and let $\mathscr{P}(X)$ be its power set. An *integer additive set-labeling* (IASL) of a graph G (see [2, 6]) is an injective function $f : V(G) \to \mathscr{P}(X) - \{\varnothing\}$ such that the induced function $f^+ : E(G) \to \mathscr{P}(X) - \{\varnothing\}$ is defined by $f^+(uv) = f(u) + f(v)\ \forall uv \in E(G)$. A graph G which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

The cardinality of the set-label of an element (vertex or edge) of a graph G is called the *set-indexing number* of that element. An element of a



(a) Mycielski graph of P_7.



(b) Mycielski graph of C_6.

Figure 1.

given graph G is said to be a *mono-indexed element* of G if its set-indexing number is 1.

A *weak integer additive set-labeling* of a graph G is an IASL $f : V(G) \to \mathscr{P}(X) - \{\varnothing\}$, where induced function $f^+ : E(G) \to \mathscr{P}(X) - \{\varnothing\}$ is defined by $f^+(uv) = f(u) + f(v)$ such that either $|f^+(uv)| = |f(u)|$ or $|f^+(uv)| = |f(v)|$, where $f(u) + f(v)$ is the sumset of $f(u)$ and $f(v)$.

Lemma 1.2. [8] An IASI $f : V(G) \to \mathscr{P}(X) - \{\varnothing\}$ of a given graph G is a weak IASI of G if and only if at least one end vertex of every edge of G is mono-indexed.

Hence, it can be seen that both end vertices of some edges of a given graph can be (must be) mono-indexed and hence those edges are also mono-indexed. The minimum number of mono-indexed edges required in a graph G so that G admits a WIASL is called the *sparing number* of G, denoted by $\varphi(G)$ (see [8]).

Note that an independence set I is said to have maximal incidence in G if maximum number of edges in G have their one end vertex in I. Then, the sparing number of any given graph can be determined using the following theorem.

Theorem 1.3. [7] Let G be a given WIASL-graph and I be an independent set in G which has the maximal incidence in G. Then, the sparing number of G is $\varphi(G) = |E(G - I)|$.

As a new graph parameter, the studies on the sparing number of graphs have been much interesting for us. Certain studies on WIASL-graphs and their sparing numbers have been done in [2,6–11]. The sparing number of certain graph powers has also been studied in [12] and a comprehensive survey on weak integer additive set-labeling of graphs and the corresponding sparing number of different graph classes have been done in [13]. The following are the relevant results, we use from these studies.

Theorem 1.4. Let G be a graph on n vertices. Then, we have the following results.

(i) If G is bipartite, then $\varphi(G) = 0$,
(ii) If G is an odd cycle, then $\varphi(G) = 1$,
(iii) If G is a complete graph, then $\varphi(G) = \frac{1}{2}(n - 1)(n - 2)$.

In this paper, we investigate the sparing number of integer powers of Mycielski graphs of certain graph classes.

2. Sparing number of Mycielski graphs

First, the relation between the sparing number of an arbitrary graph G and that of its Mycielski graph is explained in the following result.

Theorem 2.1. The sparing number of the Mycielski graph of any graph G is $\varphi(\hat{G}) = \varphi(G) + |V(G)| - |I|$, where I is an independence set in G with maximal incidence in G.

Proof. Let G be a graph with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$. Let $I = \{v'_1, v'_2, v'_3, \ldots, v'_k\}$ be an independence set with the maximal incidence in G, where $v'_i = v_j$ for some $1 \leqslant j \leqslant n$.

Let $\hat{G}$ be the Mycielski graph of a given graph G. Let V be the vertex set of G and let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and $\{u\}$ be the sets of newly introduced vertices in $\hat{G}$. Note that all mono-indexed edges in G are mono-indexed in $\hat{G}$ also.

Since I is an independent set in G, the corresponding set $I^* = \{u'_1, u'_2, u'_3, \ldots, u'_k\}$, where $u'_i = u_j$ for some $1 \leqslant j \leqslant n$, is an independent set in U such that $I \cup I^*$ is an independence set in $\hat{G}$ with maximal incidence in $\hat{G}$. Hence, all the vertices in the set $U - I^*$ must have singleton set-labels. Since some vertices in U have non-singleton set-labels, the vertex u must also have a singleton set-label. Therefore, all edges connecting the vertex u to the vertices in $U - I^*$ are mono-indexed. Hence, the sparing number of $\hat{G}$ is $\varphi(\hat{G}) = \varphi(G) + |U - I^*| = \varphi(G) + |U| - |I^*| = \varphi(G) + |V| - |I|$. $\square$

Invoking Theorem 1.1 and Theorem 1.4, we can establish the following result.

Theorem 2.2. Let G be a graph on n vertices with diameter d. Then,

(i) if $d \leqslant 2$, the sparing number of $\hat{G}^r$ is $n(2n-1)$ for any positive integer $r \geqslant 2$.

(ii) if $d = 3$, the sparing number of $\hat{G}^r$ is $n(2n-1)$ for any positive integer $r \geqslant 3$.

(iii) if $d \geqslant 4$, the sparing number of $\hat{G}^r$ is $n(2n-1)$ for any positive integer $r \geqslant 4$.

Proof. Let V be the vertex set of G and let $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and $\{u\}$ be the sets of newly introduced vertices in $\hat{G}$. Here, we have to consider the following cases.

(i) Let $d < 3$. Then, in the graph $\hat{G}$, the vertex u is at a distance 2 from every vertex of V and the distance between a vertex in V and a vertex in U is at most 2. Also, the distances between any two vertices in

V and the distances between any two vertices in U are also 2. Hence, the diameter of $\hat{G}$ is 2 and by Theorem 1.1, $\hat{G}^2$ is a complete graph. Therefore, by Theorem 1.4, we have $\varphi(\hat{G}^r) = n(2n-1)$, for any positive positive integer $r \geqslant 2$.

(ii) Let $d = 3$. Then, in the graph $\hat{G}$, as mentioned above, the vertex u is at a distance 2 from every vertex of V and the distance between any two vertices of U is also 2. The distance between a vertex in V and a vertex in U is at most 3. Hence, the diameter of $\hat{G}$ is 3 and by Theorem 1.1, $\hat{G}^3$ is a complete graph. Therefore, by Theorem 1.4, we have $\varphi(\hat{G}^r) = n(2n-1)$, for any positive positive integer $r \geqslant 3$.

(iii) Let $d \geqslant 4$. Then, in addition to the facts mentioned in (i), it can be noted that, in $\hat{G}$, the distance between any two vertices in V is at most 4. Hence, the diameter of $\hat{G}$ is 4 and by Theorem 1.1, we have $\hat{G}^4$ is a complete graph. Therefore, by Theorem 1.4, we have $\varphi(\hat{G}^r) = n(2n-1)$, for any positive integer $r \geqslant 4$. □

In view of Theorem 2.2, the diameter of the Mycielski graph of any graph with large diameter (greater than 3) is 4 and hence we need only to discuss the sparing number of squares and cubes of the Mycielski graphs of various graph classes.

3. Sparing number of powers of Mycielski graphs of paths

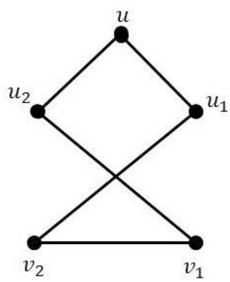
We recall that, given an integer $n \geqslant 2$, we denote by P_n the path on n vertices and by C_n the cycle on n vertices. Illustrations to different powers of the Mycielski graphs of the first few paths are given below (see Figures 2, 3, and 4).

First, we discuss the sparing number of different powers of the Mycielski graphs of paths. In view of Proposition 2.1, we have

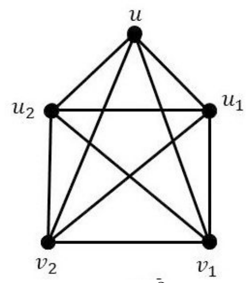
Proposition 3.1. The sparing number of $\hat{P}_n$ is given by $\varphi(\hat{P}_n) = \lfloor \frac{n}{2} \rfloor$.

Proof. We know that an independent set of P_n with maximal incidence in P_n consists of $\lceil \frac{n}{2} \rceil$ vertices. Therefore, by Theorem 2.1, $\varphi(\hat{P}_n) = \varphi(P_n) + n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$, since $\varphi(P_n) = 0$. □

Figures 2, (a), 3, (a), and 4, (a), depict the above proposition. In Figure 2, (a), the vertices $\{v_2, u_2, u\}$ must be mono-indexed and hence the edge u_2u is a mono-indexed edge in $\hat{P}_2$. In Figure 3, (a), the vertices v_1, v_3 in V and u_1, u_3 in U are in the independence set with maximal incidence and hence can have non-singleton set-labels. Therefore, the vertices v_2, u_2, u

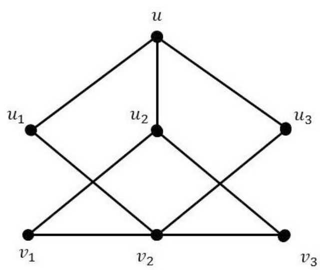


(a) $\hat{P}_2$.

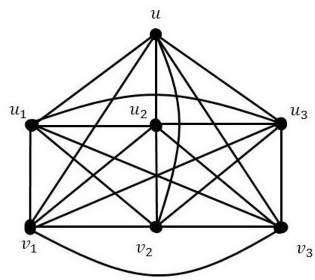


(b) $\hat{P}_2^{\,2}$.

FIGURE 2.

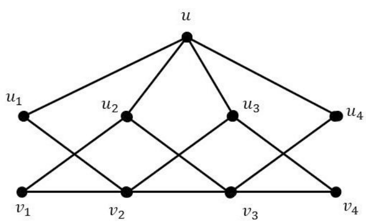


$\hat{P}_3$.

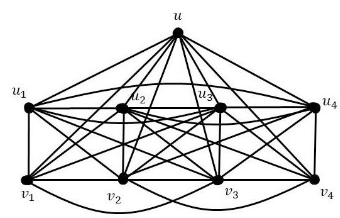


$\hat{P}_3^{\,2}$.

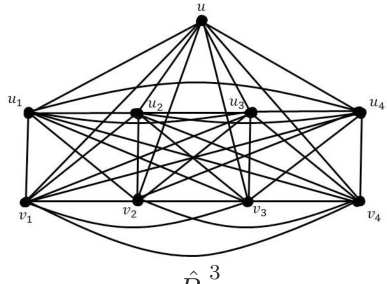
FIGURE 3.



$\hat{P}_4$.



$\hat{P}_4^{\,2}$.



$\hat{P}_4^{\,3}$.

FIGURE 4.

are mono-indexed vertices and the edge u_2u is the mono-indexed edge in $\hat{P}_3$. In Figure 4, (a), the vertices v_1, v_3 in V and u_1, u_3 in U are in the independence set with maximal incidence and hence can have non-singleton set-labels. Then, the vertices v_2, v_4 in V and u_2, u_4 in U and the vertex u must be mono-indexed. Hence, the edges uu_2 and uu_4 are mono-indexed edges in $\hat{P}_4$.

Also, note that if $n = 2, 3$, the diameter of $\hat{P}_n$ is 2 (see Figures 3, (b), and 4, (b)) and hence invoking Theorem 1.4, we have the sparing number of $\hat{P}_2^{\,2}$ is 6 and that of $\hat{P}_3^{\,2}$ is 15.

Now, it remains to investigate the sparing number of the square and cube of the Mycielski graphs of paths P_n, where $n \geqslant 5$. The following theorem discusses the sparing number of the square of $\hat{P}_n$, for $n \geqslant 4$.

Theorem 3.2. For $n \geqslant 4$, the sparing number of $\hat{P}_n^{\,2}$ is

$$\varphi(\hat{P}_n^{\,2}) = \begin{cases} \frac{1}{6}(3n^2 + 25n - 12) & \text{if } n \equiv 0 (\text{mod } 3) \\ \frac{1}{6}(3n^2 + 25n - 28) & \text{if } n \equiv 1 (\text{mod } 3) \\ \frac{1}{6}(3n^2 + 25n - 20) & \text{if } n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof. First, we have to identify the number of edges in $\hat{P}_n^{\,2}$. For this purpose, we analyse the adjacency pattern in $\hat{P}_n^{\,2}$ in the following way.

The vertex u is adjacent to every vertex in U and V. Hence, $d(u) = 2n$. Also, any two vertices u_i and u_j in U, there exists a path u_iuu_j in $\hat{P}_n$. Therefore, any two vertices in U are adjacent in $\hat{P}_n^{\,2}$. In addition to this, the adjacency between the vertices in U and V in $\hat{P}_n^{\,2}$ can be determined as follows.

(i) The vertex u_1 is adjacent to v_1, v_2 and v_3, and u_n is adjacent to v_{n-2}, v_{n-1} and v_n. Therefore, $d(u_1) = d(u_n) = n + 3$.

(ii) The vertex u_2 is adjacent to v_1, v_2, v_3 and v_4, and u_{n-1} is adjacent to $v_{n-3}, v_{n-2}, v_{n-1}$ and v_n. Therefore, $d(u_n) = d(u_{n-1}) = n + 4$.

(iii) For $3 \leqslant i \leqslant n-2$, u_i is adjacent to $v_{i-2}, v_{i-1}, v_i, v_{i+1}$ and v_{i+2}. That is, $d(u_i) = n + 5$, for $3 \leqslant i \leqslant n - 2$.

Now, we find the degree of the vertices in the set V. As mentioned earlier, we note that every element of V is adjacent to the vertex u. Remaining adjacencies of the vertices in V can be written as follows.

(i) The vertex v_1 is adjacent to v_2, v_3 in V and u_1, u_2, u_3 in U. Similarly, v_n is adjacent to v_{n-2}, v_{n-1} in V and u_{n-2}, u_{n-1}, u_n. Therefore, $d(v_1) = d(v_n) = 6$.

(ii) The vertex v_2 is adjacent to v_1, v_3, v_4 in V and u_1, u_2, u_3, u_4 in U. Similarly, v_{n-1} is adjacent to v_{n-3}, v_{n-2}, v_n in V and u_{n-3}, u_{n-2}, u_{n-1}, u_n in U. Therefore, $d(u_n) = d(u_{n-1}) = 8$.

(iii) For $3 \leqslant i \leqslant n-2$, v_i is adjacent to v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2} in V and u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2} in U. That is, $d(v_i) = 10$, for $3 \leqslant i \leqslant n-2$.

Therefore, the number of edges in $\hat{P_n}^2$ is given by

$$\begin{aligned}
|E(\hat{P_n}^2)| &= \frac{1}{2} d_{v\in \hat{P_n}^2}(v) \\
&= \frac{1}{2}\left[2n + 2\left((n+3) + (n+4) + 6 + 8\right) + ((n-4)(n+5) + 10)\right] \\
&= \frac{1}{2}\left(n^2 + 17n - 18\right).
\end{aligned}$$

Now, we need to identify the independence set I of $\hat{P_n}^2$ with maximal incidence in $\hat{P_n}^2$. We note that if $u \in I$, then no other vertex can be included in I and in this case, we have $2n$ edges with non-singleton set-labels. This satisfy the condition of an independent set with maximum incidence.

Also, note that only one vertex, say v_n from U, can be included into I, but there are some other vertices in V which are not adjacent to v_n, which can also be included into I. As $d(u_i) > d(v_i)$ for all i, we need to consider an independence set containing one element from U, rather than considering an independence set, all whose elements are taken from V.

Since independent vertices in V are at a distance 3 in $\hat{P_n}^2$, we have to consider the following cases.

Case 1. If $n \equiv 0 (\mathrm{mod}\ 3)$, then the independence set with maximal incidence, which consist of exactly one element from U is $I = \{v_3, v_6, v_9, \ldots, v_{n-3}, u_n\}$. The number of edges incident on each of the above independent set can be found out by adding the degree of vertices in them. Hence, the number of edges incident on I is given by $\epsilon_1 = 10 + 10(\frac{n-6}{3}) + (n+3) = \frac{1}{3}(13n-21)$. Therefore, I_3 is the required independence set with maximum incidence in $\hat{P_n}^2$.

Hence, all vertices in I can be labeled by distinct non-singleton subsets of the ground set X and all the $\frac{1}{3}(13n-21)$ edges incident on I have non-singleton set-labels in $\hat{P_n}^2$. Therefore, by Theorem 1.4, the sparing number of $\hat{P_n}^2$ is $\varphi(\hat{P_n}^2) = \frac{1}{2}\left(n^2+17n-18\right) - \frac{1}{3}(13n-21) = \frac{1}{6}(3n^2+25n-12)$.

Case 2. If $n \equiv 1 (\mathrm{mod}\ 3)$, note that the set $I = \{v_1, v_4, v_7, \ldots, v_{n-3}, u_n\}$ is the required independence set with maximum incidence and the number of edges incident on the set I is $\epsilon_1 = 6 + 10(\frac{n-4}{3} + (n+3)) = \frac{13}{3}(n-1)$. Therefore, by Theorem 1.4, we have $\varphi(\hat{P_n}^2) = \frac{1}{2}\left(n^2 + 17n - 18\right) - \frac{1}{3}(13n - 13) = \frac{1}{6}(3n^2 + 25n - 28)$.

Case 3. If $n \equiv 2(\text{mod } 3)$, the required independence set with maximal incidence is $I = \{v_2, v_5, v_8, \ldots, v_{n-3}, u_n\}$ and the number of edges incident with I is $\epsilon_1 = 8 + 10(\frac{n-5}{3}) + (n+3) = \frac{1}{3}(13n - 17)$. Therefore, by Theorem 1.4, $\varphi(\hat{P_n}^2) = \frac{1}{2}\left(n^2 + 17n - 18\right) - \frac{1}{3}(13n - 17) = \frac{1}{6}(3n^2 + 25n - 20)$. □

Now, we have the diameter $\hat{P_4}$ is 3. Then, $\hat{P_4}^3$ is a complete graph, (see Figure 4, (c)) and hence by Theorem 1.4, $\hat{P_4}^3$ has 28 mono-indexed edges.

The following theorem discusses the sparing number of the cube of Mycielski graph of paths on $n \geqslant 5$ vertices.

Theorem 3.3. For $n \geqslant 5$, the sparing number of $\hat{P_n}^3$ is

$$\varphi(\hat{P_n}^3) = \begin{cases} \frac{1}{4}(5n^2 + 11n - 12) & \text{if } n \equiv 0(\text{mod } 4) \\ \frac{1}{4}(5n^2 + 8n - 18) & \text{if } n \equiv 1(\text{mod } 4) \\ \frac{1}{4}(5n^2 + 9n - 18) & \text{if } n \equiv 2(\text{mod } 4) \\ \frac{1}{4}(5n^2 + 10n - 15) & \text{if } n \equiv 3(\text{mod } 4). \end{cases}$$

Proof. As in the previous theorem, first we analyse adjacency pattern in $\hat{P_n}^3$ to find its number of edges. Note that the vertex u in the graph $\hat{P_n}^3$, is adjacent to all the vertices in U and V. Since the distance between any two vertices in U is 2 and the distances between any vertex in U and V is at most 3, every vertex of U is also adjacent to all other vertices in $\hat{P_n}^3$. Hence, for any vertex v in $U \cup \{u\}$, we have $d(v) = 2n$.

Also, each vertex in V is adjacent to all vertices in U and to the vertex u. Now, what remains is to find the adjacency between the vertices in V. This can be analysed as follows.

(i) The vertex v_1 is adjacent to v_2, v_3 and v_4. Similarly, v_n is adjacent to v_{n-3}, v_{n-2} and v_{n-1}. Therefore, $d(v_1) = d(v_n) = n + 4$.

(ii) The vertex v_2 is adjacent to v_1, v_3, v_4 and v_5. Similarly, v_n is adjacent to $v_{n-4}, v_{n-3}, v_{n-2}$ and v_n. Therefore, $d(v_2) = d(v_{n-1}) = n + 5$.

(iii) The vertex v_3 is adjacent to v_1, v_2, v_4, v_5 and v_6. Similarly, v_n is adjacent $v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}$ and v_n. Therefore, $d(v_3) = d(v_{n-2}) = n + 6$.

(iv) For $4 \leqslant i \leqslant n-3$, v_i is adjacent v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2} and v_{i+3}. Hence, $d(v_i) = n + 7$, for $4 \leqslant i \leqslant n - 3$.

Hence, the number of edges in $\hat{P_n}^3$ is given by

$$
\begin{aligned}
|E(\hat{P_n}^3)| &= \frac{1}{2} d_{v\in \hat{P_n}^3}(v)\\
&= \frac{1}{2}\left[(n+1)2n + 2\left((n-4)+(n+5)+(n+6)\right)+(n-6)(n+7)\right]\\
&= \frac{1}{2}\left(3n^2+9n-12\right).
\end{aligned}
$$

Now, as in the previous theorem, we proceed to determine an independence set I of $\hat{P_n}^3$ with maximal incidence in $\hat{P_n}^3$ as explained below.

An independence set in $\hat{P_n}^3$ can have at most one vertex from $U \cup \{u\}$ and if such a vertex is in an independent set S of $\hat{P_n}^3$, then S must be a singleton set. If we label this vertex by a non-singleton set, then $2n$ edges in $\hat{P_n}^3$ will have non-singleton set-labels. Clearly, this does not satisfy the requirements of maximal incidence in $\hat{P_n}^3$. Hence, we need to choose such an independent set from the vertex set V. In this case, we need to consider the following cases.

Case 1. If $n \equiv 0 \pmod 4$, we can see that the set $I_1 = \{v_2, v_6, v_{10}, \dots, v_{n-6}, v_{n-2}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I_1 is $\epsilon_1 = (n+5) + (\frac{n-8}{4})(n+7) + (n+6) = \frac{1}{4}(n^2+7n-8)$. The set $I_2 = \{v_3, v_7, v_{11}, \dots v_{n-5}, v_{n-1}\}$ is also an independent sets with maximum incidence and with $\frac{1}{4}(n^2+7n-8)$ edges incident with it. Hence, either all the vertices in I_1 or those in I_2 can be labeled by distinct non-singleton subsets of the ground set X so that $\frac{1}{4}(n^2+7n-8)$ edges in $\hat{P_n}^3$ have non-singleton set-labels. Therefore, the sparing number of $\hat{P_n}^3$ is $\varphi(\hat{P_n}^3) = \frac{1}{2}\left(3n^2+9n-12\right) - \frac{1}{4}(n^2+7n-8) = \frac{1}{4}(5n^2+11n-12)$.

Case 2. If $n \equiv 1 \pmod 4$, note that the set $I_1 = \{v_2, v_6, v_{10}, \dots, v_{n-7}, v_{n-3}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I_1 is $\epsilon_1 = (n+5)+(\frac{n-5}{4})(n+7)+(n+6) = \frac{1}{4}(n^2+6n-15)$. The set $I_2 = \{v_4, v_8, v_{12}, \dots v_{n-5}, v_{n-1}\}$ is also an independent set with maximum incidence and with $\frac{1}{4}(n^2+6n-15)$ edges incident with it. Hence, either all the vertices in I_1 or those in I_2 can be labeled by distinct non-singleton subsets of the ground set X so that the number of edges in $\hat{P_n}^3$, having non-singleton set-labels is $\frac{1}{4}(n^2+6n-15)$. Therefore, the sparing number of $\hat{P_n}^3$ is $\varphi(\hat{P_n}^3) = \frac{1}{2}\left(3n^2+9n-12\right) - \frac{1}{4}(n^2+6n-15) = \frac{1}{4}(5n^2+8n-18)$.

Case 3. If $n \equiv 2 \pmod 4$, we have the set $I_1 = \{v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}\}$ is one of the independent sets with maximum incidence in $\hat{P_n}^3$ and the number of edges incident on I_1 is $\epsilon_1 = (n+4) + (\frac{n-6}{4})(n+7) + (n+5) = \frac{1}{4}(n^2+9n-6)$. The set $I_2 = \{v_2, v_6, v_{10}, \ldots v_{n-4}, v_n\}$ is also an independent set with maximum incidence and with $\frac{1}{4}(n^2+9n-6)$ edges incident with it. Hence, either all the vertices in I_1 or those in I_2 can be labeled by distinct non-singleton subsets of the ground set X so that the number of edges in $\hat{P_n}^3$, having non-singleton set-labels is $\frac{1}{4}(n^2+9n-6)$. Therefore, the sparing number of $\hat{P_n}^3$ is $\varphi(\hat{P_n}^3) = \frac{1}{2}\left(3n^2+9n-12\right) - \frac{1}{4}(n^2+9n-6) = \frac{1}{4}(5n^2+9n-18)$.

Case 4. If $n \equiv 3 \pmod 4$, we have the set $I_1 = \{v_1, v_5, v_9, \ldots, v_{n-6}, v_{n-2}\}$ is one of the independent sets with maximum incidence in $\hat{P_n}^3$ and the number of edges incident on I_1 is $\epsilon_1 = (n+4) + (\frac{n-7}{4})(n+7) + (n+6) = \frac{1}{4}(n^2+8n-9)$. The sets $I_2 = \{v_2, v_6, v_{10}, \ldots v_{n-5}, v_{n-1}\}$ and $I_3 = \{v_3, v_7, v_{11}, \ldots v_{n-4}, v_n\}$ are also the independent sets with maximum incidence and with $\frac{1}{4}(n^2+8n-9)$ edges incident with it. Hence, either all the vertices in I_1 or I_2 or I_3 can be labeled by distinct non-singleton subsets of the ground set X so that the number of edges in $\hat{P_n}^3$, having non-singleton set-labels is $\frac{1}{4}(n^2+8n-9)$. Therefore, the sparing number of $\hat{P_n}^3$ is $\varphi(\hat{P_n}^3) = \frac{1}{2}\left(3n^2+9n-12\right) - \frac{1}{4}(n^2+8n-9) = \frac{1}{4}(5n^2+10n-15)$. □

4. Sparing number of powers of Mycielski graphs of cycles

In this section, we discuss the sparing number of different powers of Mycielski graphs of cycles. The following result is on the sparing number of the Mycielski graphs of cycles.

Proposition 4.1. The sparing number of the Mycielski graph of a cycle C_n is

$$\varphi(\hat{C_n}) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+3}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First note that an even cycle need not have any mono-indexed edges, while odd cycle must have at least one mono-indexed edge That is, $\varphi(C_n) = 0$ for an even cycle C_n and $\varphi(C_n) = 1$ for an even cycle C_n. Also, we have any independence set with maximal incidence in a cycle C_n consists of $\lfloor \frac{n}{2} \rfloor$ elements. Then, by Theorem 2.1, we have $\varphi(\hat{C_n}) = \varphi(C_n) + n - \lfloor \frac{n}{2} \rfloor$.

Therefore, if n is even, then $\varphi(\hat{C_n}) = n - \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$ and if n is odd, then $\varphi(\hat{C_n}) = 1 + n - \lfloor \frac{n}{2} \rfloor = 1 + \lceil \frac{n}{2} \rceil = \frac{n+3}{2}$. □

Next, we proceed to determine the sparing number of different powers of the Mycielski graphs of cycles. First, recall that the diameter of a cycle C_2 is $\lfloor \frac{n}{2} \rfloor$. Hence, the square of the Mycielski graphs of the cycles C_3 and C_4 and C_5 are the complete graphs K_7, K_9 and K_11 respectively. Hence, by Theorem 1.4, we have $\varphi(\hat{C}_3^2) = 15$, $\varphi(\hat{C}_4^2) = 28$ and $\varphi(\hat{C}_5^2) = 45$.

Now, we determine the sparing number of the square of the Mycielski graphs of the cycles for $n \geqslant 6$ in the following theorem.

Theorem 4.2. For $n \geqslant 6$, the sparing number of $\hat{C}_n^2$ is

$$\varphi(\hat{C}_n^2) = \begin{cases} \frac{1}{6}(3n^2 + 25n + 30) & \text{if } n \equiv 0 (\text{mod } 3) \\ \frac{1}{6}(3n^2 + 25n + 50) & \text{if } n \equiv 1 (\text{mod } 3) \\ \frac{1}{6}(3n^2 + 25n + 70) & \text{if } n \equiv 2 (\text{mod } 3). \end{cases}$$

Proof. First, we analyse the adjacency pattern in $\hat{C}_n^2$ in the following way. In this proof, we use the convention that $v_{n+r} = v_r$, for any positive integer $r \leqslant n$.

In $\hat{C}_n^2$, the vertex u is adjacent to every vertex in $U \cup V$. Hence, $d(u) = 2n$. As explained in previous theorems, any two vertices u_i and u_j in U are adjacent in $\hat{C}_n^2$. In addition to this, each vertex u_i is adjacent to the vertices $v_{i-2}, v_{i-1}, v_i, v_{i+1}$ and v_{i+2}. That is, $d(u_i) = n+5$, where $1 \leqslant i \leqslant n$. Now, any vertex v_i is adjacent to the vertices $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ in V and $u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}$ in U. That is, $d(v_i) = 10$, where $1 \leqslant i \leqslant n$. Therefore, the number of edges in $\hat{C}_n^2$ is given by $|E(\hat{C}_n^2)| = \frac{1}{2}\left[2n + n(n+5) + 10n\right] = \frac{1}{2}(n^2 + 17n)$.

Now, we need to find out an independence set I of $\hat{C}_n^2$ with maximal incidence in $\hat{C}_n^2$. As mentioned in previous theorems, we can see that if the vertex u, belongs to the set I, then it will be a singleton set with $2n$ incidences. Also, at most one vertex of U can belong to any independent set of $\hat{C}_n^2$. Since, $d(v_i) < d(u_i)$ for all $1 \leqslant i \leqslant n$, every independent set with maximal incidence consists of one vertex from U.

Note that the independent vertices in V are at the distance 3. Then, we have the following cases.

Case 1. If $n \equiv 0 (\text{mod } 3)$, then the set $I = \{v_1, v_4, v_7, \ldots, v_{n-5}, u_{n-2}\}$ is one of the required independence set with maximal incidence. The number of edges incident with I is $\epsilon_1 = 10\left(\frac{n-3}{3}\right) + (n+5) = \frac{1}{3}(13n - 15)$. (Note that the sets $\{v_5, v_5, v_8, \ldots, v_{n-4}, u_{n-1}\}$ and $\{v_3, v_6, v_9, \ldots, v_{n-4}, u_n\}$ are also the independent sets with the same number $\frac{1}{3}(13n-15)$ of incidences). Hence, the sparing number of $\hat{C}_n^2 = \frac{1}{2}(n^2 + 17n) - \frac{1}{3}(13n - 15) = \frac{1}{6}(3n^2 + 25n + 30)$.

Case 2. If $n \equiv 1 (\text{mod } 3)$, then the set $I = \{v_1, v_4, v_7, \ldots, v_{n-6}, u_{n-3}\}$ is one of the required independence set with maximal incidence. The number of edges incident with I is $\epsilon_1 = 10\left(\frac{n-4}{3}\right) + (n+5) = \frac{1}{3}(13n-25)$. Hence, the sparing number of $\hat{C}_n^2 = \frac{1}{2}(n^2+17n) - \frac{1}{3}(13n-25) = \frac{1}{6}(3n^2+25n+50)$.

Case 3. If $n \equiv 1 (\text{mod } 3)$, then the set $I = \{v_1, v_4, v_7, \ldots, v_{n-7}, u_{n-4}\}$ is one of the required independence set with maximal incidence. The number of edges incident with I is $\epsilon_1 = 10\left(\frac{n-5}{3}\right) + (n+5) = \frac{1}{3}(13n-35)$. Hence, the sparing number of $\hat{C}_n^2 = \frac{1}{2}(n^2+17n) - \frac{1}{3}(13n-35) = \frac{1}{6}(3n^2+25n+70)$. □

Next, we proceed to determine the sparing number of the cubes of the Mycielski graphs of the cycles. We know that the cycles C_6 and C_7 have diameter 3 and hence by Theorem 1.1, the Mycielski graphs $\hat{C}_6^3$ and $\hat{C}_7^3$ are complete graphs. Hence, by Theorem 1.4, $\varphi(\hat{C}_6^3) = 66$ and $\varphi(\hat{C}_7^3) = 91$.

For $n \geqslant 8$, the Mycielski graphs $\hat{C}_n^3$ are not complete graphs and we determine the sparing number of the graphs $\hat{C}_n^3$ for $n \geqslant 8$ in the following theorem.

Theorem 4.3. For $n \geqslant 8$, the sparing number of $\hat{C}_n^3$ is

$$\varphi(\hat{C}_n^3) = \begin{cases} \frac{1}{4}(5n^2 + 11n) & \text{if } n \equiv 0 (\text{mod } 4) \\ \frac{1}{4}(5n^2 + 12n + 7) & \text{if } n \equiv 1 (\text{mod } 4) \\ \frac{1}{4}(5n^2 + 13n + 14) & \text{if } n \equiv 2 (\text{mod } 4) \\ \frac{1}{4}(5n^2 + 14n + 21) & \text{if } n \equiv 3 (\text{mod } 4). \end{cases}$$

Proof. In $\hat{C}_n^3$, vertex u is adjacent to all vertices in $U \cup V$. Since the distance between any two vertices in U is 2 and the distances between a vertex in U and a vertex in V is at most 3,every vertex of U is also adjacent to all other vertices in $\hat{C}_n^3$. Hence, for any vertex v in $U \cup \{u\}$, we have $d(v) = 2n$. Also, each vertex in V is also adjacent to every vertex in $U \cup \{u\}$. In addition to this, for $1 \leqslant i \leqslant n$, the vertex v_i is adjacent to the vertices $v_{i-3}, v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$ and v_{i+3}. Hence, $d(v_i) = n + 7$ for all $v_i \in V$. Hence, the number of edges in $\hat{C}_n^3$ is given by $|E(\hat{C}_n^3)| = \frac{1}{2} = 2n(n+1) + n(n+7) = \frac{1}{2}(3n^2 + 9n)$.

As pointed out in earlier results, the independence set containing the vertex u is a singleton set with $2n$ incidences, which does not meet the requirements of an independence set with maximum incidence. As, at most one vertex of U can belong to any independent set of $\hat{C}_n^2$ and $d(v_i) < d(u_i)$ for all $1 \leqslant i \leqslant n$, every independent set with maximal incidence consists of exactly one vertex from U and all other vertices (which are independent in $U \cup V$) from V. Now, note that, in $\hat{C}_n^3$, the independent vertices are at the distance 4. Hence we need to consider the following cases.

Case 1. If $n \equiv 0 \,(\mathrm{mod}\ 4)$, then the set $I = \{v_1, v_5, v_9, \ldots, v_{n-3}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I is $\epsilon_1 = (\frac{n}{4})(n+7) = \frac{1}{4}(n^2+7n)$. (Also, note that the sets $\{v_2, v_6, v_{10}, \ldots, v_{n-2}\}$, $\{v_3, v_7, v_{11}, \ldots, v_{n-1}\}$ and $\{v_4, v_8, v_{12}, \ldots, v_n\}$ are also the independence sets the same number of incidences). Hence, all vertices in I can be labeled by distinct non-singleton subsets of the ground set X so that $\frac{1}{4}(n^2+7n)$ edges in $\hat{C}_n^3$ have non-singleton set-labels. Therefore, in this case, the sparing number of $\hat{C}_n^3$ is $\varphi(\hat{C}_n^3) = \frac{1}{2}\left(3n^2+9n\right) - \frac{1}{4}(n^2+7n) = \frac{1}{4}(5n^2+11n)$.

Case 2. If $n \equiv 1 \,(\mathrm{mod}\ 4)$, then the set $I = \{v_1, v_5, v_9, \ldots, v_{n-4}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I is $\epsilon_1 = (\frac{n-1}{4})(n+7) = \frac{1}{4}(n^2+6n-7)$. Hence, all vertices in I can be labeled by distinct non-singleton subsets of the ground set X so that $\frac{1}{4}(n^2+6n-7)$ edges in $\hat{C}_n^3$ have non-singleton set-labels. Therefore, in this case, the sparing number of $\hat{C}_n^3$ is $\varphi(\hat{C}_n^3) = \frac{1}{2}\left(3n^2+9n\right) - \frac{1}{4}(n^2+6n-7) = \frac{1}{4}(5n^2+12n+7)$.

Case 3. If $n \equiv 2 \,(\mathrm{mod}\ 4)$, then the set $I = \{v_1, v_5, v_9, \ldots, v_{n-5}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I is $\epsilon_1 = (\frac{n-2}{4})(n+7) = \frac{1}{4}(n^2+5n-14)$. Hence, all vertices in I can be labeled by distinct non-singleton subsets of the ground set X so that $\frac{1}{4}(n^2+5n-14)$ edges in $\hat{C}_n^3$ have non-singleton set-labels. Hence, in this case, the sparing number of $\hat{C}_n^3$ is $\varphi(\hat{C}_n^3) = \frac{1}{2}\left(3n^2+9n\right) - \frac{1}{4}(n^2+5n-14) = \frac{1}{4}(5n^2+13n+14)$.

Case 4. If $n \equiv 3 \,(\mathrm{mod}\ 4)$, then the set $I = \{v_1, v_5, v_9, \ldots, v_{n-6}\}$ is one of the independent sets with maximum incidence and the number of edges incident on I is $\epsilon_1 = (\frac{n-3}{4})(n+7) = \frac{1}{4}(n^2+4n-21)$. Hence, if we label all vertices in I by distinct non-singleton subsets of the ground set X, then $\frac{1}{4}(n^2+4n-21)$ edges in $\hat{C}_n^3$ have non-singleton set-labels. Therefore, in this case, the sparing number of $\hat{C}_n^3$ is $\varphi(\hat{C}_n^3) = \frac{1}{2}\left(3n^2+9n\right) - \frac{1}{4}(n^2+4n-21) = \frac{1}{4}(5n^2+14n+21)$. $\square$

5. Sparing number of powers of the Mycielski graphs of some related graphs

Another two interesting graph classes related to paths and cycles are fan graphs and wheel graphs. A *fan graph* F_{n+1} is the graph obtained by drawing edges from all vertices of a path P_n to an external vertex, while a *wheel graph* W_n is the graph obtained by drawing edges from all vertices of a cycle C_n to an external vertex. Then, we have the following results.

Proposition 5.1. For a fan graph F_{n+1}, we have

(i) $\varphi(\hat{F}_{n+1}) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$

(ii) $\varphi(\hat{F}^2_{n+1}) = 2n^2 + 3n + 1.$

Proof. (i) Note that $\varphi(F_{n+1}) = \lfloor \frac{n}{2} \rfloor$ (see [13]). Also, we have the independence set with maximal incidence in a fan graph F_{n+1} is $\lceil \frac{n}{2} \rceil$ and hence by Theorem 2.1, we have

$$\begin{aligned} \varphi(\hat{F}_{n+1}) &= \varphi(F_{n+1}) + |V(F_{n+1})| - |I| = \lfloor \frac{n}{2} \rfloor + n + 1 - \lceil \frac{n}{2} \rceil \\ &= \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(ii) We know that the diameter of the fan graph F_{n+1} is 2 and hence $\hat{F}^2_{n+1}$ is a complete graph on $2n+3$ vertices. Therefore, by Theorem 1.4, $\varphi(\hat{F}^2_{n+1}) = 2n^2 + 3n + 1$. □

In a similar way, the sparing number of the wheel graph W_{n+1} is also determined in the following proposition.

Proposition 5.2. For a wheel graph W_{n+1}, we have

(i) $\varphi(\hat{W}_{n+1}) = \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n+2 & \text{if } n \text{ is odd.} \end{cases}$

(ii) $\varphi(\hat{W}^2_{n+1}) = 2n^2 + 3n + 1.$

Proof. (i) Note that $\varphi(W_{n+1}) = \lceil \frac{n}{2} \rceil$ (see [13]). Also, we have the independence set with maximal incidence in a wheel graph W_{n+1} is $\lceil \frac{n}{2} \rceil$ and hence by Theorem 2.1, we have

$$\begin{aligned} \varphi(\hat{W}_{n+1}) &= \varphi(W_{n+1}) + |V(W_{n+1})| - |I| = \lceil \frac{n}{2} \rceil + (n+1) - \lfloor \frac{n}{2} \rfloor \\ &= \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n+2 & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

(ii) We know that the diameter of the wheel graph W_{n+1} is also 2 and hence $\hat{W}^2_{n+1}$ is a complete graph on $2n+3$ vertices. Therefore, by Theorem 1.4, $\varphi(\hat{W}^2_{n+1}) = 2n^2 + 3n + 1$. □

6. Conclusion

In this paper, we study sparing number of the Mycielski graphs of paths and cycles. This study can be extended to many path and cycle related graphs like sun graphs, sunlet graphs, web graphs, helm graphs, dragon graphs etc. Further studies on the sparing number of the Mycielski graphs of several other well-known graph classes remain open.

Studies on the sparing number of the Mycielski graphs of the line graphs and total graphs of certain standard graphs classes seem to be promising for future investigations. Similar studies on other associated graphs such as the subdivisions, super-subdivisions, homeomorphic graphs etc. of some graph classes are also possible.

Further studies on many other parameters of the different parameters of the Mycielski graphs of different graph classes are also interesting and challenging. All these facts highlight the scope for further studies in this area.

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