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RESEARCH ARTICLE

Continuous limits of tilting modules

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Abstract. We provided a constructive argument to obtain an infinite generated tilting module from a family of tilting modules satisfying some hypotheses. We also applied the result over a hereditary algebra to get the Lukas tilting module.

1. Introduction

Tilting theory and tilted algebras were introduced in the context of finitely generated modules over finite dimensional algebras by Happel-Ringel [12] based on the work of Brenner-Butler [8] and Auslander-Platzeck-Reiten [3], see also [2]. The theory was generalized in the last decades, by relaxing the homological dimensions in the definition of tilting modules, or by considering not necessarily finitely generated modules, or by considering arbitrary rings. Also tilting theory over abelian and triangulated categories were considered recently. In [7], inspired by a construction given by Buan and Solberg [9], we looked at the situation where a direct limit of a direct system of tilting modules is still a tilting module. In this work our intent is generalize the result from [7] to the context of not necessarily enumerable family of tilting modules and rebuild that construction with few additional hypotheses. In the last section, we apply this approach to compute an

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infinite generated tilting module equivalent (in sense of Add subcategories) to the "Lukas Tilting Module".

2. Preliminaries results

Throughout this paper, k will denote a fixed field. By an algebra, we mean an associative finite dimensional k-algebra with unity and by a ring, a unitary associative ring. Given an algebra A (or a ring), we shall denote by $\mathrm{Mod}A$ the category of all left A-modules and by $\mathrm{mod}A$ its subcategory whose objects consist of finitely generated modules. Given $X \in \mathrm{Mod}A$, we denote by $\mathrm{pd}_A(X)$ the projective dimension of X and for M and N in $\mathrm{Mod}A$, $\mathrm{Hom}_A(M,N)$ and $\mathrm{Ext}(M,N)$ mean the abelian groups obtained respectively by applying the bifunctors Hom and Ext to the pair (M,N) of A-modules.

Definition 2.1. *Let $(I,\leqslant)$ be a direct set. A direct system of A-modules is a set $\mathcal{L}$ of A-modules, indexed by a set I, and a family $\mathcal{F}$ of A-morphisms satisfying: For $i \leqslant j$, there exist a morphism $f_j^i : M_i \to M_j$ in $\mathcal{F}$ such that $f_i^i : M_i \to M_i$ is the identity map for all $i \in I$ and if $i \leqslant j \leqslant l$, then $f_l^i = f_l^j \circ f_j^i$.*

Definition 2.2. *Let $\{M_i, f_j^i\}$ be a direct system in $\mathrm{Mod}\,A$. The direct limit of this system is an A-module, denoted by $\varinjlim_{i\in I} M_i$, and a family of morphisms $\alpha_i : M_i \to \varinjlim_{i\in I} M_i$ such that:*

(a) *$\alpha_i = \alpha_j f_j^i$ if $i \leqslant j$;*

(b) *For each A-module X such that there exist a family of maps $f_i : M_i \to X$, with $\varphi_i = \varphi_j f_j^i$, if $i \leqslant j$, there exist a unique morphism $\beta : \varinjlim_{i\in I} M_i \to X$ with $\varphi_i = \beta \circ \alpha_i$.*

A direct limit of a direct system of A-modules always exist ([15]). For a class $\mathcal{C} \subseteq \mathrm{Mod}\,A$ we set:

$$\mathcal{C}^{\perp_i} = \{X \in \mathrm{Mod}\,A \mid \mathrm{Ext}_A^i(Y,X) = 0\ \forall Y \in \mathcal{C}\},$$
$$^{\perp_i}\mathcal{C} = \{X \in \mathrm{Mod}\,A \mid \mathrm{Ext}_A^i(X,Z) = 0\ \forall Z \in \mathcal{C}\},$$

and then

$$\mathcal{C}^{\perp} = \bigcap_{i\geqslant 1} \mathcal{C}^{\perp_i} \quad \text{and} \quad {}^{\perp}\mathcal{C} = \bigcap_{i\geqslant 1} {}^{\perp_i}\mathcal{C}.$$

If $\mathcal{C} = \{M\}$ we will use the notation $M^{\perp}$ $({}^{\perp}M)$ to denote the class $\mathcal{C}^{\perp}$ $({}^{\perp}\mathcal{C})$.

Definition 2.3. *Let $\mathcal{C}$ be a class of modules. A $\mathcal{C}$-preenvelope for an A-module X is a pair (M, f) where $M \in \mathcal{C}$ and $f : X \longrightarrow M$ satisfy the condition: for all $h : X \longrightarrow Y$ with $Y \in \mathcal{C}$ factors through M. Equivalently, the group morphism $\operatorname{Hom}_A(X, M) \xrightarrow{f_*} \operatorname{Hom}_A(X, Y)$ is an epimorphism.*

Furthermore, if f is a monomorphism and $\operatorname{Coker}(f) \in^{\perp} \mathcal{C}$, then the pair (M, f) is called a special preenvelope.

A class $\mathcal{C}$ of A-modules is a preevelope class if all A-module admit a $\mathcal{C}$-special preenvelope.

The concept of precover class and special precover is dual; see [11] for more details.

The notions of preenvelopes and precover classes were introduced independently by Enochs to the category of modules over a ring ([11]) and by Auslander and Smalo in the context of category of modules finitely generated over a Artin algebra ([5],[4]). In the Auslander and Smalo approach preenvelopes and precover classes are called covariant and contravariant subcategories respectively.

3. Some facts on tilting modules

A module $T \in \operatorname{Mod} A$ is called *n-tilting* if it satisfies the following conditions:

(a) $\operatorname{pd} T \leqslant n$;
(b) $\operatorname{Ext}^i_A(T, T^{(I)}) = 0$ for each $i \geqslant 1$ and any index set I;
(c) There exist an exact sequence

$$0 \longrightarrow A \longrightarrow T_0 \rightarrow \quad \dots \longrightarrow T_r \longrightarrow 0$$

where $T_j \in \operatorname{Add} T$ for each $0 \leqslant j \leqslant r$.

If the number “n” is irrelevant to reader understanding, we will write only "tilting module".

For a n-tilting A-module T, the orthogonal class $T^{\perp}$ is a preenvelope class and $\operatorname{Add} T = T^{\perp} \cap^{\perp} (T^{\perp})$.

Two n-tilting A-modules T and $T^{'}$ are equivalents if $T^{\perp} = (T^{'})^{\perp}$. From this equality is easy to deduce that $\operatorname{Add} T = \operatorname{Add} T^{'}$.

Proposition 3.1 (Angeleri-Hügel and Coelho [1]). *Let U, $T \in \operatorname{Mod} A$ n-tilting modules satisfying the property $U \in T^{\perp}$. Then $U^{\perp} \subseteq T^{\perp}$ and* $\operatorname{pd}T \leqslant \operatorname{pd}U$.

Proposition 3.2 (Bazzoni and Stovicek [6]). *Let T be a n-tilting A-module. Then $T^{\perp}$ is closed by direct limits.*

Proposition 3.3 (Braga and Coelho[7]). *Let T be a n-tilting A-module. Then there exist an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f_0"] & T_0 \arrow[r, "f_1"] & \dots \arrow[r, "f_k"] & T_k \arrow[r] & 0 \end{tikzcd} \tag{1}$$

where $T_i \in \operatorname{Add} T$ and such that:

a) $k \leqslant n$;

b) *Each f_i is obtained by the composition of $coker(f_{i-1})$ with a special $T^{\perp}$-preenvelope map of* $\operatorname{Coker}(f_{i-1})$;

c) $\operatorname{Add}(\bigoplus_{i=1}^{k} T_i) = \operatorname{Add} T$.

The exact sequence in (1) will called a T-coresolution of the ring A.

Let U and T be tilting modules (eventually for different $n \in \mathbb{N}$) which $U \in T^{\perp}$. Assume that $\operatorname{pd}(U) = r$ for some $r \in \mathbb{N}$ and consider a T-coresolution

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f_0"] & T_0 \arrow[r, "f_1"] & T_1 \arrow[r, "f_2"] & \dots \arrow[r] & T_s \arrow[r] & 0 \end{tikzcd}$$

of the A-module A.

Then $s \leqslant r$ and there exist an U-coresolution of A

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g_0"] & U_0 \arrow[r, "g_1"] & U_1 \arrow[r, "g_2"] & \dots \arrow[r] & U_r \arrow[r] & 0 \end{tikzcd}$$

commuting the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f_0"] \arrow[d, equal] & T_0 \arrow[r] \arrow[d, "f_0"] & \dots \arrow[r] & T_s \arrow[rr] \arrow[d, "f_s"] & & 0 \arrow[d] & & \\ 0 \arrow[r] & A \arrow[r, "g_0"] & U_0 \arrow[r, "g_1"] & \dots \arrow[r] & U_s \arrow[rr] & & U_{s+1} \dots \arrow[r] & U_r \arrow[r] & 0 \end{tikzcd}$$

The pairs (U_i, f_i) in the vertical maps are special $U^{\perp}$-preenvelopes of T_i for each i and the pair $(\bigoplus_{i=0}^{r} U_i, D)$ is a special $U^{\perp}$-preenvelope of $\bigoplus_{i=0}^{r} T_i$ (see [7] Lemma 2.4).

The map $D : \bigoplus_{i=0}^{r} T_i \to \bigoplus_{i=0}^{r} U_i$ is given by the matrix

$$D = \begin{pmatrix} f_0 & 0 & 0 & \dots & 0 \\ 0 & f_1 & 0 & \dots & 0 \\ 0 & 0 & f_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f_r \end{pmatrix}$$

where $T_j = 0$ and $f_j = 0$ if $s < j \leqslant r$, obviously, when such a j exist.

4. A continuous limit of tilting modules

If the set I, in the definition 2.2, is the set of ordinal numbers and "$\leqslant$" is the usual order, we say that a direct system $(M_\alpha | \alpha \leqslant \mu)$ of modules is *continuous* if $M_0 = 0$ e $M_\alpha = \varinjlim_{\gamma<\alpha} M_\gamma$ for each limit ordinal $\alpha < \mu$.

Lemma 4.1 ([7] and [10]). *Let $(M_\alpha | \alpha \leqslant \mu)$ be a sequence of modules and $(f_{\alpha\beta} | \alpha \leqslant \beta \leqslant \mu)$ a sequence monomorphisms such that $\{(M_\alpha, f_{\alpha\beta}) | \alpha \leqslant \beta \leqslant \mu\}$ is a continuous direct system.*

Let C be an A-module such that $\operatorname{Ext}^i_A(M_{\alpha+1}/f_{\alpha\alpha+1}(M_\alpha), C) = 0$ for all $\alpha + 1 \leqslant \mu$ and $i \geqslant 1$. Then $\operatorname{Ext}^i_A(M_\mu, C) = 0$.

Now we turn our attention to build a continuous direct system of tilting modules. To proceed this, we first consider a family of A-modules $\{T_\alpha\}_{\alpha<\mu}$ satisfying the following hypotheses:

1) μ a limit ordinal;
2) $\operatorname{Add} T_{\gamma+1} \neq \operatorname{Add} T_{\delta+1}$ if $\gamma + 1 \neq \delta + 1$ and $T_{\delta+1} \in (T_{\gamma+1})^{\perp}$ if $\gamma + 1 < \delta + 1$;
3) $\operatorname{pd} T_{\alpha+1} \leqslant n$ for each ordinal $\alpha + 1 < \mu$.

We start by defining $\overline{T}_0 = 0$ and, for each ordinal $\alpha+1 < \mu$, considering a commutative diagram

$$\begin{tikzcd}
& \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & T^0_\alpha \arrow[r] \arrow[d, "f^0_\alpha"] & T^1_\alpha \arrow[r] \arrow[d, "f^1_\alpha"] & T^2_\alpha \arrow[r] \arrow[d, "f^2_\alpha"] & \ldots \arrow[r] & T^n_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & T^0_{\alpha+1} \arrow[r] \arrow[d, "f^0_{\alpha+1}"] & T^0_{\alpha+1} \arrow[r] \arrow[d, "f^1_{\alpha+1}"] & T^2_{\alpha+1} \arrow[r] \arrow[d, "f^2_{\alpha+1}"] & \ldots \arrow[r] & T^n_{\alpha+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & T^0_{\alpha+2} \arrow[r] \arrow[d, "f^0_{\alpha+2}"] & T^1_{\alpha+2} \arrow[r] \arrow[d, "f^1_{\alpha+2}"] & T^2_{\alpha+2} \arrow[r] \arrow[d, "f^2_{\alpha+2}"] & \ldots \arrow[r] & T^n_{\alpha+2} \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{2}$$

obtained by successive applications of the result at the end of section 3. Then, for each $T^i_\alpha \neq 0$, the pair $(T^i_{\alpha+1}, f^i_\alpha)$ is a $(T_{\alpha+1})^{\perp}$-preevelope of T^i_α.

Following the argument, for each $\alpha+1$, we define $\overline{T}_{\alpha+1} = \bigoplus_i^n T^i_{\alpha+1}$ and the morphisms $f_{\alpha+1,\alpha+2}$ are given by the diagonal matrix

$$\begin{pmatrix} f^0_{\alpha+1} & 0 & 0 & \dots & 0 \\ 0 & f^1_{\alpha+1} & 0 & \dots & 0 \\ 0 & 0 & f^2_{\alpha+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & f^n_{\alpha+1} \end{pmatrix}$$

If the ordinal α can be obtained from γ by finite steps, then

$$f_{\gamma\alpha} = f_{(\alpha-1)\alpha} \circ \dots \circ f_{(\gamma+1)(\gamma+2)} \circ f_{\gamma(\gamma+1)}.$$

If α is a limit ordinal $\overline{T}_\alpha = \varinjlim_{\gamma<\alpha} \overline{T}_\gamma$ and the morphism is $f_{\beta\alpha} = \varinjlim_{\gamma<\alpha}(f_{\beta\gamma})$.

5. The main theorem

Theorem 5.1. *Let A be a ring and $\{T_\alpha\}_{\alpha<\mu}$ a class of n-tilting A-modules such that:*

a) $\operatorname{pd} T_{\alpha+1} \leqslant n$;
b) $\operatorname{Add} T_{\gamma+1} \neq \operatorname{Add} T_{\delta+1}$ *if* $\gamma+1 \neq \delta+1$;
c) $T_{\delta+1} \in (T_{\gamma+1})^\perp$ *if* $\gamma+1 < \delta+1$.

Then there exist another class of n-tilting A-modules $\{\overline{T}_\alpha\}_{\alpha<\mu}$ constituting a continuous direct system satisfying the conditions a), b) and c) above, with $\operatorname{Add}\overline{T}_\alpha = \operatorname{Add} T_\alpha$ and whose its direct limit is a $(n+1)$-tilting A-module.

Proof. Under these hypotheses, the direct system $(\overline{T}_\alpha, f_{\alpha\beta})$ is continuous, as the one obtained in diagram 2.

For each limit ordinal α, put $\overline{T}_\alpha = \varinjlim_{\gamma<\alpha}(\overline{T}_\gamma)$ and for each column on Diagram (2) set the limits $\overline{T}^j_\alpha = \varinjlim_{\alpha<\lambda} T^j_\alpha$. Then, we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & \overline{T}^0_\alpha \arrow[r] & \overline{T}^2_\alpha \arrow[r] & \dots \arrow[r] & \overline{T}^n_\alpha \arrow[r] & 0 . \end{tikzcd}$$

The exactness of above sequence is a consequence of the exactness of functor direct limit (see [15]). Moreover, direct limits commutes with finite direct sums, then $\bigoplus_{j=0}^n \overline{T}^j_\alpha \in \operatorname{Add}\overline{T}_\alpha$. So, the A-module A admit a coresolution in $\operatorname{Add}\overline{T}_\alpha$. Therefore $\overline{T}_\alpha$ satisfy the condition c) from n-tilting module definition.

Now, by induction hypothesis, we suppose for any $\gamma < \alpha$ that the A-module $\overline{T}_\gamma$ is a n-tilting module and want to prove that $\mathrm{Ext}^i_A(\overline{T}_\alpha, \overline{T}^{(I)}_\alpha) = 0$. To do this, we only need to verify if $\overline{T}_\alpha$ satisfies the hypotheses of Lemma 4.1 for $C = \overline{T}^{(I)}_\alpha$.

For a fixed ordinal number $\delta < \alpha$, as $\overline{T}_\delta$ is a tilting module, the class $(\overline{T}_\delta)^\perp$ is closed by arbitrary direct sums. Also $(\overline{T}_\delta)^\perp = S_\delta^\perp$ where the class $S_\delta \subset \mathrm{FP}_\infty(A)$ (see[6]). Then for each $X \in S_\delta$ we get

$$\mathrm{Ext}^i_A(X, \overline{T}_\alpha) = \mathrm{Ext}^i_A(X, \varinjlim_{\gamma<\alpha} \overline{T}_\gamma) \cong \varinjlim_{\gamma<\alpha} \mathrm{Ext}^i_A(X, \overline{T}_\gamma)$$

for all $i > 0$.

By induction hypothesis, for each $\delta \leqslant \gamma < \alpha$, $\overline{T}_\gamma \in \overline{T}^\perp_\delta$, so by the Proposition 3.1, $\overline{T}^\perp_\gamma \subset \overline{T}^\perp_\delta$. Then for each $X \in S_\delta$, $\mathrm{Ext}^i_A(X, \overline{T}_\gamma) = 0$. Therefore $\overline{T}_\alpha \in S_\delta^\perp$ and $\mathrm{Ext}^i_A(\overline{T}_\delta, \overline{T}_\alpha) = 0$. By a similar argument, $\mathrm{Ext}^i_A(\overline{T}_\delta, \overline{T}^{(I)}_\alpha) = 0$ for all $i > 0$. That is, $\overline{T}^{(I)}_\alpha \in \overline{T}^\perp_\delta$ for all $\delta < \alpha$.

Now consider the exact sequence

$$0 \longrightarrow \overline{T}_\delta \xrightarrow{f^\delta_{\delta+1}} \overline{T}_{\delta+1} \longrightarrow \mathrm{Coker} f^\delta_{\delta+1} \longrightarrow 0. \tag{3}$$

Applying the functor $\mathrm{Hom}_A(\_, \overline{T}^{(I)}_\alpha)$ to the sequence (3), we get the long exact sequence of homology

$$\mathrm{Hom}_A(\overline{T}_{\delta+1}, \overline{T}^{(I)}_\alpha) \xrightarrow{(f^\delta_{\delta+1})^*} \mathrm{Hom}_A(\overline{T}_\delta, \overline{T}^{(I)}_\alpha) \longrightarrow \mathrm{Ext}^1_A(\mathrm{Coker} f^\delta_{\delta+1}, \overline{T}^{(I)}_\alpha) \ \ldots \tag{4}$$

with $(f^\delta_{\delta+1})^*$ onto. Since the next term in 4 is zero, then

$$\mathrm{Ext}^1_A(\mathrm{Coker} f^\delta_{\delta+1}, \overline{T}^{(I)}_\alpha) = 0.$$

For $i \geqslant 2$, as $\mathrm{Ext}^i_A(\overline{T}^\delta, \overline{T}^{(I)}_\alpha) = \mathrm{Ext}^{i+1}_A(\overline{T}^{\delta+1}, \overline{T}^{(I)}_\alpha) = 0$, then

$$\mathrm{Ext}^i_A(\mathrm{Coker} f^\delta_{\delta+1}, \overline{T}^{(I)}_\alpha) = 0.$$

By the Lemma 4.1, we have $\mathrm{Ext}^i_A(\overline{T}_\alpha, \overline{T}^{(I)}_\alpha) = 0$. This prove the condition b) from n-tilting module definition.

Now, in order to prove that $\mathrm{pd}\overline{T}_\alpha \leqslant n+1$, consider X an A-module and $\delta < \alpha$ an ordinal number. Since $\overline{T}_\delta$ is a n-tilting module, $\mathrm{pd}(\overline{T}_\delta) \leqslant n$, therefore $\mathrm{Ext}^i_A(\overline{T}_\delta, X) = 0$ for all $i > n$ and for all $\delta < \alpha$.

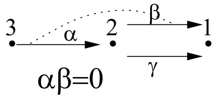
From the exactness of sequence (3), we have $\mathrm{Ext}^i_A(\mathrm{Coker} f^{\delta}_{\delta+1}, X) = 0$ for $i > n+1$ and $\mathrm{Ext}^i_A(\overline{T}_\alpha, X) = 0$ for all $i > n+1$. Then $\mathrm{pd}(\overline{T}_\alpha) < n+1$ and $\overline{T}_\alpha$ is a $n+1$-tilting module.

Now, by $\overline{T}_\alpha \in \overline{T}_\delta^\perp$ for $\delta < \alpha$, is not hard to see that $\overline{T}_\alpha \in \bigcap_{\gamma<\alpha} T_\gamma^\perp$ and, as $\overline{T}_\alpha$ is a tilting module, $\overline{T}_\alpha^\perp \subseteq \bigcap_{\gamma<\alpha} T_\gamma^\perp$. Reciprocally, if $Y \in \bigcap_{\gamma<\alpha} T_\gamma^\perp$, then $\mathrm{Ext}^i_A(\overline{T}_\gamma, Y) = 0$ for all $i > 0$ and for all $\gamma < \alpha$. Using the exact sequence (3) and the Lemma 4.1 we get $\mathrm{Ext}^i_A(\overline{T}_\alpha, Y) = 0$. This implies that $\bigcap_{\gamma<\alpha} T_\gamma^\perp \subseteq \overline{T}_\alpha^\perp$ and then $\overline{T}_\alpha^\perp = \bigcap_{\gamma<\alpha} T_\gamma^\perp$ for all $\alpha < \mu$. Here, also for a limit ordinal, we have $\overline{T}_\alpha \in \overline{T}_\gamma^\perp$ if $\gamma < \alpha$.

Finally, for each limit ordinal $\alpha < \mu$, $\overline{T}_{\alpha+1} \in \overline{T}_{\gamma+1}^\perp$ for all $\gamma < \alpha$. Then $\overline{T}_{\alpha+1} \in \overline{T}_\alpha^\perp$. Since both tilting modules, by Proposition 3.1, $\mathrm{pd}\,\overline{T}_\alpha \leqslant \mathrm{pd}\,\overline{T}_{\alpha+1} \leqslant n$. ☐

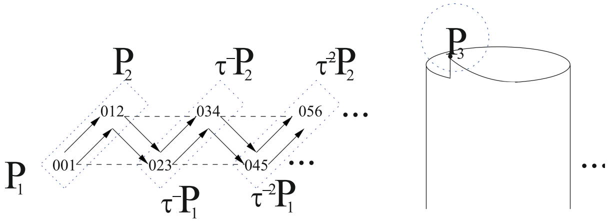
Unfortunately for the ordinal μ, although the projective dimension of the class of tilting modules $\{T_\alpha\}_{\alpha<\mu}$ is limited by n, we can only be sure that $\mathrm{pd}\, T_\mu \leqslant n+1$.

Example 5.2. *Consider the path algebra A, over a algebrically closed field, given by the quiver with relations $\alpha\beta = 0$.*



Its Auslander-Reiten quiver has a postprojective component (corresponding to the postprojective component of the Kronecker algebra given by the vertices 1 and 2). Also, the projective P_3 lies in a ray tube.

The post projective and semiregular componentes are pictured below.



The sequence of A-modules $(T_i)_{i\in\mathbb{N}}$ where for each $i \in \mathbb{N}$, $T_i = \tau^{-i}P_1 \oplus \tau^{-i}P_2 \oplus P_3$ is a sequence of 1-tilting modules. In this case $\mathrm{pd}(\varinjlim_{i\in\mathbb{N}} T_i) = 1$.

We have been looking for an exemple where $\mathrm{pd}\, T_\mu = n+1$, but we have not found it yet.

6. A special case on hereditary algebras

In this section we will look at the case where we have a sequence of tilting modules over a post projective component of a hereditary algebra.

Obviously, over a hereditary algebra all tilting modules are 1-tilting.

Theorem 6.1. *Let A be a hereditary k-algebra and $\{T_1, T_2, \ldots\}$ a countable direct system of tilting A-modules pairwise not equivalent such that $T_{i+1} \in (T_i)^{\perp}$. Then the direct limit $\varinjlim_{i\in\mathbb{N}} T_i$ is a 1-tilting module.*

Proposition 6.2. *Let $\{T_1, T_2, \ldots\}$ be a sequence of tilting A-modules. Then*

$$\left(\varinjlim_{i\in\mathbb{N}} T_i\right)^{\perp} = \bigcap_{i\in\mathbb{N}} (T_i)^{\perp} = (\{T_1, T_2, \ldots\})^{\perp}$$

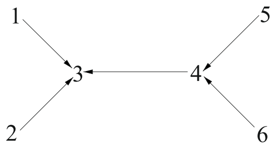
In [14], F. Lukas described a process to obtain a countable $\mathcal{P}$-filtered tilting A-module L such that $\mathrm{L}^{\perp} = \mathcal{P}^{\perp}$ (where $\mathcal{P}$ denote the post projective component of ind$A$).

Proposition 6.3 ([13]). *Let A be a hereditary algebra and $\mathcal{S} \subset \mathrm{add}(\mathcal{P})$ a infinite set. Then $\mathcal{S}^{\perp} = \mathcal{P}^{\perp} = \mathrm{L}^{\perp}$.*

Corollary 6.4. *Let $(T_i)_{i\in\mathbb{N}} \subset \mathrm{add}(\mathcal{P})$ be a sequence of tilting A-modules. Then $\varinjlim_{i\in\mathbb{N}} T_i$ is equivalent to* L.

The above corollary allows us to calculate the Lukas tilting module L by using tilting sequences.

Example 6.5. *Consider the path algebra A, given by the quiver*



The start of the post projective component of the Auslander-Reiten quiver of A is pictured bellow.

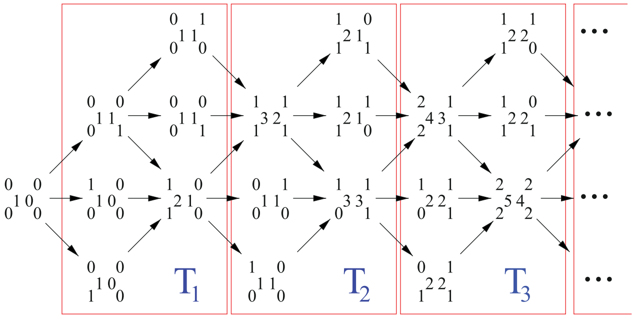


FIGURE 1. The sequence of tilting A-modules.

The sequence of modules, obtained from the direct sums of indecomposable modules into each rectangle, is a sequence of tilting modules. Its direct limit is equivalent to the Lukas Tilting Module L.

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