Algebra and Discrete Mathematics
Volume **28** (2019). Number 1, pp. 75–93 RESEARCH ARTICLE

Cohen–Macaulay modules over the plane curve singularity of type T_{44}, II

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To the memory of Volodymyr Kyrychenko

Abstract. We accomplish the classification of Cohen–Macaulay modules over the curve singularities of type T_{44} and the description of the corresponding matrix factorizations, started in [8].

Introduction

The plane curve singularities of types T_{pq}, given by the relation

$$X^p + Y^q + aX^2Y^2 = 0,$$

where $1/p+1/q \leqslant 1/2$, play an important role in the theory of singularities. They are "serial" unimodal singularities in the Arnold classification [1]. They also are the "critical" Cohen–Macualay tame curve singularities [7]. The simplest of them is the singularity of type T_{44}, which can be also given by the relation

$$XY(X-Y)(X-\lambda Y) = 0 \quad (\lambda \notin \{0,1\}).$$

A classification of Cohen–Macaulay modules over this singularity in terms of their Auslander–Reiten quiver was given by Dieterich [5]. Another approach, using cluster tilting, was suggested in [4]. Nevertheless, neither of them gave an explicit description of modules by generators and relations, or, equivalently, the corresponding *matrix factorizations* [9].

2010 MSC: Primary 13C14; Secondary 14H20.
Key words and phrases: Cohen–Macaulay modules, matrix factorizations, curve singularities.

In the paper [8], the authors used for this purpose the technique of *matrix problems* and classified a part of such modules, called the *modules of the first level*, together with the corresponding matrix factorizations. In this paper we will accomplish the classification of Cohen–Macaulay modules over the singularity of type T_{44} and of the corresponding matrix factorizations.

1. Matrix problem and modules of the first level

We recall the results of [8] on Cohen–Macaulay modules over the T_{44}-singularity, i.e. the ring $\boldsymbol{R} = \boldsymbol{S}/(F)$, where $\boldsymbol{S} = \Bbbk[[X, Y]]$, $\Bbbk$ is an algebraically closed field, and $F = XY(X-Y)(X-\lambda Y)$ ($\lambda \in \Bbbk\backslash\{0, 1\}$). We denote by $\mathsf{CM}(\boldsymbol{R})$ the category of maximal Cohen–Macaulay $\boldsymbol{R}$-modules. We consider $\boldsymbol{R}$ as the subring of the direct product $\boldsymbol{R}_1 \times \boldsymbol{R}_2 \times \boldsymbol{R}_3 \times \boldsymbol{R}_4$, where all $\boldsymbol{R}_i = \Bbbk[[t]]$, generated by the elements $x = (t, 0, t, \lambda t)$ and $y = (0, t, t, t)$. We denote by $\boldsymbol{R}_{ij}$ the projection of $\boldsymbol{R}$ to $\boldsymbol{R}_i \times \boldsymbol{R}_j$. All rings $\boldsymbol{R}_{ij}$ are isomorphic to $\Bbbk[[X, Y]]/(XY)$, hence all indecomposable $\boldsymbol{R}_{ij}$-modules are $\boldsymbol{R}_i$, $\boldsymbol{R}_j$ and $\boldsymbol{R}_{ij}$. Let $\boldsymbol{K}_i \simeq \Bbbk((t))$ be the field of fractions of $\boldsymbol{R}_i$, $\boldsymbol{K}_{ij} = \boldsymbol{K}_i \times \boldsymbol{K}_j$. Every Cohen–Macaulay $\boldsymbol{R}$-module M embeds into $\boldsymbol{K} \otimes_{\boldsymbol{R}} M$. Denote by N the image of M under the projection $\boldsymbol{K} \otimes_{\boldsymbol{R}} M \to \boldsymbol{K}_{12} \otimes_{\boldsymbol{R}} M$ and by L be the kernel of the surjection $M \to N$. Then $N \in \mathsf{CM}(\boldsymbol{R}_{12})$ and $L \in \mathsf{CM}(\boldsymbol{R}_{34})$, the exact sequence $0 \to L \to M \to N \to 0$ defines an element $\chi(M) \in \operatorname{Ext}^1_{\boldsymbol{R}}(N, L)$ and the following result holds [8].

Theorem 1.1. *The map $M \mapsto \chi(M)$ induces an equivalence of the category $\mathsf{CM}(\boldsymbol{R})$ of maximal Cohen–Macaulay modules over $\boldsymbol{R}$ and the category $\mathbb{E}$ of elements of the $\mathsf{CM}(\boldsymbol{R}_{12})$-$\mathsf{CM}(\boldsymbol{R}_{34})$-bimodule $\operatorname{Ext}^1_{\boldsymbol{R}}$ (in the sense of [6]).*

If $\boldsymbol{I}_s$ denotes the kernel of the projection $\boldsymbol{R} \to \boldsymbol{R}_s$ ($s \in \{1, 2, 12\}$) and $L \in \mathsf{CM}(\boldsymbol{R}_{34})$, then $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_s, L) \simeq L/\boldsymbol{I}_s L$, which gives a list of generators of the vector spaces $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_s, \boldsymbol{R}_r)$ for $s \in \{1, 2, 12\}$, $r \in \{3, 4, 34\}$ presented in Table 1.

Here 1_s is the unit of the ring $\boldsymbol{R}_s$, and $t_s = t1_s$. Thus an object of the category $\mathbb{E}$ is given by a block matrix

$$\mathrm{X} = \begin{pmatrix} X_3^1 & X_3^2 & X_3^{12} \\ X_4^1 & X_4^2 & X_4^{12} \\ X_{34}^1 & X_{34}^2 & X_{34}^{12} \end{pmatrix},$$

where X_r^s is a matrix with elements from $\operatorname{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_s, \boldsymbol{R}_r)$.

Table 1.

	$\boldsymbol{R}_1$	$\boldsymbol{R}_2$	$\boldsymbol{R}_{12}$
$\boldsymbol{R}_3$	1_3	1_3	$1_3,\ t_3$
$\boldsymbol{R}_4$	1_4	1_4	$1_4,\ t_4$
$\boldsymbol{R}_{34}$	1_{34}, $t_3=-t_4$	1_{34}, $t_3=-\lambda t_4$	$1_{34}, t_3,\ t_4$, $t_3^2=-\lambda t_4^2$

Two matrices X and X′ of this form describe isomorphic modules if and only if $S\mathrm{X}=\mathrm{X}'T$, where $S=(S_r^s)$ $(s,r\in\{3,4,34\})$ and $T=T_r^s$ $(s,r\in\{1,2,12\})$ are block matrices of the appropriate size such that[1]

- $S_r^s=0$ if $\{s,r\}=\{3,4\}$ and $T_r^s=0$ if $\{s,r\}=\{1,2\}$;
- the elements of S_r^r or T_r^r are from $\boldsymbol{R}_r$;
- the elements of T_{12}^s are from $\boldsymbol{R}_s$ if $s\in\{1,2\}$ and elements of S_r^{34} are from $\boldsymbol{R}_r$ if $r\in\{3,4\}$;
- the elements of T_1^{12} are from $x\boldsymbol{R}_{12}$ and the elements of T_2^{12} are from $y\boldsymbol{R}_{12}$;
- the elements of S_{34}^s $(s\in\{3,4\}$ are from $t_s\boldsymbol{R}_{34}$.

In [8] a description was given of the *modules of the first level*, i.e. those corresponding to the matrices X whose elements are from $\Bbbk$ (no terms with t_i). A list of indecomposable matrices of this kind is given in Table 2.

Remark 1.2. Note that in the representations $X_3(n)^+$, $X_5(n)^+$, $X_8(n)$, $X_9(n)$ we can set the vector $\mathbf{e}_n$ either in the last row of the first horizontal stripe or in the last row of the second horizontal stripe, obtaining equivalent representations. Just in the same way, we can set the vector $\mathbf{e}_1^\top$ either in the last column of the first vertical stripe or in the last column of the second vertical stripe.

Actually, a complete list of indecomposable matrices is obtained from this table if we also consider the transposed matrices and also replace some superscripts + by −. The latter operation means that we interchange the first two stripes (horizontal or vertical). Namely, in $\mathrm{X}_1(n)$ we interchange both vertical and horizontal stripes, while in other cases we interchange the stripes which are of unequal sizes. Note that transposing the matrices X_i for $i\in\{0,1,6,9,10\}$ gives isomorphic objects from $\mathbb{E}$.

[1] Note that the description of these matrices in [8] is inaccurate. It does not imply the results of [8], since only modules of the first kind were considered there.

TABLE 2.

$$\mathrm{X}_0(n,\mu)=\left(\begin{array}{c|c} I_n & J_n(\mu) \\ \hline I_n & I_n \end{array}\right) \quad (\mu\in\Bbbk\setminus\{0,1\});$$

$$\mathrm{X}_1(n)^{++}=\left(\begin{array}{c|c} I_m & J_m(0) \\ \hline I_m & I_m \end{array}\right) \quad \text{if } n=2m;$$

$$\mathrm{X}_1(n)^{++}=\left(\begin{array}{cc|c} I_m & 0 & J_m(0) \\ 0 & 1 & \mathbf{e}_m \\ \hline I_m & 0 & I_m \end{array}\right) \text{ if } n=2m+1; \quad \mathrm{X}_2(n)^{+}=\left(\begin{array}{c|c} I_n & J_n(1) \\ 0 & \mathbf{e}_n \\ \hline I_n & I_n \end{array}\right);$$

$$\mathrm{X}_3(n)^{+}=\left(\begin{array}{cc|cc|c} I_n & 0 & I_n & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline J_n(1) & \mathbf{e}_1^\top & I_n & 0 & 0 \\ \mathbf{e}_n & 0 & 0 & 0 & 1 \end{array}\right); \quad \mathrm{X}_4(n)^{+}=\left(\begin{array}{cc|c} I_n & 0 & J_n(1) \\ 0 & 1 & \mathbf{e}_n \\ \hline I_n & 0 & I_n \\ 0 & 1 & 0 \end{array}\right);$$

$$\mathrm{X}_5(n)^{+}=\left(\begin{array}{cc|c|c} I_n & \mathbf{e}_1^\top & J_n(1) & 0 \\ 0 & 0 & \mathbf{e}_n & 1 \\ \hline I_n & 0 & I_n & 0 \\ 0 & 0 & 0 & 1 \end{array}\right); \quad \mathrm{X}_6(n)=\left(\begin{array}{c|c} I_n & J_n(1) \\ \hline I_n & I_n \end{array}\right);$$

$$\mathrm{X}_7(n)=\left(\begin{array}{ccc|ccc} I_m & 0 & 0 & J_m(1) & 0 & 0 \\ 0 & I_k & 0 & 0 & I_k & 0 \\ 0 & (1-\varepsilon)\mathbf{e}_k & 0 & \mathbf{e}_m & 0 & 0 \\ \hline I_m & 0 & 0 & I_m & 0 & 0 \\ 0 & J_k(1) & \mathbf{e}_1^\top & 0 & I_k & 0 \\ 0 & \mathbf{e}_k & 0 & \varepsilon\mathbf{e}_m & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right),$$

where $m=[n/2]$, $k=n-m-1, \varepsilon=0$ if n is odd and $\varepsilon=1$ if n is even;

$$\mathrm{X}_8(n)=\left(\begin{array}{c|c|c} I_n & J_n(1) & 0 \\ 0 & \mathbf{e}_n & 1 \\ \hline I_n & I_n & 0 \\ 0 & 0 & 1 \end{array}\right); \quad \mathrm{X}_9(n)=\left(\begin{array}{cc|cc|c} I_n & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline J_n(1) & \mathbf{e}_1^\top & I_n & 0 & 0 \\ \mathbf{e}_n & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \end{array}\right);$$

$$\mathrm{X}_{10}=\left(1\right).$$

(The matrix X_{10} only has the entry from $\mathrm{Ext}^1_{\boldsymbol{R}}(\boldsymbol{R}_{12},\boldsymbol{R}_{34})$; it corresponds to the regular $\boldsymbol{R}$-module).

2. Reduced matrix problem. First step

Suppose now that an object from $\mathbb{E}$ is presented by a matrix $\mathrm{M} = \mathrm{X} + t\mathrm{Y}$, where X has elements from $\Bbbk$ and is a direct sum of canonical indecomposable matrices from Table 2 and, maybe, zero matrices:

$$\mathrm{X} = \begin{pmatrix} \mathrm{C}_1 & 0 & \dots & 0 & 0 \\ 0 & \mathrm{C}_2 & \dots & 0 & 0 \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots} \\ 0 & 0 & \dots & \mathrm{C}_m & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

where each C_k is one of the matrices $\mathrm{X}_i(n)$ or $\mathrm{X}_i^\top(n)$ (perhaps with superscripts + or −). Respectively, Y becomes subdivided into blocks Y_{kl} $(1 \leqslant k, l \leqslant m+1)$. If we replace M by $(I+tS)\mathrm{M}$ or by $\mathrm{M}(I+tT)$, the part X does not change. Taking for S and T elementary matrices, we can add the block C_k to the blocks Y_{kl} and Y_{lk}. Using these transformations, we get the following results.

1) If $\mathrm{C}_k = \mathrm{X}_0(n,\mu)$ and $\mu \neq \lambda^{-1}$ or C is one of the representations $\{\,\mathrm{X}_1(n)^\pm, \mathrm{X}_6(n), \mathrm{X}_8(n), \mathrm{X}_9(n), \mathrm{X}_{10}\,\}$ or their transposed, we can make zero all blocks Y_{kl} and Y_{lk}. Here one must take into account that the matrix X_4^1 multiplied by x becomes $\lambda t_4 \mathrm{X}_4^1$ and in the part X_{34}^2 we have $t_3 = -\lambda t_4$, while in the part X_{34}^1 we have $t_3 = -t_4$. Moreover, if $\mathrm{C}_k = \mathrm{X}_8(n)$ or $\mathrm{C}_k = \mathrm{X}_9(n)$, we can do zero the rows containing 1 in the (12)-column, using endomorphisms of $\boldsymbol{R}_{12}$ given by multiplications by t_1 and t_2. Analogously, if $\mathrm{C}_k = \mathrm{X}_8(n)^\top$ or $\mathrm{C}_k = \mathrm{X}_9(n)$, we can make zero all columns containing 1 in the (34)-row.

Therefore, these representations are actually direct summands of the whole matrix M. So we can further suppose that they do not occur in X.

2) If $\mathrm{C}_k = \mathrm{X}_0(n,\lambda^{-1})$, we only can make zero all rows of Y_{kl} except one of then (the first one in either Y_3-part or in Y_4-part), as well as all columns of X_{lk}, except one of them (the first one in either Y^1-part or in Y^2-part). We denote this row and this column, respectively, by $a_0(n)$ and $a^0(n)$.

3) If $\mathrm{C}_k \in \{\,\mathrm{X}_2(n)^\pm, \mathrm{X}_4(n)^\pm\,\}$, its columns are linear independent, so we can make zero all blocks Y_{lk} of the same vertical stripe. On the other hand, we can make zero all rows of the matrices Y_{kl} of the same horizontal stripe, except one of them (for definiteness, the first one). We denote this row, respectively, by $a_2(n)^\pm$ and $a_4(n)^\pm$.

4) In the same way, if $\mathrm{C}_k \in \{\, \mathrm{X}_2^\top(n)^\pm, \mathrm{X}_4^\top(n)^\pm \,\}$, we can make zero all blocks Y_{kl} and all columns of the matrices Y_{lk} except one of them (for definiteness, the first one), denoted, respectively, by $a^2(n)^\pm$ or $a^4(n)^\pm$.

5) If $\mathrm{C}_k \in \{\, \mathrm{X}_3(n)^\pm, \mathrm{X}_5(n)^\pm \,\}$, it is non-degenerate. So we can make zero all blocks Y_{kl}. We can also make zero all columns of the blocks Y_{lk}, except the last one, since the latter can contain both t_3 and t_4, while we can only delete one of them. We denote this column, respectively, by $a^3(n)^\pm$ or $a^5(n)^\pm$ and suppose that it only contains t_3 and does not contain t_3^2, since $t_3^2 = -\lambda t_4^2$.

6) In the same way, if $\mathrm{C}_k \in \{\, \mathrm{X}_3^\top(n)^\pm, \mathrm{X}_5^\top(n)^\pm \,\}$, we can make zero all blocks Y_{lk} and all rows of Y_{kl}, except the last one, which we denote, respectively, by $a_3(n)^\pm$ or $a_5(n)^\pm$ and suppose that it only contains t_3 and does not contain t_3^2.

7) Finally, if $\mathrm{C}_k = \mathrm{X}_7(n)$, it is of size $(2n+1) \times 2n$ and of rank $2n$. Hence we can make zero all blocks Y_{lk}. The same observations as above show that we can make zero all rows of Y_{kl}, except 2 of them, namely, the last row and one more row (for definitness, the first one). We denote them, respectively, by $\tilde{a}_7(n)$ and $a_7(n)$ and suppose that $\tilde{a}_7(n)$ only contains t_3 and does not contain t_3^2.

8) In the same way, if $\mathrm{C}_k = \mathrm{X}_7^\top(n)$, we can make zero all blocks Y_{kl} and all columns of Y_{lk}, except 2 of them, namely, the last column and one more column (for definitness, the first one). We denote them, respectively, by $\tilde{a}^7(n)$ and $a^7(n)$ and suppose that $\tilde{a}^7(n)$ only contains t_3 and does not contain t_3^2.

We gather all rows, as well as all columns with the same names. Then we obtain a block matrix $\bar{\mathrm{Y}}$ whose horizontal stripes are those labelled by the symbols $a_i(n)^\pm$ $(2 \leqslant i \leqslant 5)$, $a_7(n)$, $\tilde{a}_7(n)$ and stripes O_s consisting of zero rows of the stripe X_s, while the vertical stripes are those labelled by the symbols $a^j(n)^\pm$ $(2 \leqslant j \leqslant 5)$, $a^7(n)$, $\tilde{a}^7(n)$ and stripes O^r consisting of zero columns of the stripe X^r. We denote by O_s^r the block on the intersection of the horizontal stripe O_s and the vertical stripe O^r. It is convenient to suppose that $\mathrm{O}^1, \mathrm{O}^2, \mathrm{O}^{12}$ are the last among vertical stripes and $\mathrm{O}_3, \mathrm{O}_4, \mathrm{O}_{34}$ are the last among horizontal stripes. Note that the block O_s^r is zero if both $r \neq 12$ and $s \neq 34$.

Obviously, one can make elementary transformations within each stripe with the following restrictions:

- the transformations inside the stripes $a_7(n)$ and $\tilde{a}_7(n)$ are the same;
- the transformations inside the stripes $a^7(n)$ and $\tilde{a}^7(n)$ are the same;
- the transformations inside the stripes $a_0(n)$ and $a^0(n)$ are contragradient.

The transformations between different stripes are obtained from the morphisms of the corresponding matrices X and X′, i.e. the pairs of matrices (S,T) of the form described in Section 1, but without t-part, such that $S\mathrm{X} = \mathrm{X}'T$. Namely, denote by y_i (y'_i) the rows of Y (respectively, of Y′) and by y^i (y'^i) the columns of Y (respectively, of Y′). If $S = (\sigma_{ij})$ and $T = (\tau_{ij})$, we can replace each row y'_i by $y'_i + \sum_j \sigma_{ij} y_j$ and each column y^i by $y^i + \sum_j \tau_{ji} y'^j$.

3. Reduced matrix problem. The O-part

Now we consider the blocks O^r_s. The block O^{12}_{34} is of the form $t_3A_3 + t_4A_4 + t_3^2B$, where A_1, A_2, B are scalar matrices. Using automorphisms of N and L, we can replace the triple (A_4, A_4, B) by $(SA_4T^{-1}, SA_4T^{-1}, SBT^{-1})$, where S, T are arbitrary invertible matrices. Therefore, we can consider the pair (A_3, A_4) as a pencil of matrices [10, Chapter XII] or, the same, a representation of the *Kronecker quiver* [11, Sec. 3.2]. So we can use the Kronecker classification of indecomposable pencils (ibid.), they are the pairs

$$\mathrm{P}^q(n) = (I_n, J_n(q)), \quad \mathrm{P}^\infty(n) = (J_n(0), I_n),$$
$$\mathrm{P}^+(n) = (U, V), \quad \mathrm{P}^-(n) = (U^\top, V^\top),$$

where I_n is the unit $n \times n$ matrix, $J_n(q)$ $(q \in \Bbbk)$ is the $n \times n$ Jordan cell with the eigenvalue q,

$$U = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

(both of size $n \times (n+1)$).

We can add the rows of the stripe O_{34} to the rows of all other stripes. It allows to make zero all columns over each pair $\mathrm{P}^q(n)$ $(q \in \Bbbk \cup \{\,\infty, +\,\})$, except, maybe, the first column over $\mathrm{P}^0(n)$ (only containing t^4) or over P^∞_0 (only containing t_3). Over the pair $\mathrm{P}^-(n)$ we can make zero all columns of the matrix X^{12}_3 except the first one and all columns of X^{12}_4 except the last one. Analogously, one can add the columns of the stripe O^{12} to the columns of all other stripes. Since $t_3 = -t_4$ in the block X^1_{34} and $t_3 = -\lambda t_4$ in the block X^2_{34} (see Table 1), it allows to make zero all rows to the left of each pair $\mathrm{P}^q(n)$ $(q \in \Bbbk \cup \{\,\infty, -\,\})$, except the last row for the pair $\mathrm{P}^1(n)$ in the block O^1_{34} or for the pair $\mathrm{P}^\lambda(n)$ in the block O^2_{34}. To the left of the pair $\mathrm{P}^+(n)$ we can make zero all rows of the matrix X^1_{34} except the first one and all rows of X^2_{34} except the last one. Moreover, we can delete all terms with t_i^2 in these rows. Therefore, all pairs $\mathrm{P}^q(n)$ with $q \in \Bbbk \setminus \{0,1,\lambda\}$ become direct summands of the whole matrix M. So we can further suppose that they do not occur in the decomposition of O^{12}_{34}.

We denote the remaining rows by $p_q(n)$ $(q \in \{\,1,\lambda,+\,\})$ and $\tilde{p}_-(n)$ so that $p_1(n)$ and $\tilde{p}_+(n)$ are in the stripe Y^1, while $p_\lambda(n)$ and $\tilde{p}_+(n)$ are in the stripe Y^2. We also denote the remaining columns by $p^q(n)$ $(q \in \{\,0,\infty,-\,\}$ and $\tilde{p}^-(n)$ so that $p^0(n)$ and $p^-(n)$ are in the stripe Y^4, while $p^\infty(n)$ and $\tilde{p}^-(n)$ are in the stripe Y^3. Again we gather all rows and columns with a fixed label and denote the resulting stripe with the same label. The stripe of the matrix $\bar{\mathrm{Y}}^i$ $(i=1,2)$ consisting of the rows which are zero in O^{12}_{34} is denoted by o_i and the stripe of the matrix $\bar{\mathrm{Y}}_i$ $(i=3,4)$ consisting of the columns which are zero in O^{12}_{34} is denoted by o^i. We call the stripes $p_q(n)$ and $p^q(n)$ *extra*.

Using the description of morphisms between indecomposable pencils (see [11, Sec. 3.2]), we can see the effect of these morphisms on the new stripes. It is convenient to present them using an order on the corresponding symbols: $p < p'$ means that we can add rows (columns) of the stripe p to those of the stripe p' (one can see that then there is no converse transformation). Here is the resulting order:

- $p_q(n) < p_q(n')$ if $q \in \{1,\lambda\}$ and $p^q(n) < p^q(n')$ if $q \in \{0,\infty\}$, $d > d'$;
- $p_+(n) < p_+(n')$, $\tilde{p}_+(n) < \tilde{p}_+(n')$ and $p^-(n) < p^-(n')$, $\tilde{p}^-(n) < \tilde{p}^-(n')$ if $d < d'$;
- $p_+(n) < p_1(n')$, $\tilde{p}_+(n) < p_\lambda(n')$ and $p^-(n) < p^\infty(n')$, $\tilde{p}^-(n) < p^0(n')$ for all d, d';
- $o_i \leqslant p$ and $o^i \leqslant p'$ for every horizontal stripe p and every vertical stripe p'.

Hence the remaining part of the matrices O^r_s can be considered as a representation of a bunch of chains $\mathcal{N}$ in the sense of [2] or [3, Appendix B] (we use the formulations of the second paper). Namely, we have the pairs of chains:

$$\mathcal{E}_1 = \{\, o_1, p_1(n), p_-(n) \mid d \in \mathbb{N}\,\}, \quad \mathcal{F}_1 = \{f^1\},$$
$$\mathcal{E}_2 = \{\, o_2, p_\lambda(n), \tilde{p}_-(n) \mid d \in \mathbb{N}\,\}, \quad \mathcal{F}_2 = \{f^2\},$$
$$\mathcal{E}_3 = \{e_3\}, \quad \mathcal{F}_3 = \{\, o^3, p^\infty(n), p^+(n) \mid d \in \mathbb{N}\,\},$$
$$\mathcal{E}_4 = \{e_4\}, \quad \mathcal{F}_4 = \{\, o^4, p^0(n), \tilde{p}^+(n) \mid d \in \mathbb{N}\,\},$$

with the order defined above and the relation $\sim$ defined as follows:

$$p^{+}(n) \sim \tilde{p}^{+}(n) \quad \text{and} \quad p_{-}(n) \sim \tilde{p}_{-}(n).$$

The description of representations of bunches of chains imply that in this case every indecomposable representation has at most one non-zero element in every row and in every column. Except "trivial," i.e. containing no non-zero elements, they correspond to the following words:

$$\begin{gathered} f^1 - p_-(n) \sim \tilde{p}_-(n), \quad p_-(n) \sim \tilde{p}_-(n) - f^2, \quad f^1 - p_-(n) \sim \tilde{p}_-(n) - f^2, \\ e_3 - p^+(n) \sim \tilde{p}^+(n), \quad p^+(n) \sim \tilde{p}^+(n) - e_4, \quad e_3 - p^+(n) \sim \tilde{p}^+(n) - e_4, \\ f^1 - p_1(n), \quad f_2 - p_\lambda(n), \quad e_3 - p^\infty(n), \quad e_4 - p^0(n), \\ o_1 - f^1, \quad o_2 - f^2, \quad o^3 - e_3, \quad o^4 - e_4. \end{gathered}$$

where the relation '$-$' show the places, where the element is non-zero (actually, it equals 1). We denote the corresponding representations (in the same order) by

$$\begin{gathered} \mathrm{P}_1^-(n), \quad \mathrm{P}_2^-(n), \quad \mathrm{P}_{12}^-(n), \\ \mathrm{P}_3^+(n), \quad \mathrm{P}_4^+(n), \quad \mathrm{P}_{34}^+(n), \\ \mathrm{P}_1^1(n), \quad \mathrm{P}_2^\lambda(n), \quad \mathrm{P}_3^\infty(n), \quad \mathrm{P}_4^0(n), \\ \mathrm{F}^1, \quad \mathrm{F}^2, \quad \mathrm{E}_3, \quad \mathrm{E}_4. \end{gathered} \tag{3.1}$$

For instance, the representation $\mathrm{P}_{12}^-(n)$ is $\left(\mathbf{e}_1^\top \mid \mathbf{e}_n^\top \mid P^-(n) \right)$, where $\mathbf{e}_1^\top$ is in the block O_{34}^1 and $\mathbf{e}_n^\top$ is in the block O_{34}^2. The representation $\mathrm{P}_3^\infty(n)$ is $\left(\frac{\mathbf{e}_1}{\mathrm{P}^\infty(n)} \right)$, where $\mathbf{e}_1$ is in the block O_3^{12}. The representation F^2 is the 1×1 matrix (1) in the block O_{34}^2.

Note that the terms with $t_3^2 = -\lambda t_4^2$ can be deleted from all rows and columns except those which become zero after the above decompositions of X and O. Therefore, such terms can only occur as direct summands of the whole matrix M of the form (t_3^2) (or, the same, (t_4^2)). So we can suppose that there are no such terms at all.

One can easily reconstruct the matrix factorizations which describe the presentations of the corresponding Cohen–Macaulay modules in terms of generators and relations. For instance, the representation $\mathrm{P}_{12}^-(n)$ considered above gives the element of the bimodule $\mathrm{Ext}^1_{\boldsymbol{R}}$:

$$\mathrm{M} = \left(\begin{array}{c|c|ccccc} t_3 & 0 & t_3 & 0 & \dots & 0 \\ 0 & 0 & t_4 & t_3 & \dots & 0 \\ 0 & 0 & 0 & t_4 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & t_3 \\ 0 & t_3 & 0 & 0 & \dots & t_4 \end{array}\right)$$

It means that in the corresponding module M there are $2n+1$ generators $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_{n+1}$, where $u_k \in \boldsymbol{R}_{34}$, while v_k is a preimage of a generating element from $\boldsymbol{R}_{12}$ if $1 \leqslant k \leqslant n-1$, from $\boldsymbol{R}_1$ if $k = n$ and from $\boldsymbol{R}_2$ if $k = n+1$. Thus $(x-y)(x-\lambda y)u_k = 0$. The columns of the matrix M give the following relations for v_k:

$$xyv_k = (1-\lambda)^{-1}(x-\lambda y)u_k + (1-\lambda)^{-1}(y-x)u_{k+1} \quad \text{for } 1 \leqslant k \leqslant n-1,$$
$$xv_n = yu_1,$$
$$yv_{n+1} = xu_{n-1}.$$

Here we use the fact that the projection of $x - \lambda y$ on $\boldsymbol{R}_{34}$ equals $((1-\lambda)t, 0) = (1-\lambda)t_3$, while the projection of $y - x$ equals $(0, (1-\lambda)t) = (1-\lambda)t_4$. Replacing v_k by $(\lambda - 1)v_k$, we obtain the matrix factorization

$$\begin{pmatrix} z_3z_4 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & z_3z_4 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & z_3z_4 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & z_3z_4 & 0 & \dots & 0 & 0 & 0 \\ -z_4 & z_3 & \dots & 0 & 0 & z_1z_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -z_4 & z_3 & 0 & \dots & z_1z_2 & 0 & 0 \\ -z_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & z_2 & 0 \\ 0 & 0 & \dots & 0 & -z_2 & 0 & \dots & 0 & 0 & z_1 \end{pmatrix}$$

Here, like in [8], we set $z_1 = y$, $z_2 = x$, $z_3 = x - y$, $z_4 = x - \lambda y$. They are generators of the kernels of the projections $\boldsymbol{R} \to \boldsymbol{R}_i$. Just in the same way one obtains the modules and matrix factorizations corresponding to the other representations of the bunch of chains $\mathcal{N}$.

Proposition 3.1. *Every indecomposable direct summand of the block* O *is actually a direct summand of the wholle matrix* M.

Proof. The claim is obvious for all summands except $\mathrm{P}^{\mu}(n)$, where $\mu \in \{+,-,0,\infty,1,\lambda\}$ and those from the list (3.1). The main ingredient of the proof in these exceptional case is the following result.

Lemma 3.2. 1) *Let the matrix* M *be of the form:*

$$\mathrm{M} = \left(\begin{array}{c|c|c} 1 & 1 & 0 \\ \hline a_1t_4 & a_2t_4 & t_3 \\ a_3t_4 & a_4t_4 & t_4 \end{array}\right),$$

where the first two column are in the M^1 *and in the* M^2 *stripes, the third column is in the* M^{12} *stripe, the first row is in any horizontal stripe and the other two rows are in the* M_{34} *stripe. Replacing* M *by* $S\mathrm{M}T$*, where* S *and* T *are invertible matrices of the form described on page 77, one can transform it to the form*

$$\left(\begin{array}{c|c|c} 1 & 1 & 0 \\ \hline 0 & 0 & t_3 \\ 0 & 0 & t_4 \end{array}\right).$$

2) *Let the matrix* M *be of the form:*

$$\mathrm{M} = \left(\begin{array}{c|c|c} 1 & a_1t_3 & a_2t_3 \\ \hline 1 & a_3t_4 & a_4t_4 \\ 0 & t_3 & t_4 \end{array}\right),$$

where the first two rows are in the M_3 *and in the* M_4 *stripes, the third row is in the* M_{34} *stripe, the first column is in any vertical stripe and the other two columns are in the* M^{12} *stripe. Replacing* M *by* $S\mathrm{M}T$*, where* S *and* T *are invertible matrices of the form described on page 77, one can transform it to the form*

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & t_3 & t_4 \end{array}\right).$$

Proof. (1) We suppose that the first row is in stripe M_3 (the case M_4 is analogous and the case M_{34} easier). Set $S = \begin{pmatrix} 1 & 0 & 0 \\ x_1t_3 & 1 & 0 \\ x_2t_3 & 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y_1 & y_2 & 1 \end{pmatrix}$. Since $t_3 = -t_4$ in the M^1 stripe and $t_3 = -\lambda t_4$ in the M^2

stripe, we see that the claim means that

$$\begin{aligned} a_1 &= x_1 + y_1, \\ a_2 &= \lambda x_1 + \lambda y_2, \\ a_3 &= x_2 - y_1, \\ a_4 &= \lambda x_2 - y_2. \end{aligned}$$

One esily sees that the determinant of this set of linear equastions is $\lambda^2 - \lambda \neq 0$, since $\lambda \notin \{0, 1\}$.

(2) is proved analogously, the calculations being even easier. □

Now, let our summand be $\mathrm{P}^-(n)$. We have seen that all columns intersecting this block can be made zero. Let H be any block from Table 2 such that not all columns intersecting it has already been made zero. From Table 2 one sees that there are two columns and a row in the block H such that their non-zero coefficients just form the configuration $(\,1 \mid 1\,)$ from the claim (1) of the lemma and one of these columns is the same column which remained non-zero. Then, using the lemma, one can make both these columns zero. All other cases are quite analogous. □

4. Reduced matrix problem. The final step

Proposition 3.1 shows that we can now suppose that $\mathrm{O} = 0$ in the matrix M, so

$$\mathrm{M} = \begin{pmatrix} \mathrm{X} + \mathrm{Y} & \mathrm{Y}^0 \\ \mathrm{Y}_0 & 0 \end{pmatrix},$$

where the coefficients of the matrices $\mathrm{Y}, \mathrm{Y}_0, \mathrm{Y}^0$ are of the form $at_3 + bt_4$. The nonzero elelments of Y can only be on the intersections of stripes $a_i(n)$ (maybe $a_i(n)^\pm$) with the stripes $a^j(n)$ (maybe $a^j(n)^\pm$).

The following of these stripes and the corresponding representations are called the *extra*:

- horizontal stripes $a_3(n)^\pm$, $a_5(n)^\pm$, $\tilde{a}_7(n)$ (they are in the stripe Y^{12});
- vertical stripes $a^3(n)^\pm$, $a^5(n)^\pm$, $\tilde{a}^7(n)$ (they are in the stripe Y_{34}).

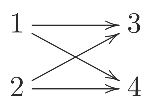
We denote by o_s the part of the matrix Y_0 consisting of its intersections with the extra vertical stripes and by o_u the remaining part of this matrix. Analogously, we denote by o^s the part of the matrix Y^0 consisting of its intersections with the extra horizontal stripes and by o^u its remaning part. We also call the stripes o_s and o^s *extra*.

Let now α be one of the indices $a_i(n)^\pm, a_i(n), \tilde{a}_7(n), o_s, o_u$, β be one of the indices $a^i(n)^\pm, a^i(n), \tilde{a}^7(n), o^s, o^u$. We denote by Y_α^β the block on the intersection of the horizontal stripe α and the vertical stripe β. Note that the cases when both $\alpha \in \{o_s, o_u\}$ and $\beta \in \{o^s, o^u\}$ are impossible.

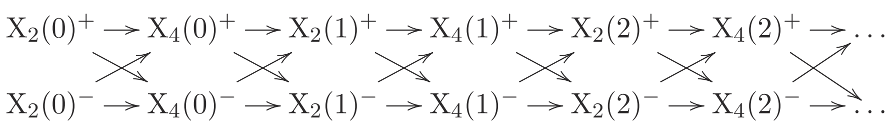
Using transformations of the matrix Y we can make zero some of these blocks. For instance, if $\mathrm{C}_k = \mathrm{X}_2(n)^{\pm}$ and $\mathrm{C}_l = \mathrm{X}_3(m)^{\pm}$, we can make $Y_{kl} = 0$. Indeed, before we could make zero all rows except the first one and all columns except the last one. But using the last row of $\mathrm{X}_3(n)^{\pm}$, we can make zero the last remaning element. Therefore, we can suppose that $Y_\alpha^\beta = 0$ if $\alpha = a_2(n)^{\pm}$ and $\beta = a^3(n)^{\pm}$. Just in the same way, one can make zero the blocks Y_α^β in all cases when one of the indices α, β is extra while the other one is not.

Now we have to see what transformations of the remaining blocks are induced by the morphisms between different direct summands of the part X. Obviously, we can also add the rows (columns) of the stripe Y_0 (respectively, Y^0) to all other horizontal (respectively, vertical) stripes.

The representations of the first kind containing no horizontal stripe X_{34} and no vertical stripe X^{12} are indeed representations of the quiver Γ of type $\tilde{A}_3$, namely,



Namely, the representations of types $\mathrm{X}_2(n)^{\pm}$ and $\mathrm{X}_4(n)^{\pm}$ constitute the *prepojective part* of its Auslander–Reiten quiver [11], which is



Note that the map $\mathrm{X}_2(n)^+ \to \mathrm{X}_4(n)^+$ idenity $\mathrm{X}_2(n)^+$ with the part of $\mathrm{X}_4(n)^+$ obtained by deleting the last column of the X^1-stripe (analogously for $\mathrm{X}_2(n)^-$ and $\mathrm{X}_4(n)^-$ with the X^2-stripe instead). Note that if we delete the last row and the last column of the representation $\mathrm{X}_7(n+1)$, we obtain the representation equivalent to $\mathrm{X}_4(n)$. Therefore, there are morphisms $\mathrm{X}_2(n)^{\pm} \to \mathrm{X}_7(n+1)$ and $\mathrm{X}_7(n+1) \to \mathrm{X}_4(n)^{\pm}$.

The representations $\mathrm{X}_0(n, \lambda^{-1})$ form a *homogenious tube* from the regular part of the Auslander–Reiten quiver. As there are morphisms from any preprojective representation to any representation of a homogenious tube, there are morphisms from $\mathrm{X}_2(n)^{\pm}$, $\mathrm{X}_4(n)^{\pm}$ and $\mathrm{X}_7(n)$ to $\mathrm{X}_0(m, \lambda^{-1})$ for any m. There are also morphisms

$$\mathrm{X}_0(n, \lambda^{-1}) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{X}_0(m, \lambda^{-1})$$

for arbitrary n, m. Moreover, one can easily check that if $m > n$ the map α does not imply the non-zero row and the non-zero column of the corresponding part of Y.

It gives the order for the stripes $a_0(n)$, $a_2(n)^{\pm}$, $a_4(n)^{\pm}$ and $a_7(n)$ as described below ($a < a'$ means that the rows of the stripe a can be added to those of the stripe a'):

$$a_2(n)^{\pm} < a_7(n+1) < a_4(n)^{\pm} < a_2(n+1)^{\pm} < a_0(m+1) < a_0(m)$$

for any n, m and any combination of $+, -$. Note that $\mathrm{X}_2(n)^+$ and $\mathrm{X}_2(n)^-$ are incomparable, as well as $\mathrm{X}_4(n)^+$ and $\mathrm{X}_4(n)^-$. Certainly, the stripe o_u is the least among non-extra horizontal stripes.

For the transposed representations $\mathrm{X}_2^{\top}(n)^{\pm}$, $\mathrm{X}_4^{\top}(n)^{\pm}$ (which constitute the *preinjective component* of the Auslander–Reiten quiver) and $\mathrm{X}_7^{\top}(n)$, together with the representations $\mathrm{X}_0(n, \lambda^{-1})$ we have the analogous morphisms going in the opposite directions. Nevertheless, since the transformations of columns are contragradient to those of rows, it gives the same order:

$$o^u < a^2(n)^{\pm} < a^7(n+1) < a^4(n)^{\pm} < a^2(n+1)^{\pm} < a^0(m+1) < a^0(m)$$

for any n, m and any combination of $+, -$.

We consider now the extra stripes. Note that an extra representation S has either a row which is non-zero only at the (12)-place or a column which is non-zero only at the (34)-place. Suppose that $\mathrm{S} \neq \mathrm{X}_9(n)$. If we delete this row and the last column or, respectively, this column and the last row, we obtain a non-extra representation. Namely, if we delete a row, we obtain (by types)

- $X_1(2n)^{\mp+}$ from $X_3(n)^{\pm}$;
- $X_1(2n+1)^{+\pm}$ from $X_5(n)^{\pm}$;
- $X_4^{\top}(n+1)^{+}$ from $X_7^{\top}(n)$.

We call these representations *truncated*.

Note that the representations of type X_1 form a *special tube* of representations of the quiver Γ, so there are morphisms to these representations from any of representations of type X_2 or X_4 and morphisms from these representations to any of representations of type $\mathrm{X}_2^{\top}$ or $\mathrm{X}_4^{\top}$.

If we add the (12)-column of one of the preceding representations to the (12)-column of another one, we obtain two non-zero elements in the latter column. One of them can be deleted using the row with a unique non-zero element, which is in the last column. If we delete the other non-zero element using another row, the result becomes as if we add a row of the second truncated representation to a row of the first one. So it is possible if there is a non-zero morphism of these truncated representations. Therefore, we obtain the following partial order for extra stripes:

$$o^s < \tilde{a}^7(n) < \tilde{a}^7(n+1) < a^3(m)^{\pm} < a^5(m)^{\pm} < a^3(m+1)^{\pm}$$

for any n, m and any combination of $+, -$.

Analogously, for extra horizontal stripes we obtain the order

$$o_s < \tilde{a}_7(n) < \tilde{a}_7(n+1) < a_3(m)^{\pm} < a_5(m)^{\pm} < a_3(m+1)$$

for any n, m and any combination of $+, -$.

One can also check that there are no transfromations between extra and non-extra stripes.

Therefore, for the remaining part of the matrices Y, Y_0, Y^0 we obtain a problem which can also be formulated in terms of a bunch of chains $\mathcal{Y}$. Namely, $\mathcal{Y}$ consists of two pairs of chains

$$\mathcal{E}_1 = \{ o_u, a_0(n), a_2(n), a_4(n), a_7(n) \},$$
$$\mathcal{F}_1 = \{ o^u, a^0(n), a^2(n), a^4(n), a^7(n) \};$$

and

$$\mathcal{E}_2 = \{ o_s, a_3(n), a_5(n), \tilde{a}_7(n) \}, \quad \mathcal{F}_2 = \{ o^s, a^3(n), a^5(n), \tilde{a}^7(n) \}$$

with the order

$$o_u < a_2(n) < a_7(n+1) < a_4(n) < a_2(n+1) < a_0(m+1) < a_0(m),$$
$$o^u < a^2(n) < a^7(n+1) < a^4(n) < a^2(n+1) < a^0(m+1) < a^0(m),$$
$$o_s < \tilde{a}_7(n) < \tilde{a}_7(n+1) < a_3(m) < a_5(m) < a_3(m+1),$$
$$o^s < \tilde{a}^7(n) < \tilde{a}^7(n+1) < a^3(m) < a^5(m) < a^3(m+1)$$

for any n, m, and the relation $\sim$ given by the rules

$$o_u \sim o_s \text{ and } o^u \sim o^s;$$
$$a_i(n) \sim a_i(n) \text{ and } a^i(n) \sim a^i(n) \text{ for } i \in \{ 2, 3, 4, 5 \};$$
$$a_7(n) \sim \tilde{a}_7(n) \text{ and } a^7(n) \sim \tilde{a}^7(n);$$
$$a^0(n) \sim a_0(n).$$

We call the elements of these chains the $\mathcal{Y}$-*letters*.

So we obtain now the following complete description of isomorphism classes for the category $\mathbb{E}$, hence for Cohen–Macaulay modules over the T_{44} singularity $\boldsymbol{R}$.

Theorem 4.1. *Cohen–Macaulay modules over the ring* $\boldsymbol{R}$ *are those of the first level [8], modules from the list 3.1 and modules corresponding to representations of the bunch of chains* $\mathcal{Y}$.

5. Generators and relations

One can interpret this answer in terms of generators and relations (or matrix factorizations) analogously to the procedure described in [8]. As matrix factorizations for the modules of the first level and those of the list 3.1 are known, we only have to consider indecomposable modules that are defined by indecomposable representations of the bunch of chains $\mathcal{Y}$. This time there is a vast variety of $\mathcal{Y}$-words, hence of representations of this bunch of chains, we just explain how to construct generators and relations and present examples.

So, let B be an indecomposable representation of the bunch of chains $\mathcal{Y}$. It is defined by one of the combinatorial data:

1) *usual word* w, i.e. a $\mathcal{Y}$-word such that it does not begin with $a - b$ and does not end with $b - a$, where $a \sim a$;
2) a pair (w, δ), where $\delta \in \{+, -\}$ and w is a *special word* w, i.e. a $\mathcal{Y}$-word such that either it begins with $a - b$ or it ends with $b - a$, where $a \sim a$, but not both;
3) a quadruple $(w, \delta_0, \delta_1, m)$, where $m \in \mathbb{N}$, $\delta_0, \delta_1 \in \{+, -\}$ and w is a *byspecial word* w, i.e. a $\mathcal{Y}$-word such that it begins with $a - b$ and ends with $b' - a'$, where $a \sim a$ and $a' \sim a'$;
4) a triple (w, μ, m), where $\mu \in \Bbbk$, $m \in \mathbb{N}$ and w is a cyclic word.

There are some restrictions on these data (see [2,3]), but we do not precise them here, since they do nor imply the procedure of the construction of generators and relations.

The ends a and a' of a special or a bispecial word are called *special ends.* If the word w has a part $a \sim a$, we replace the first a by a^+ and the second one by a^-. If a is a special end of a special word, we replace it by a^δ. If a and a' are the ends of a bispecial word, we replace them by a^{δ_0} and ${a'}^{\delta_1}$. The resulting word is denoted by $\tilde{w}$.

Let now M be the Cohen–Macaulay defined by one of the described combinatorial data, $\tilde{w} = x_1 r_1 x_2 r_2 \dots r_{n-1} x_n$, Υ be the matrix describing the corresponding representation of the bunch of chains $\mathcal{Y}$. Every letter $x_i \notin \{\, o_u, o_s, o^u, o^s \,\}$ corresponds to a matrix X_i from the Table 2, hence to an $\boldsymbol{R}$-module M_i of the first lever. Moreover, we have chosen above a fixed generator g_i of this module, corresponding to the chosen row or column of the matrix X_i. Note that this row (column) is in the $\boldsymbol{R}_{34}$-stripe (respectively, in the $\boldsymbol{R}_{12}$-stripe) if x_i denotes an extra stripe. If $x_i \in \{ o^u, o^s \}$, we set $M_i = \boldsymbol{R}_{12}$ and if $x_i \in \{ o_u, o_s \}$, we set $M_i = \boldsymbol{R}_{34}$. In these cases M_i has one generator which we also denote by g_i. We call g_i *marked generators.* If $x_i \in \{ o^u, o^s \}$, the relation for the marked generator is $z_{12}g_i = 0$ and $x_i \in \{ o_u, o_s \}$, the relation for the marked generator is $z_{34}g_i = 0$. If w contains a subword $x_i \sim x_{i+1}$, where $x_{i+1} \neq x_i$ and $x_i \in \{ o_u, o_s, o^u, o^s \}$, we call $i+1$ a *non-essential index.* In this case we set $g_{i+1} = g_i$. All other indices are called *essential.* We denote by $|w|$ the set of essential indices. Now the procedure of constructing generators and relations for M is the following.

Procedure 5.1. 1) We consider the module of the first level $\tilde{M} = \bigoplus_{i\in|w|} M_i^m$, where $m = 1$ if w is a usual or a special word. The set of generators for M is the union of the sets of generators for M_i^m. If $m > 1$, we denote the copies of marked generators g_i by g_{ij} $(1 \leqslant j \leqslant m)$.

2) The relations for non-marked generators of M remain the same as in $\tilde{M}$ (taken from the matrices of [8, Table 2]).
3) Also the relations for the marked generators g_i remain the same as in $\tilde{M}$ if $x_i \in \mathcal{E}_1 \cup \mathcal{E}_2$ (i.e. corresponds to a row).
4) If $x_i \in \mathcal{F}_1 \cup \mathcal{F}_2$ (i.e. corresponds to a column), we add to the relations for the marked generators g_i the terms $\gamma_{ij} z_s u_j$, where γ_{ij} are the coefficients of the i-th column of the matrix Υ and u_j are the marked generators of $\tilde{M}$ from the $\boldsymbol{R}_s$-part, where $s \in \{3, 4\}$. If u_j comes from an extra stripe, it must be replaces by $z_r h_j$ $(r \in \{1, 2\})$, where h_j is the generator of the representation M_j such that $z_r h_j = g_j$ (it comes from the column of the form $\mathbf{e}_m^\top$). Then the corresponding term is $\gamma_{ij} z_s z_r h_j$.

Note that it is convenient to write first all representations corresponding to the letters with "lower" indices (like $a_7(n)$ or o_u). Then the matrix Φ will be lower triangular, like most matrices in [8, Table 2].

Example 5.2. Let M corresponds to the pair $(w, -)$, where $w = o_s \sim o_u - a^2(1) \sim a^2(1) - a_4(2)$, thus $\tilde{w} = o_s \sim o_u - a^2(1)^+ \sim a^2(1)^- - a_4(2)^-$ and

$$\Upsilon = \left(\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \end{array} \right)$$

Then the matrix Φ from the matrix factorization of F is

$$\begin{pmatrix} z_3z_4 & 0 & 0 & 0 \\ 0 & \Phi_4(2)^- & 0 & 0 \\ \Upsilon_1 & 0 & \Phi_2^*(1)^+ & 0 \\ \Upsilon_2 & \Upsilon_3 & 0 & \Phi_2^*(1)^- \end{pmatrix}$$

where $\Phi_2^*(1)^\pm$ and $\Phi_4(2)^-$ are the matrices from [8, Table 2], $\Upsilon_1 = \Upsilon_2 = \begin{pmatrix} z_1 & 0 & 0 \end{pmatrix}$ and Υ_3 is 6×3 matrix with z_1 at the (11)-place and 0 elsewhere.

Example 5.3. Let M corresponds to the triple $(w, \mu, 2)$, where $w = a^3(2) \sim a^3(2) - a_5(1) \sim a_5(1)$. Then

$$\Upsilon = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1+\mu & 0 & 1 & 0 \\ 1 & 1+\mu & 0 & 1 \end{array}\right)$$

and

$$\Phi = \begin{pmatrix} \Phi_5^*(1)^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Phi_5^*(1)^- & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Phi_5^*(1)^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Phi_5^*(1)^- & 0 & 0 & 0 & 0 \\ \tilde{\Upsilon} & 0 & (\mu+1)\tilde{\Upsilon} & \tilde{\Upsilon} & \Phi_3(2)^+ & 0 & 0 & 0 \\ 0 & \tilde{\Upsilon} & 0 & (\mu+1)\tilde{\Upsilon} & 0 & \Phi_3(2)^+ & 0 & 0 \\ \tilde{\Upsilon} & 0 & \tilde{\Upsilon} & 0 & 0 & 0 & \Phi_3(2)^- & 0 \\ 0 & \tilde{\Upsilon} & 0 & \tilde{\Upsilon} & 0 & 0 & 0 & \Phi_3(2)^- \end{pmatrix},$$

where $\tilde{\Upsilon}$ is 7×4 matrix having z_2z_3 at the (74)-place and 0 elsewhere.

Just in the same way we obtain matrix factorization in all cases.

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Received by the editors: 15.05.2019.