Algebra and Discrete Mathematics
Volume **28** (2019). Number 2, pp. 171–183

RESEARCH ARTICLE

Strongly prime submodules and strongly 0-dimensional modules

Zehra Bilgin, Suat Koç, and Neslihan Ayşen Özkirişci

Communicated by R. Wisbauer

Abstract. In this work, we study strongly prime submodules and strongly 0-dimensional modules. We give some equivalent conditions for being a strongly 0-dimensional module. Besides we show that the quasi-Zariski topology on the spectrum of a strongly 0-dimensional module satisfies all separation axioms and it is a metrizable space.

Introduction

Prime ideals have a distinguished place in commutative ring theory. Their generalization to module theory, namely prime submodules are one of the useful tools in understanding the structure of modules over commutative rings. Let R be a commutative ring, and M an R-module. A submodule P of M is called a prime submodule if whenever $rm \in P$ for some $m \in M$ and $r \in R$, either $m \in P$ or $r \in (P : M)$. It is still an appealing problem to extend properties of prime ideals to prime submodules.

One of the well-known properties of prime ideals is that: For a commutative ring R and a prime ideal P of R, if P contains the intersection of a finite family of ideals, it contains at least one of those ideals. In [8], Gilmer examined that when this property is valid for an infinite family of ideals.

2010 MSC: Primary 13A15, 13A99, 13C05, 13C99; Secondary 13E15, 13E99.
Key words and phrases: strongly prime submodule, strongly 0-dimensional module, multiplication module, quasi-Zariski topology.

In [12], Jayaram et al. named prime ideals satisfying this property for any infinite family of ideals as strongly prime ideals. That is, a prime ideal P is called strongly prime if whenever an infinite intersection of a family of ideals is contained in P, at least one of the ideals in that family is in P. Jayaram et al. called a ring strongly 0-dimensional if every prime ideal of the ring is strongly prime. They proved that strongly 0-dimensional rings are zero dimensional and examined some properties of these rings including their relation with von Neumann regular, Artinian and Noetherian rings. In [9], Gottlieb conducted a further study on strongly prime ideals and strongly 0-dimensional rings. He gave some equivalent conditions for being a strongly 0-dimensional ring, and determined a class of strongly 0-dimensional rings, namely strongly n-regular rings.

We note that there is another type of ideals in commutative ring theory named strongly prime ideals defined by Hedstrom and Houston in [10]. According to that a prime ideal of a domain R with quotient field K is called strongly prime if $x, y \in K$ and $xy \in P$ imply that $x \in P$ or $y \in P$. This concept is unrelated to strongly prime ideals that are considered in this paper.

An R-module M is called a multiplication module if every submodule of M can be written in the form IM for some ideal I of R. Multiplication modules are investigated by many authors, for detailed information see [2], [5], [1] and [16]. In [15], Oral et al. generalized the concept of strongly 0-dimensional rings to the class of multiplication modules as follows: Let R be a commutative ring and M an R-module. A prime submodule P of M is said to be a strongly prime submodule if whenever an intersection of a family of submodules is contained in P, at least one of the submodules in the family is in P. If every prime submodule of M is strongly prime then M is called a strongly 0-dimensional module. Among other things, Oral et al. investigated relations between strongly 0-dimensional modules, von Neumann regular modules and Q-modules.

In this work, we examine some further properties of strongly prime submodules and strongly 0-dimensional modules. All rings are assumed to be commutative with identity and all modules are unitary and multiplication. In [2], Ameri defined a product for submodules of multiplication modules as follows: Let $N = IM$ and $K = JM$ be two submodules of a multiplication R-module M where I and J are ideals of R. The product of N and K is defined as $(IJ)M$. For elements m and m' of M, the product of m and m' is defined as $RmRm'$ where Rm is the cyclic submodule of M generated by m. Among other things, Ameri obtained a characterization of radical of a submodule of a multiplication module in terms of elements of M. We note that the radical of a submodule of a module is defined as the intersection of the prime submodules containing that submodule. Ameri proved that for a submodule N of a multiplication module M, the radical of N, denoted as $\mathrm{rad}(N)$, is

$$\mathrm{rad}(N) = \{m \in M : m^k \subseteq N \text{ for some } k \in \mathbb{N}\}.$$

Using that multiplication we extend the notion of descending chain condition for principal powers, first introduced in [9], to multiplication modules: A module M satisfies the descending chain condition (DCC) on principal powers if every chain $m \supseteq m^2 \supseteq m^3 \supseteq \cdots$ stops, i.e., $m^n = m^{n+1}$ for some $n \in \mathbb{N}$. Using DCC on principal powers we give some equivalent conditions for being a strongly 0-dimensional module (See Theorem 2 and Theorem 4). A module is called quasi-semi-local if it has finitely many maximal submodules. In Corollary 4, we prove that a finitely generated module is strongly 0-dimensional if and only if it is a zero dimensional quasi-semi-local module. After examining strongly prime submodules and strongly 0-dimensional modules in Section 2, we investigate strongly prime and strongly 0-dimensional property for idealization of M in R in Section 3. The ring

$$R(+)M = \{(r, m) : r \in R, m \in M\}$$

with component-wise addition and multiplication defined as

$$(r, m)(s, n) = (rs, rn + sm)$$

is called the idealization of M in R. The reader may consult [3] and [11] for further information and the ideal structure of an idealization of M.

Finally, in Section 4, we examine topological structure of the space of prime submodules of a strongly 0-dimensional module. Let M be a module and $\mathrm{Spec}(M)$ the set of prime submodules of M. For a submodule N of M set $V(N) = \{P \in \mathrm{Spec}(M) : N \subseteq P\}$. The family $\{V(N) : N \text{ a submodule of } M\}$ determines a topology on $\mathrm{Spec}(M)$ as closed sets if and only if it is closed under finite unions. In that case, this topology is called quasi-Zariski topology and M is called a top module, for details, see [13]. Note that every multiplication module is a top module. In Proposition 2, we characterize all finitely generated strongly 0-dimensional modules in terms of its quasi-Zariski topology. In view of this result, $\mathrm{Spec}(M)$ satisfies all separation axioms if M is strongly 0-dimensional. Furthermore, a finitely generated module M is strongly 0-dimensional if and only if $\mathrm{Spec}(M)$ is finite T_1-space (see Theorem 10).

1. Strongly prime submodules and strongly 0-dimensional modules

Throughout this study all rings are assumed to be commutative with nonzero identity and all modules are unitary multiplication. Let R denote such a ring and M denote such an R-module. In this section we examine some properties of strongly prime submodules and strongly 0-dimensional modules.

Definition 1. [15, Definition 2.1] A prime submodule P of an R-module M is called strongly prime if $\bigcap_{i\in J} N_i \subseteq P$ implies that $N_j \subseteq P$ for some $j \in J$. An R-module M is called strongly 0-dimensional if all prime submodules are strongly prime.

Lemma 1. *Let P be a strongly prime submodule of M. Then $(P : M)$ is a maximal ideal of R.*

Proof. Assume that $r \notin (P : M)$. Then there exists $m \in M$ such that $rm \notin P$. Since $rm \notin P$, we have $Rm \nsubseteq P$. Set $K = \bigcap_{Q\nsubseteq P} Q$. Thus we have $K \nsubseteq P$ since P is a strongly prime submodule. Then there exists $m' \in K - P$. Since $Rm' \nsubseteq P$, we have $Rm' = K$. As P is prime and $r \notin (P : M)$, we have $rm' \notin P$. So, we get $Rrm' \nsubseteq P$ and thus $Rrm' \subseteq Rm' = K$. This implies $Rrm' = Rm'$. Then there exists $r' \in R$ such that $r'rm' = m'$ and hence $(1 - rr')m' = 0 \in P$. Since $m' \notin P$ we get $1 - rr' \in (P : M)$. Consequently $(P : M)$ is a maximal ideal of R. □

Corollary 1. *If P is a strongly prime submodule of M, then P is a maximal submodule of M.*

Proof. By Lemma 1, $(P : M)$ is a maximal ideal of R and so $P = (P : M)M$ is a maximal submodule of M. □

Lemma 2. *If $(P : M)$ is a strongly prime ideal of R, then P is a strongly prime submodule of M.*

Proof. If $(P : M)$ is a strongly prime ideal of R, then the ideal $(P : M)$ is maximal by [9, Proposition 1.2]. So, the submodule P is a maximal submodule of M. Assume that $\bigcap_{i\in J} N_i \subseteq P$ for a family of submodules $\{N_i\}_{i\in J}$. Then we have $(\bigcap_{i\in J} N_i : M) = \bigcap_{i\in J}(N_i : M) \subseteq (P : M)$. Since $(P : M)$ is a strongly prime ideal, we conclude that $(N_j : M) \subseteq (P : M)$ for some $j \in J$. This implies $(N_j : M)M = N_j \subseteq (P : M)M = P$ and this completes the proof. □

Corollary 2. *If R is a strongly 0-dimensional ring, then M is a strongly 0-dimensional R-module.*

Next, we give a chain condition for submodules generated by powers of a single element.

Definition 2. An R-module M satisfies the descending chain condition (DCC) on principal powers if, for any $m \in M$ the chain

$$m \supseteq m^2 \supseteq m^3 \supseteq ...$$

stops, i.e, $m^n = m^{n+1}$ for some n.

The following condition is defined by Bilgin and Oral in [6].

Definition 3. A family $\{N_i\}_{i\in J}$ of submodules of M satisfies $(*)$ property if for all $x \in M$ there exists $n \in \mathbb{N}$ such that $x \in \text{rad}(N_i)$ implies $x^n \subseteq N_i$.

Bilgin and Oral proved that a family satisfies $(*)$ property if and only if intersection and radical operation commutes for this family as can be seen in the following result:

Theorem 1 ([6, Lemma 4.4.]). *A family $\{N_i\}_{i\in I}$ of submodules of M satisfies $(*)$ property if and only if for each subset $J \subseteq I$,*

$$\text{rad}\left(\bigcap_{i\in J} N_i\right) = \bigcap_{i\in J} \text{rad}(N_i).$$

We prove in the following lemma that DCC on principal powers is equivalent to the condition that every family satisfies $(*)$ property.

Lemma 3. *M satisfies DCC on principal powers if and only if every family of submodules satisfies $(*)$ property.*

Proof. $(\Rightarrow)$: Let $\{N_i\}$ be a family of submodules and $x \in \bigcap_{i\in J} \text{rad}(N_i)$. Let

$$x \supseteq x^2 \supseteq x^3 \supseteq \cdots$$

be a descending chain of principal powers. Then, by assumption, we have $x^n = x^{n+1}$ for some n. Since $x \in \text{rad}(N_i)$ for each i, we have $x^{t_i} \subseteq N_i$. If $t_i \geqslant n$, then $x^n = x^{t_i} \subseteq N_i$, so $x^n \subseteq N_i$. If $t_i \leqslant n$, then $x^n \subseteq x^{t_i} \subseteq N_i$, and thus $x^n \subseteq N_i$.

$(\Leftarrow)$: Let $x \supseteq x^2 \supseteq x^3 \supseteq ...$ be a descending chain of principal powers. Observe that $\text{rad}(x) = \text{rad}(x^i)$ for each i: Assume that $Rx = IM$. Then

$$\text{rad}(IM) = \text{rad}(\sqrt{I}M) = \text{rad}(\sqrt{I^i}M) = \text{rad}(I^iM)$$

and thus $\operatorname{rad}(x) = \operatorname{rad}(x^i)$. As $x \in \operatorname{rad}(x) = \operatorname{rad}(x^i)$ for each i, by $(*)$ property, there exists $n \in \mathbb{N}$ such that $x^n \subseteq x^i$ for all i, and so $x^n = x^{n+1}$. □

Now, we give one of the main results of this paper. The following theorem gives some equivalent conditions for being a strongly 0-dimensional module.

Theorem 2. *M is strongly 0-dimensional if and only if the following conditions hold:*
(i) *M satisfies DCC on principal powers.*
(ii) *For every family $\{P_i\}_{i\in J}$ of prime submodules and any prime submodule P of M, the inclusion $\bigcap_{i\in J} P_i \subseteq P$ implies $P_j \subseteq P$ for some $j \in J$.*

Proof. Let M be a strongly 0-dimensional module, $\{N_i\}_{i\in J}$ a family of submodules and P a prime submodule containing $\bigcap_{i\in J} N_i$. Then $N_j \subseteq P$ for some $j \in J$, so $\operatorname{rad}(N_j) \subseteq P$. Then $\bigcap_{i\in J} \operatorname{rad}(N_i) \subseteq \operatorname{rad}(N_j) \subseteq P$, hence $\operatorname{rad}(\bigcap_{i\in J} N_i) \supseteq \bigcap_{i\in J} \operatorname{rad}(N_i)$. Since the opposite inclusion always holds, M satisfies DCC on principal powers by Lemma 3. The condition (ii) is clear. Now, assume (i) and (ii) hold. Let P be a prime submodule and $\bigcap_{i\in J} N_i \subseteq P$ for any family of submodules $\{N_i\}_{i\in J}$ of M. Then, by (i), we have $\operatorname{rad}(\bigcap_{i\in J} N_i) = \bigcap_{i\in J} \operatorname{rad}(N_i) = \bigcap_{N_i \subseteq P_{i_k}} P_{i_k} \subseteq P = \operatorname{rad}(P)$. This implies $N_j \subseteq P_{j_k} \subseteq P$ for some $j \in J$ by (ii). Consequently, M is a strongly 0-dimensional module. □

The following theorem gives an equivalent condition for a finitely generated module to satisfy DCC on principal powers.

Theorem 3. *Let M be a finitely generated R-module. M satisfies DCC on principal powers if and only if, for every $x \in M$, $I + \operatorname{Ann}(x^n) = R$ for some $n \in \mathbb{N}$, where $Rx = IM$.*

Proof. $(\Rightarrow)$: Let $x \in M$. Then $Rx = IM$ for some finitely generated ideal I of R. Since M satisfies DCC on principal powers, $I^nM = I^{n+1}M$ for some $n \in \mathbb{N}$. Then I^nM is finitely generated, because M and I are finitely generated. By [4, Corollary 2.5], there is an $r \in I$ such that $(1-r)I^nM = 0$. Then $1 - r \in \operatorname{Ann}(I^nM) = \operatorname{Ann}(x^n)$. Hence $I + \operatorname{Ann}(x^n) = R$.

$(\Leftarrow)$: Let $x \in M$ and $I + \operatorname{Ann}(x^n) = R$. Then we have $I^nM = (I + \operatorname{Ann}(x^n))I^nM = I^{n+1}M$. □

We conclude that if a finitely generated module satisfies DCC on principal powers, then its Krull dimension is zero.

Corollary 3. *Let M be a finitely generated module. If M satisfies DCC on principal powers, then M is zero dimensional.*

Proof. Assume that $P_1 \subsetneq P_2$ are prime submodules of M. Let $x \in P_2 - P_1$. Then $x^n \nsubseteq P_1$ for all $n \in \mathbb{N}$. This implies $\mathrm{Ann}(x^n) \subseteq (P_1 : M) \subseteq (P_2 : M)$ and so $I + \mathrm{Ann}(x^n) \subseteq (P_2 : M) \neq R$. Thus M does not satisfy the DCC on principal powers. □

In the following theorem, we give some further equivalent conditions for a finitely generated module to be strongly 0-dimensional.

Theorem 4. *Let M be a finitely generated module. Then M is strongly 0-dimensional if and only if the following two conditions hold:*

(i) *No maximal submodule of M contains the intersection of the other maximal submodules, and*

(ii) *M satisfies the DCC on principal powers.*

Proof. Assume that M is strongly 0-dimensional. Since M is zero dimensional, (i) is clear and (ii) follows from Theorem 2. For the converse, assume that M satisfies (i) and (ii). Then M is zero dimensional by Corollary 3. Let K be a maximal submodule of M and $\{N_i\}_{i \in J}$ a family of submodules such that $K \supseteq \bigcap_{i \in J} N_i$. Then $K \supseteq \mathrm{rad}(\bigcap_{i \in J} N_i)$. Hence, $K \supseteq \bigcap_{i \in J} \mathrm{rad}(N_i)$ by (ii). For each $i \in J$, the submodule $\mathrm{rad}(N_i)$ is an intersection of maximal submodules as M is zero dimensional. Thus K must be one of these maximal submodules. Then $K \supseteq N_j$ for some $j \in J$. Consequently M is strongly 0-dimensional. □

A module M is called quasi-semi-local if it has only finitely many maximal submodules. The following theorem shows that a finitely generated strongly 0-dimensional module is quasi-semi-local.

Theorem 5. *Let M be a finitely generated R-module. If M is a strongly 0-dimensional module, then M is quasi-semi-local.*

Proof. Let $\Omega = \{N_i : i \in J\}$ be the set of all distinct maximal submodules of M. Assume that Ω is an infinite set. Since all N_i's are distinct, $\Omega' = \{(N_i : M) : N_i \in \Omega \text{ for all } i \in J\}$ is an infinite set of distinct maximal ideals (not necessarily the set of all maximal ideals) of R. Then, by [9, Proposition 1.9], either we have $\bigcap_{j \neq i}(N_j : M) \subseteq (N_i : M)$ for some $i \in J$ or there exists a maximal ideal K of R such that $\bigcap_{j \in J}(N_j : M) \subseteq K$, where $K \neq (N_j : M)$. If $\bigcap_{j \in J}(N_j : M) \subseteq K$, then $(0 : M) \subseteq K$ and so $N = KM$ is a maximal submodule, by assumption, we have $N = KM = N_k$ for some $k \in J$. This implies that $(N : M) = (KM : M) = K = (N_k : M)$ which is a contradiction. Now assume that $\bigcap_{j \neq i}(N_j : M) \subseteq (N_i : M)$ for $i \in J$. Then we get $(\bigcap_{j \neq i}(N_j : M))M \subseteq (N_i : M)M$ and this yields

$$\bigcap_{j \neq i}((N_j : M)M) = \bigcap_{j \neq i} N_j \subseteq N_i.$$

Since N_i is a strongly prime submodule, we have $N_j \subseteq N_i$ for some $j \in J$, a contradiction. □

It can be easily seen that there is a one-to-one correspondence between maximal submodules of a finitely generated multiplication R-module M and maximal ideals of the ring $R/(0 : M)$. Therefore, such a module M is zero-dimensional if and only if the ring $R/(0 : M)$ is zero dimensional. As a consequence, we have the following result:

Lemma 4. *Suppose that M is a finitely generated R-module. Then M is a zero dimensional module if and only if*

$$\sqrt{\bigcap_{j \in J} I_j} = \bigcap_{j \in J} \sqrt{I_j}$$

for each family $\{I_j\}_{j \in J}$ of ideals of R such that $(0 : M) \subseteq I_j$.

Lemma 5. *Suppose that M is a finitely generated R-module. If M is zero dimensional, then M satisfies DCC on principal powers.*

Proof. It is sufficient to show that

$$\operatorname{rad}\left(\bigcap_{i \in J} N_i\right) = \bigcap_{i \in J} \operatorname{rad}(N_i).$$

Assume that $\bigcap_{i \in J} N_i \subseteq P$ for some prime submodule P of M. Then

$$\left(\bigcap_{i \in J} N_i : M\right) = \bigcap_{i \in J}(N_i : M) \subseteq (P : M).$$

By Lemma 4, we have

$$\sqrt{\bigcap_{i \in J}(N_i : M)} = \bigcap_{i \in J} \sqrt{(N_i : M)} \subseteq (P : M).$$

Thus $(\bigcap_{i\in J}(\text{rad}\,(N_i) : M))M \subseteq (P : M)M$ by [14, Lemma 2.4]. This implies that

$$\bigcap_{i\in J}[\text{rad}\,(N_i) : M)M] = \bigcap_{i\in J}\text{rad}(N_i) \subseteq P.$$

Then we get $\bigcap_{i\in J}\text{rad}(N_i) \subseteq \text{rad}(\bigcap_{i\in J} N_i)$ which completes the proof. □

Let M be a finitely generated R-module. If M is a zero dimensional quasi-semi-local R-module, then M satisfies DCC on principal powers by Lemma 5. In this case, no maximal submodule contains the intersection of other maximal submodules. Thus, by Theorem 4, M is a strongly 0-dimensional module. Therefore, all finitely generated strongly 0-dimensional modules are exactly zero dimensional quasi-semi-local modules.

Corollary 4. *Let M be a finitely generated R-module. Then M is strongly 0-dimensional if and only if M is zero dimensional quasi-semi-local.*

Combining all these results, we have the following corollary:

Corollary 5. *Let M be a finitely generated R-module. Then M is a strongly 0-dimensional module if and only if $R/(0 : M)$ is a strongly 0-dimensional ring.*

2. When the idealization of a module is strongly 0-dimensional?

The idealization of M in R is defined as the ring

$$R(+)M = \{(r, m) : r \in R, m \in M\}$$

with component-wise addition and multiplication

$$(r, m)(s, n) = (rs, rn + sm)$$

for $(r, m), (s, n) \in R(+)M$. It is a commutative ring with identity $(1, 0)$. The maximal and prime ideals of $R(+)M$ are characterized by Anderson and Winders in [3] as follows:

Theorem 6 ([3, Theorem 3.2])**.** *The prime (resp., maximal) ideals of $R(+)M$ have the form $\mathfrak{P}(+)M$ where $\mathfrak{P}$ is a prime (resp., maximal) ideal of R. Hence,*

$$\dim(R(+)M) = \dim(R).$$

Here we determine strongly prime ideals of $R(+)M$:

Lemma 6. *Let P^* be a strongly prime ideal of $R(+)M$. Then $P^* = P(+)M$ for some strongly prime ideal P of R.*

Proof. Assume that P^* is a strongly prime ideal of $R(+)M$. Since P^* is also prime, we have $P^* = P(+)M$ for some prime ideal P of R. Assume that $\bigcap_{i\in J} I_i \subseteq P$ for some family of ideals $\{I_i\}_{i\in J}$ of R. Then we have

$$\bigcap_{i\in J} I_i(+)M = \bigcap_{i\in J}(I_i(+)M) \subseteq P(+)M.$$

Since $P(+)M$ is a strongly prime ideal of $R(+)M$, we have $I_j(+)M \subseteq P(+)M$, and thus $I_j \subseteq P$ for some $j \in J$. As a consequence, P is a strongly prime ideal of R. □

Brewer and Richman [7] give an equivalent condition for a ring to be zero dimensional as follows:

Theorem 7 ([7, Theorem 2.2])**.** *A ring R is zero dimensional if and only if there exists n such that $Rx^n = Rx^{n+1}$ for each $x \in R$.*

This lemma indicates that R is zero dimensional if and only if R satisfies the DCC on principal powers. In [9], Gottlieb gives the following theorem as a different characterization of strongly 0-dimensional rings.

Theorem 8 ([9, Theorem 1.8])**.** *R is strongly 0-dimensional if and only if the following conditions hold:*
- (i) *No maximal ideal of R contains the intersection of the other maximal ideals.*
- (ii) *R satisfies the DCC on principal powers.*

As it mentioned in the introduction, a strongly 0-dimensional ring is always zero dimensional by [12, Theorem 2.9]. By combining the previous two theorems, it can be easily seen that the converse is true when the condition (i) of Theorem 8 is satisfied.

The next theorem shows that strongly 0-dimensional property of $R(+)M$ depends only on strongly 0-dimensional property of R.

Theorem 9. *Let M be an R-module. Then R is a strongly 0-dimensional ring if and only if $R(+)M$ is a strongly 0-dimensional ring.*

Proof. The necessary condition follows from Lemma 6. For the sufficiency, suppose that R is a strongly 0-dimensional ring. Then it is also zero dimensional. Thus, $R(+)M$ is also zero dimensional by Theorem 6 and satisfies DCC on principal powers by Theorem 7. Now, it is enough to check that if $R(+)M$ satisfies the condition in Theorem 8 (i). Assume that $\bigcap_{i\in J}(\mathfrak{M}_i(+)M) \subseteq \mathfrak{M}_j(+)M$ where $\mathfrak{M}_i, \mathfrak{M}_j$ are maximal ideals of R for all $i \in J$ and $i \neq j$. Then we have

$$\bigcap_{i\in J} \mathfrak{M}_i(+)M \subseteq \mathfrak{M}_j(+)M$$

that is, $\bigcap_{i\in J} \mathfrak{M}_i \subseteq \mathfrak{M}_j$, a contradiction. □

3. The spectrum of a strongly 0-dimensional module

In this section we will examine topological structure of the set of all prime submodules Spec(M) of a strongly 0-dimensional module M.

Let N be a submodule of a module M and set

$$V(N) = \{P \in \mathrm{Spec}(M) : N \subseteq P\}.$$

The set Spec(M) is equipped with the quasi-Zariski topology if and only if the family $\{V(N) : N \subseteq M\}$ is closed under finite unions. In this case M is called a top module, see [13]. Since all modules are assumed to be multiplication in this article, they are also top modules.

Theorem 10. *Let M be a finitely generated R-module. Then M is a strongly 0-dimensional module if and only if* Spec(M) *is a finite T_1-space.*

Proof. Let M be a strongly 0-dimensional module. Then $\mathrm{Spec}(M) = Max(M)$ and also M is quasi-semi-local. This implies that Spec(M) is a finite T_1-space. Conversely, assume that Spec(M) is a finite T_1-space. Since Spec(M) is a T_1-space, every prime submodule is maximal and hence M is 0-dimensional. Also note that M is quasi-semi-local since Spec(M) is finite topological space. Thus M is a strongly 0-dimensional module by Corollary 4. □

Note that if M is a strongly 0-dimensional module which is not quasi-local, Spec(M) is not a connected space since all finite T_1-spaces are equipped with discrete topology. Also note that the quasi-Zariski topology on a strongly 0-dimensional module satisfies all separation axioms. In particular, Spec(M) is metrizable for a strongly 0-dimensional module M.

References

[1] Abd El-Bast Z, Smith PF, Multiplication modules, *Comm Algebra*, **16**, (1988), 755-779.

[2] Ameri R, On the prime submodules of multiplication modules, *Int J Math Sci*, **27**, (2003), 1715-1724.

[3] Anderson DD, Winders M, Idealization of a module, *J Commut Algebr*, **56(1)**, (2009), 3-56.

[4] Atiyah MF, MacDonald IG. *Intorduction to Commutative Algebra*, Addison-Wesley Publishing Company, Massachusetts, 1994.

[5] Barnard A, Multiplication modules, *J Algebra*, **71(1)**, (1988), 174-178.

[6] Bilgin Z, Oral KH, Coprimely structured modules, *Palestine J Math*, **7**, (2018), 161–169.

[7] Brewer J, Richman F, Subrings of 0-dimensional rings, *Multiplicative Ideal Theory in Commutative Rings*, (2006), 73-88.

[8] Gilmer RW, An intersection condition for prime ideals, *Lect Notes Pure Appl*, **189**, (1997), 327-331.

[9] Gottlieb C, On strongly prime ideals and strongly zero-dimensional rings, *J. Algebra Appl.*, **16(10)**, (2017), 1750191-1-1750191-9.

[10] Hedstrom JR, Houston EG, Pseudo-valuation domains, *Pacific J Math* , **75(1)**, (1978), 137-147.

[11] Huckaba JA, *Commutative rings with zero divisors*, New York Monographs and Textbooks in Pure and Applied Mathematics 117, Marcel Dekker, Inc., 1988.

[12] Jayaram C, Oral KH, Tekir Ü, Strongly 0-dimensional rings, *Comm Algebra*, **41(6)**, (2013), 2026-2032.

[13] McCasland RL, Moore ME, Smith PF, On the spectrum of a module over a commutative ring, *Comm Algebra*, **25(1)**, (1997), 79-103.

[14] Mostafanasab H, Yetkin E, Tekir Ü, Darani AY, On 2-Absorbing primary submodules over commutative rings, *An. St. Univ. Ovidius Constanta*, **24(1)**, (2016), 335-351.

[15] Oral KH, Özkirişci NA, Tekir Ü, Strongly 0-dimensional module, *Canad Math Bull*, **57(1)**, (2014), 159-165.

[16] Smith PF, Some remarks on multiplication modules, *Arch Math*, **50(3)**, (1988), 223-235.

CONTACT INFORMATION

Z. Bilgin Philosophy Department, Istanbul Medeniyet University 34000, Kadikoy, Istanbul, Turkey
E-Mail(s): zehrabilgin.zb@gmail.com

S. Koç Department of Mathematics, Faculty of Science and Arts, Marmara University 34722, Istanbul, Turkey
E-Mail(s): suat.koc@marmara.edu.tr

N. A. Özkirişci Department of Mathematics, Faculty of Science and Arts, Yildiz Technical University 34210, Istanbul, Turkey
E-Mail(s): aozk@yildiz.edu.tr

Received by the editors: 29.03.2018
and in final form 18.09.2018.