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RESEARCH ARTICLE

A simplified proof of the reduction point crossing sign formula for Verma modules

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Abstract. The Unitary Dual Problem is one of the most important open problems in mathematics: classify the irreducible unitary representations of a group. That is, classify all irreducible representations admitting a definite invariant Hermitian form. Signatures of invariant Hermitian forms on Verma modules are important to finding the unitary dual of a real reductive Lie group. By a philosophy of Vogan introduced in [Vog84], signatures of invariant Hermitian forms on irreducible Verma modules may be computed by varying the highest weight and tracking how signatures change at reducibility points (see [Yee05]). At each reducibility point there is a sign ε governing how the signature changes. A formula for ε was first determined in [Yee05] and simplified in [Yee19]. The proof of the simplification was complicated. We simplify the proof in this note.

1. Introduction

In the 1930s, I.M. Gelfand introduced a broad programme in abstract harmonic analysis that is a grand generalization of Fourier analysis. The programme permitted the solution of problems in areas ranging from topology to mathematical physics by algebraic means. Associate to a difficult problem an algebraic object (eg. a space of functions) and translate the problem to an algebraic problem. Decompose the algebraic problem

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into smaller simpler problems, solve the simpler problems, reassemble the solutions into a solution to the algebraic problem, and then transfer the solution to a solution to the original problem. To realize Gelfand's programme, the Unitary Dual Problem must be solved.

In [Mac58], Mackey showed how to construct unitary representations of a group G from unitary representations of a normal subgroup N and G/N. In [Duf82], Duflo described the unitary dual of an algebraic Lie group in terms of unitary duals of smaller reductive Lie groups. Thus we wish to solve the Unitary Dual Problem for real reductive Lie groups, which is equivalent to classifying irreducible unitary Harish-Chandra modules. Harish-Chandra modules may be constructed from Verma modules. The Unitary Dual Problem in the case of real reductive Lie groups is still unsolved in general.

In the 1970s, Knapp and Zuckerman classified Hermitian representations of a real reductive group (those admitting a non-degenerate invariant Hermitian form). Thus the approach to the Unitary Dual Problem has been the following: calculate signatures of invariant Hermitian forms on Hermitian representations and determine when the forms are definite.

In [Vog84], Vogan developed a philosophy for computing signatures of invariant Hermitian forms. The philosophy depended on computing some signs which were unknown at the time. The signs were first computed in [Yee05]. Vogan's philosophy in the case of Verma modules is the following (for a full account, see [Yee05] and for a more detailed summary than this introduction see [Yee19]). As the highest weight varies, Verma modules may be thought of as realized on the same vector space. Invariant Hermitian forms on Hermitian Verma modules are unique up to a real scalar. The radical of the invariant Hermitian form is the unique maximal proper submodule of the Verma module. Thus if you vary the highest weight analytically, if the invariant Hermitian forms remain non-degenerate, the signatures of the forms cannot change. Let $\mathfrak{h}$ and $\mathfrak{b}$ be the Cartan subalgebra and Borel subalgebra, respectively, from which the Verma modules are constructed, let $\mathfrak{g}$ be the Lie algebra for which the Verma module is a representation, and let $\Delta^+(\mathfrak{g},\mathfrak{h})$ be the positive roots corresponding to $\mathfrak{b}$. Let ρ be one half the sum of the positive roots and let $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ for $\alpha \in \Delta(\mathfrak{g},\mathfrak{h})$. Let $\lambda \in \mathfrak{h}^*$ and let $\mathbb{C}_{\lambda-\rho}$ be the $\mathfrak{h}$-module of weight $\lambda - \rho$. Then the Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$ is reducible if and only if $(\lambda, \alpha^\vee) \in \mathbb{Z}^+$ for some positive root α. Thus Verma modules are reducible when the highest weight lies on a reducibility hyperplane $H_{\alpha,n} := \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha^\vee) = n\}$ where $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$ and $n \in \mathbb{Z}^+$. Thus the reducibility hyperplanes partition the highest weights into regions where the signature cannot change. These regions may be broken up into alcoves parameterized by the (dual) affine Weyl group. The antidominant Weyl chamber is a large region not containing any reducibility hyperplanes. By an asymptotic argument, Wallach was able to determine the signature of Hermitian forms in this region in [Wal84]. Vogan's philosophy is now the following. Cross reducibility hyperplanes one at a time. Then the signature changes by the signature of the radical of the form as one crosses a single reducibility hyperplane. The radical is another Verma module that you can arrange to be "closer" to Wallach's region. The radical being a Verma module, the signature on the radical is unique up to a real scalar. Thus the signature character (see [Yee05] for a definition) of the radical is either equal to the signature character of the canonical form on the Verma module or is equal to its negative. This is the sign associated with the reduction point crossing. In particular, for adjacent alcoves C and C' separated by the reducibility hyperplane $H_{\alpha,n}$ and for $\lambda \in C$ and $\lambda' \in C'$ we have ([Yee05], Lemma 4.3)

$$ch_s M(\lambda) = e^{\lambda-\lambda'} ch_s M(\lambda') + 2\varepsilon(C, C') ch_s M(\lambda - n\alpha)$$

where $\varepsilon(C, C') = \pm 1$. By induction, the signature character of an arbitrary Verma module may be expressed as a sum of products of crossing signs and powers of 2 times the signature in Wallach's region with a translation (see [Yee05], Theorem 4.6). The signature character formula was simplified in [Yee19] and [LY18] where it was shown that the signatures can be expressed as sums of Hall-Littlewood polynomial summands evaluated at $q = -1$ times a version of the Weyl denominator. Key to simplifying the signature character formula was a simplification of the reduction point crossing sign formula. The proof of the sign simplification was somewhat complicated. We will simplify the proof in the next section.

2. A simplification of the proof of the reduction point sign formula

First, we restrict ourselves to computing signs when the real form with respect to which our forms are invariant is the compact real form. For other forms, the signs may be related to corresponding signs for the compact real form easily (see Theorem 3.17 of [Yee19]).

The following notation will be consistent throughout what follows:

- $\mathfrak{g}$ is a complex semisimple Lie algebra with compact real form $\mathfrak{g}_0$;

- $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$ with complexification $\mathfrak{h}$;
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$;
- $\Delta(\mathfrak{g}, \mathfrak{h})$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$, $\Pi = \{\alpha_i\}_{1 \leqslant i \leqslant n}$ is the base corresponding to $\mathfrak{b}$, and s_i is the reflection corresponding to α_i;
- $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is the system of positive roots in $\Delta(\mathfrak{g}, \mathfrak{h})$ with respect to Π and ρ is one half the sum of the positive roots;
- for $\lambda \in \mathfrak{h}^*$, $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$ is the Verma module of highest weight $\lambda - \rho$;
- W is the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{h})$;
- $H_{\gamma,n}$ is the hyperplane $\{\mu \in \mathfrak{h}_0^* : (\mu, \gamma^\vee) = n\}$;
- for $w \in W, \Delta(w) := \{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{h}) : w(\alpha) < 0\}$;
- $\mathcal{C}_0$ is the antidominant Weyl chamber.

In order to explain the purpose of this proof accurately, a few definitions are also required:

Definition 1 (see [Yee05], Theorem 5.3.4)**.** Let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$, and let $\gamma = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$ be an expression with the property that for all integers $1 \leqslant j \leqslant k-1$:

$$\mathrm{ht}(s_{i_j} s_{i_{j+1}} \cdots s_{i_{k-1}} \alpha_{i_k}) > \mathrm{ht}(s_{i_{j+1}} s_{i_{j+2}} \cdots s_{i_{k-1}} \alpha_{i_k}).$$

Then let $w_\gamma = s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$.

(Note that w_γ is in general not unique, but all theorems below hold for any choice of w_γ).

Definition 2. Let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$,$w_\gamma$ be defined as in definition 1, and $w \in W$. Then we define

$$S_1 := \{\beta \in \Delta(w_\gamma^{-1}) : |\beta| = |\gamma|,\ \beta \neq \gamma,\ \text{and } \beta, s_\beta \gamma \in \Delta(w^{-1})\}$$
$$S_2 := \{\beta \in \Delta(w_\gamma^{-1}) : |\beta| \neq |\gamma|,\ \text{and } \beta, -s_\beta s_\gamma \beta \in \Delta(w^{-1})\}.$$

In [Yee05], it was shown that for a given reducibility hyperplane, the reduction point crossing sign only depends on the Weyl chamber containing the point of crossing, so we denote by $\varepsilon(H_{\gamma,n}, w)$ the crossing sign from $H^+_{\gamma,n} := \{\lambda \in \mathfrak{h}_0^* : (\lambda, \gamma^\vee) > n\}$ to $H^-_{\gamma,n} := \{\lambda \in \mathfrak{h}_0^* : (\lambda, \gamma^\vee) < n\}$ in the Weyl chamber $w\mathcal{C}_0$ where $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$, $n \in \mathbb{Z}^+$, and $w \in W$. The reduction point crossing sign when $\mathfrak{g}_0$ is the compact real form of $\mathfrak{g}$ is given as follows:

Theorem 1 ([Yee05], Theorem 5.3.4). *Let $\mathfrak{g}_0$ be the compact real form of $\mathfrak{g}$. Let γ be a positive root that does not form a type G_2 root system with other roots in $\Delta(\mathfrak{g}, \mathfrak{h})$. If γ hyperplanes are positive on $w\mathcal{C}_0$,then*

$$\varepsilon(H_{\gamma,n}, w) = (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1}):|\beta|=|\gamma|,\beta\neq\gamma \text{ and } \beta, s_\beta\gamma\in\Delta(w^{-1})\}} \times (-1)^{\#\{\beta\in\Delta(w_\gamma^{-1}):|\beta|\neq|\gamma| \text{ and } \beta,-s_\beta s_\gamma\beta\in\Delta(w^{-1})\}}. \tag{1}$$

In the notation of definition 2, this takes the following more compact, form:

$$\varepsilon(H_{\gamma,n}, w) = (-1)^{\#S_1+\#S_2}. \tag{2}$$

The simplification of this formula in [Yee19] is as follows:

Theorem 2 ([Yee19], Theorem 4.11). *Let $\mathfrak{g}_0$ be the compact real form of $\mathfrak{g}$ and let γ be a positive root such that γ hyperplanes are positive on $w\mathcal{C}_0$. Then:*

$$\varepsilon(H_{\gamma,n}, w) = (-1)^{\frac{\ell(w)-\ell(s_\gamma w)-1}{2}}. \tag{3}$$

We develop a shorter, alternative proof of this simplification. It is easy to verify that the equation holds for type G_2. We thus prove that the exponents equations (2) and (3) are equal when γ does not form a type G_2 root system with any other roots in $\Delta(\mathfrak{g}, \mathfrak{h})$; that is:

$$\ell(w) - \ell(s_\gamma w) = 2(\#S_1 + \#S_2) + 1. \tag{4}$$

Now we provide a proof of equation (4). First, the following Lemma will allow us to simplify the sets S_1 and S_2:

Lemma 1 ([Yee19], Lemma 4.4). *Let $\Delta(g, \mathfrak{h})$ be a root system not containing any components of type G_2 and let $\beta, \gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$.*

- *If $|\beta| = |\gamma|$, then $s_\beta\gamma = -s_\gamma\beta$.*
- *If $|\beta \neq |\gamma|$, then $-s_\beta s_\gamma(\beta) = -s_\gamma(\beta)$.*

Observing Lemma 1 and the definitions of S_1 and S_2, all conditions except for the length of β are the same in both sets. Therefore we can express them as a single set by removing the length condition:

Definition 3. $S := \{\beta \in \Delta(w_\gamma^{-1}) : \beta, -s_\gamma(\beta) \in \Delta(w^{-1})\}$.

Lemma 2. *Let $\Delta(\mathfrak{g}, \mathfrak{h})$ be a root system not containing any components of type G_2, let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$, w_γ be defined as in 1 and let $w \in W$ be an element with $w^{-1}\gamma < 0$ (i.e. γ hyperplanes are positive on $w\mathcal{C}_0$). Then $S = (S_1 \cup S_2)\dot{\cup}\{\gamma\}$.*

Proof. By using the formulas from 1 we can rewrite S_1 and S_2 as

$$S_1 := \{\beta \in \Delta(w_\gamma^{-1}) : |\beta| = |\gamma|,\ \beta \neq \gamma,\ \text{and } \beta, -s_\gamma\beta \in \Delta(w^{-1})\},$$
$$S_2 := \{\beta \in \Delta(w_\gamma^{-1}) : |\beta| \neq |\gamma|,\ \text{and } \beta, -s_\gamma\beta \in \Delta(w^{-1})\}.$$

Then the only root which might possibly be an element of S but not of S_1 or S_2 is γ. Since $\gamma = w_\gamma s_{i_k}(\alpha_{i_k})$, we have $w_\gamma^{-1}(\gamma) = -\alpha_{i_k}$. Therefore $\gamma \in \Delta(w_\gamma^{-1})$. Then since $\gamma \in \Delta(w_\gamma^{-1})$ and $-s_\gamma\gamma = \gamma \in \Delta(w^{-1})$, $\gamma \in S$. $\square$

The purpose of the following Lemma is to prove that if $\alpha \in S$ then $-s_\gamma(\alpha) \notin S$, unless $\alpha = \gamma$. In the proof of Theorem 3 we will require this fact to ensure that no double-counting occurs when we use S to count a certain set of roots. We recall it from [Yee19] without proof.

Lemma 3 ([Yee19], Lemma 4.7)**.** *Let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$ and w_γ be as defined in Definition 1. Then*

$$\Delta(w_\gamma^{-1}) \cap -s_\gamma\Delta(w_\gamma^{-1}) = \{\gamma\}.$$

From Lemma 2 and a well-known equivalent definition of the length function, it follows that equation (4) is equivalent to the equation

$$\#\Delta(w^{-1}) - \#\Delta(w^{-1}s_\gamma) = 2\#S - 1. \tag{5}$$

We now begin the proof of equation (5). Our method will be to first construct an injection $f : \Delta(w^{-1}s_\gamma) \hookrightarrow \Delta(w^{-1})$, and then use S to count the points outside of $\operatorname{im} f$. (It will turn out that $\beta \in S$ corresponds to $\beta, -s_\gamma\beta$ outside of $\operatorname{im} f$.) First we construct f:

Lemma 4. *Let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h})$ and $w \in W$ with $w^{-1}\gamma < 0$ (i.e. so that γ hyperplanes are positive on $w\mathcal{C}_0$). Define*

$$f : \Delta(w^{-1}s_\gamma) \to \Delta(w^{-1}) : f(\beta) = \begin{cases} \beta & \text{if } \beta \in \Delta(w^{-1}) \\ s_\gamma(\beta) & \text{if } \beta \notin \Delta(w^{-1}). \end{cases} \tag{6}$$

Then $\operatorname{im} f \subseteq \Delta(w^{-1})$ *and f is injective.*

Proof. First we prove inclusion. Let $\beta \in \Delta(w^{-1}s_\gamma)$. If $\beta \in \Delta(w^{-1})$ then trivially $f(\beta) \in \Delta(w^{-1})$, so consider the case that $\beta \notin \Delta(w^{-1})$.

Since β is in the domain, $w^{-1}(s_\gamma(\beta)) < 0$, so $s_\gamma(\beta) \in \Delta(w^{-1})$ is equivalent to $s_\gamma(\beta) > 0$. We expand $w^{-1}s_\gamma(\beta)$ into the form

$$w^{-1}(\beta) - \frac{2(\beta, \gamma)}{(\gamma, \gamma)} w^{-1}(\gamma) < 0. \tag{7}$$

Since $\beta \notin \Delta(w^{-1}), w^{-1}(\beta) > 0$, and by assumption $w^{-1}(\gamma) < 0$, therefore inequality (7) requires that $(\beta, \gamma) < 0$, from which it follows that $s_\gamma(\beta) > \beta > 0$ as required.

Next we prove injectivity. Let $\beta_1, \beta_2 \in \Delta(w^{-1}s_\gamma)$. In the cases that $f(\beta_1) = \beta_1, f(\beta_2) = \beta_2$ and $f(\beta_1) = s_\gamma(\beta_1), f(\beta_2) = s_\gamma(\beta_2)$ it is obvious that $f(\beta_1) = f(\beta_2)$ implies that $\beta_1 = \beta_2$.

For the remaining case, without loss of generality assume that $f(\beta_1) = \beta_1$ and $f(\beta_2) = s_\gamma(\beta_2) = \beta_1$. This generates an immediate contradiction since by the conditions of f, $\beta_2 \notin \Delta(w^{-1})$ but $w^{-1}s_\gamma(\beta_1) = w^{-1}(\beta_2) < 0$. Therefore f is injective. $\square$

With this lemma we are prepared to prove the main theorem:

Theorem 3. *Let $\gamma \in \Delta^+(\mathfrak{g}, \mathfrak{h}), w_\gamma = s_{i_1}s_{i_2}\cdots s_{i_k}$ be defined as in Definition 1 and $w \in W$ have the property that $w^{-1}\gamma < 0$ (i.e. γ hyperplanes are positive on $w\mathcal{C}_0$). Let S be defined as in Definition 3. Then*

$$\#\Delta(w^{-1}) = \#\Delta(w^{-1}s_\gamma) + 2\#S - 1$$

so that if $\mathfrak{g}_0$ is the compact real form, then

$$\varepsilon(H_{\gamma,n}, w) = (-1)^{\frac{\ell(w)-\ell(s_\gamma w)-1}{2}}.$$

Proof. By Lemma 4, $\Delta(w^{-1}s_\gamma) \xhookrightarrow{f} \Delta(w^{-1})$, so $\#\Delta(w^{-1}s_\gamma) = \#\operatorname{im} f$. Now let us consider the set $\Delta(w^{-1})\setminus\operatorname{im} f$. We will show that for every α in this set, $-s_\gamma(\alpha)$ is contained in the set as well, and one of α or $-s_\gamma(\alpha) \in S$. Since Lemma 3 implies that with the exception of $\alpha = \gamma$ exactly one of $\alpha, -s_\gamma\alpha \in S$, this procedure will establish that $\#(\Delta(w^{-1}) \setminus \operatorname{im} f) = 2\#S - 1$.

If $\alpha \in \Delta(w^{-1})\setminus\operatorname{im} f$, then we must have $\alpha \notin \Delta(w^{-1}s_\gamma)$, since otherwise $f(\alpha) = \alpha$ is in $\operatorname{im} f$. We must also have that $s_\gamma(\alpha) \notin \Delta(w^{-1}s_\gamma)$. If it were possible that $s_\gamma\alpha \in \Delta(w^{-1}s_\gamma)$, then we would have either $f(s_\gamma\alpha) = s_\gamma\alpha$, which contradicts the fact that $\alpha \notin \Delta(w^{-1}s_\gamma)$, or $f(s_\gamma\alpha) = \alpha$, which contradicts the assumption that $\alpha \notin \operatorname{im} f$. Therefore $s_\gamma\alpha \notin \Delta(w^{-1}s_\gamma)$ as well.

Since $w^{-1}s_\gamma(s_\gamma(\alpha)) = w^{-1}(\alpha) < 0$, the condition $s_\gamma(\alpha) \notin \Delta(w^{-1}s_\gamma)$ requires $s_\gamma(\alpha) < 0$. Then since $\alpha \notin \Delta(w^{-1}s_\gamma)$ we have $w^{-1}(-s_\gamma(\alpha)) < 0$, and $-s_\gamma(\alpha) > 0$ so $-s_\gamma(\alpha) \in \Delta(w^{-1})$.

Thus we have shown that if $\alpha \in \Delta(w^{-1})\setminus\operatorname{im} f$, then $-s_\gamma(\alpha) \in \Delta(w^{-1})$. In fact, $-s_\gamma\alpha \in \Delta(w^{-1}) \setminus \operatorname{im} f$ since $s_\gamma(-s_\gamma\alpha) = -\alpha \notin \Delta(w^{-1}s_\gamma)$ and $-s_\gamma\alpha \notin \Delta(w^{-1}s_\gamma)$ since $w^{-1}s_\gamma(-s_\gamma\alpha) = -w^{-1}\alpha > 0$. The remaining step in the proof is to show that if $\alpha \in \Delta(w^{-1}) \setminus \operatorname{im} f$ then either $\alpha \in \Delta(w_\gamma^{-1})$ or $-s_\gamma(\alpha) \in \Delta(w_\gamma^{-1})$.

If $\alpha \in \Delta(w_\gamma^{-1})$ and $\alpha \neq \gamma$, then $\alpha \in S$ and $-s_\gamma\alpha \notin S$ by Lemma 3. Otherwise assume that $\alpha \notin \Delta(w_\gamma^{-1})$. We can expand $w_\gamma^{-1}(-s_\gamma(\alpha))$ as:

$$w_\gamma^{-1}(-s_\gamma(\alpha)) = -w_\gamma^{-1} w_\gamma s_{i_k} w_\gamma^{-1}\alpha = -s_{i_k} w_\gamma^{-1}\alpha.$$

Since $w_\gamma^{-1}\alpha > 0$ and the only positive root sent to a negative root by s_{i_k} is α_{i_k}, this expression is negative unless $w_\gamma^{-1}\alpha = \alpha_{i_k}$. However this is impossible since w_γ^{-1} is a bijection and $w_\gamma^{-1}(-\gamma) = \alpha_{i_k}$, but $\alpha \in \Delta(w^{-1})$ so α cannot be a negative root. Therefore $-s_\gamma(\alpha) \in \Delta(w_\gamma^{-1})$ and so $-s_\gamma(\alpha) \in S$ while $\alpha \notin S$.

We have established a correspondence $\beta \leftrightarrow \{\beta, -s_\gamma\beta\}$ between roots in S and roots in $\Delta(w^{-1}) \setminus \operatorname{im} f$. Since exactly one of $\beta, -s_\gamma\beta \in S$ unless $\beta = \gamma$ by Lemma 3, the set S counts $2\#S - 1$ roots. Therefore

$$\#\Delta(w^{-1}) = \#\operatorname{im} f + \#(\Delta(w^{-1}) \setminus \operatorname{im} f) = \#\Delta(w^{-1}s_\gamma) + 2\#S - 1$$

from which the formula for $\varepsilon(H_{\gamma,n}, w)$ follows. $\square$

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